Homework 2: Due October 21

1. Your answers should be complete, clear and concise.
2. You are strongly encouraged to work together as this maximizes the amount of mathematics that you learn. However, after discussions with others, you should write up the solutions yourself.
3. You should write down whom you have discussed the problems with (whether they are in the class or not).

1. (a) Find an example of a connected graph G and some value of p such that percolation with parameter p on G is such that the number of infinite clusters is 1 with positive probability and is 2 with positive probability.
(b) Does there exist a graph connected G and some value of p such that percolation with parameter p on G is such that the number of infinite clusters is 1 with positive probability and is 0 with positive probability?

2. (a). Let G be a graph which is a tree. Show that for every $p < 1$, the number of infinite clusters is either 0 a.s. or infinite a.s. If you can't do this for general trees, try to do it where you assume that every vertex has the same degree.
(b). Why does the Burton Keane proof not go through here?

3. It is the case that the critical value for site percolation on Z^3 is less than $1/2$ which you can assume. Consider the following non-i.i.d. edge

percolation model. Assign each site the value 0 or 1 independently with prob. 1/2 each. Now place an edge between pairs of nearest neighbor sites which have the same value.

(a). Explain why this edge model is not i.i.d.

(b). How many infinite clusters are there in this edge percolation model?

(c). Where does the Burton-Keane proof break-down here?

4. Find a rooted tree T such that the number of vertices at distant n from the root is at least 2^n for every n but such that $p_c = 1$. The tree is not required to have the form that it had in the first homework exercises where everyone at a certain level had the same number of children.

5. (a) Find an example of random variables $X_1, \ldots, X_n$ such that $P(X_i = 1) = 1/2 = P(X_i = 0)$ for all i but where the FKG inequality fails for some pair of increasing events A and B. (The X_i's are not necessarily i.i.d.).

(b). Where does the proof of the FKG inequality fail?

(c). Find an example of random variables $X_1, \ldots, X_n$ such that $P(X_i = 1) = 1/2 = P(X_i = 0)$ for all i but where the BK inequality fails for some pair of increasing events A and B. (The X_i's are not necessarily i.i.d.).

6. CHALLENGE PROBLEM!! (Try at your own risk.)

Try to find an event A in $\{0,1\}^n$ such that $P_{1/2}(A) \in (1/4, 3/4)$ and where

$$\sup_{i \in \{1,\ldots,n\}} P_{1/2}(i \text{ is pivotal})$$

is quite small. How small can you make this?