Homework 3: Due November

1. Your answers should be complete, clear and concise.
2. You are strongly encouraged to work together as this maximizes the amount of mathematics that you learn. However, after discussions with others, you should write up the solutions yourself.
3. You should write down whom you have discussed the problems with (whether they are in the class or not).

1. Here is a fun problem which will in fact connect to what we have been doing. 3 people, working as a team, will be put in a room tomorrow, and a random colored hat (red or blue, independently, each with probability 1/2) will be placed on their heads. They can see the other hats but not their own. Then, with no communication, each one needs to either (1) guess that their own hat is red, (2) guess that their own hat is blue or (3) abstain from guessing. They can agree today on a "joint strategy" that they will use tomorrow (but once they are in the room tomorrow, no further communication is allowed). The team wins if nobody makes an incorrect guess and a legal strategy is one which has the property that in all cases at least one person makes a real guess.
Goal: Find the a legal strategy which maximizes the probability that the team wins.

Hint (1): One strategy is that player 1 always guesses red and the other players always abstain. This clearly has a probability of winning of exactly 1/2.

Hint (2): Since the rules are such that at least one person guesses in all cases, we conclude, by independence of the colors, that if a person guesses, he/she will be right exactly 1/2 the time. Hence one cannot do better than 1/2.

(a). Do you believe the above argument?

(b). Can you do better than 1/2?

(c). If you can do better, identify the mistake in the above argument. (If you find a better strategy, it does NOT suffice to simply say that the above argument must be wrong since the conclusion is wrong; you need to find the actual mistake in the argument.)

2. In the written proof that $p_c = 1/2$, we argued on pages 6-7 that with probability at least ϵ_1, there is a left to right crossing of $[0, 2n] \times [0, n]$ and that with probability at least ϵ_2 there is a top-to-bottom open dual crossing of $[1/2, n - 1/2] \times [-1/2, 2n + 1/2]$. Obviously, we can't have both events. The following is more or less claimed in the notes.

Claim: With probability at least $\epsilon_1\epsilon_2$, there a left to right crossing τ of $[0, 2n] \times [0, n]$ and a "top-to-bottom" open dual crossing in $[1/2, n - 1/2] \times [-1/2, 2n + 1/2]$ from the top of $[1/2, n - 1/2] \times [-1/2, 2n + 1/2]$ to one step above τ.

"Proof": We know with probability at least ϵ_1, there is some left to right crossing τ of $[0, 2n] \times [0, n]$. Now if we condition on a particular (not necessarily open) left to right crossing τ actually being open, by independence there will be a "top-to-bottom" open dual crossing in $[1/2, n - 1/2] \times$

$[-1/2, 2n+1/2]$ from the top of $[1/2, n-1/2] \times [-1/2, 2n+1/2]$ to one step above τ with probability at least ϵ_2. Since this is true for any fixed (not necessarily open) left to right crossing τ, this completes the proof. "qed"

(a). Is this proof valid?

(b). Is there some relationship between the issue arising in this "proof" and the "proof" in the hint of problem 1 above?

(c). How would you, in some detail, remedy the above "proof"?