Percolation

Lecture 1

Notes by Jeffrey Steif, transcribed by Oskar Sandberg

The goal of the first lecture is to introduce the fundamental elements of percolation, and to show that bond percolation on $\mathbb{Z}^d$ has a non-trivial critical value.

Graph Theoretic Basics

As usual, we let $\mathbb{Z}^d = \{(x_1, x_2, \dots, x_n) : x_i \in \mathbb{Z}\}$. On this we will use two different norms, the L^1 norm:

$$|x| = \sum_{i=1}^{d} |x_i|$$

and the L^∞ norm:

$$||x|| = \max_{i=1\dots d}\{|x_i|\}.$$

$\mathbb{Z}^d$ becomes a graph when we place edges between all $x, y \in \mathbb{Z}^d$ with $|x - y| = 1$ (that is, we $2d$ edges from each point to its neighbors following each axis). We call this graph:

$$L^d = (\mathbb{Z}^d, E^d).$$

Definition 1. *A path is an alternating sequence of vertices and edges*

$$x_0, e_0, x_1, e_1, \dots, x_{n-1}, e_{n-1}, x_n$$

where all the x_i are distinct, and each $e_i = \{x_i, x_i + 1\}$. Paths may also be infinite.

Definition 2. *A circuit is a sequence:*

$$x_0, e_0, x_1, e_1, \dots, x_{n-1}, e_n, x_0$$

such that $x_0, e_0, \dots, x_{n-1}$ is a path, and $e_n = \{x_{n-1}, x_0\}$.

Additionally, let the boundary of a subset $A \in \mathbb{Z}^d$, denoted by ∂A, be

$$\partial A = \{x \in A : \exists y \in A^c \text{ with } \{x, y\} \in E^d\}$$

and:

$$\begin{aligned} B(n) &= [-n, n]^d = \{x : \|x\| \leq n\} \\ S(n) &= \{x : |x| \leq n\} \end{aligned}$$

The first is denoted by B for Ball, but is, in fact, a square in the plane (and a cube elsewhere). The second is denoted by S for sphere. Both sets are filled - their respective boundaries are denoted by ∂B and ∂S.

Bond Percolation

For some parameter $p \in [0, 1]$, bond percolation assigns each edge in L^d as open (a.k.a. retained) independently with probability p. Otherwise it is closed (removed). An open

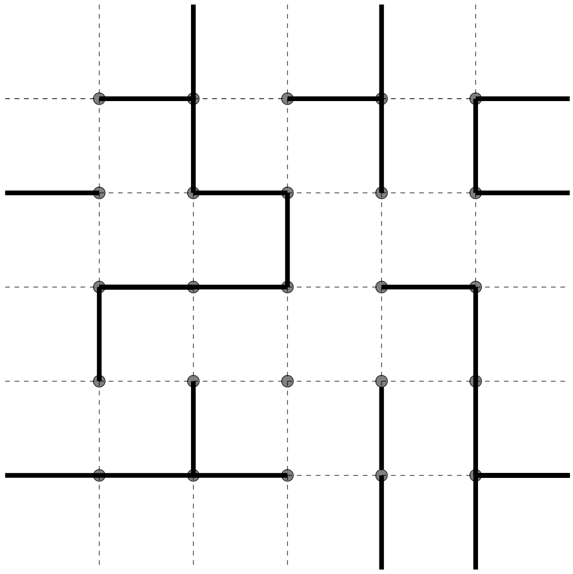

Figure 1: Bond percolation on L^2.

edge is also said to have state 1, a closed edge has state 0. Z^d together with the set of open edges forms a new, random, graph, whose properties we study.

Short Formality

The sample space for percolation is:

$$\Omega = \prod_{e \in E^d} \{0, 1\}.$$

An element of Ω is of the form:

$$\omega = \{\omega(e) : e \in E^d\}$$

where $\omega(e) = \{0, 1\}$. The probability measure that allows edges to be open or closed independently is given by the product measure over all the edges:

$$P_p = \prod_{e \in E^d} (p\delta_1 + (1-p)\delta_0).$$

There is a one-to-one correspondence between elements of Ω and subsets of $\mathbf{E}^d$ given by:

$$\omega \in \Omega \leftrightarrow K(\omega) = \{e \in E^d : \omega(e) = 1\}.$$

That is $K(\omega)$ is simply the set of edges which are open in ω (the rest are closed).

We now fix $p \in [0, 1]$. Consider the random subgraph of L^d consisting of vertices Z^d and the open edges. The components of this graph are called (open) clusters.

For $A, B \subseteq \mathbb{Z}^d$ we let $A \leftrightarrow B$ denote the event that there exists an open path from some vertex in A to some vertex in B.

Definition 3. *For $x \in \mathbb{Z}^d$, $C(x)$ is the cluster containing x. That is:*

$$C(x) = \{y \in \mathbb{Z}^d : \{x\} \leftrightarrow \{y\}\}.$$

Note that $C(x)$ is random.

Clearly, $C(x) = \{x\}$ iff all of the edges adjacent to x are closed. The distribution of the size of $C(x)$ does not depend on x, and so we concentrate on studying $C(0)$.

The Object of Principle Interest

Definition 4. *The percolation function, $\theta(p)$ is given by:*

$$\theta(p) = \mathbf{P}_p(|C(0)| = \infty).$$

It is clear that $\theta(0) = 0$ (since $C(0)$ is always $\{0\}$ in this case), and $\theta(1) = 1$ (since then $C(0) = \mathbb{Z}^d$). Also $\theta(p) < 1$ if $p < 1$.

Exercise 1: Show that $\theta(p)$ is non-decreasing in p.

A bigger question is: Is $\theta(p)$ continuous? This question is not completely resolved, and is the main open question in the field.

Since $\theta(p)$ is increasing, and since $\theta(0) = 0$ and $\theta(1) = 1$, there must be a critical value, p_c, after which $\theta(p)$ takes positive values. That is:

$$\exists p_c(d) : \theta(p) = \begin{cases} 0 & \text{for } p < p_c(d) \\ > 0 & \text{for } p > p_c(d) \end{cases}$$

We can thus also define the critical value by:

$$\begin{aligned} p_c &= \sup\{p : \theta(p) = 0\} \\ &= \inf\{p : \theta(p) > 0\}. \end{aligned}$$

It is of course possible that $p_c = 0$ or that $p_c = 1$, but that would not make for a very interesting subject of study. In fact, this is not the case:

Theorem 1. $\forall d \geq 2$ *the critical value for bond percolation in non-trivial:*

$$p_c(d) \in (0, 1).$$

Exercise 2: $\theta(p)$ is non-decreasing in d, which implies that $p_c(d)$ is non-decreasing in d.

Proof. (Of Theorem 1.) We must show two things:

1. For sufficiently small $p > 0$, $\theta(p) = 0$.

2. For sufficiently large $p < 1$, $\theta(p) > 0$.

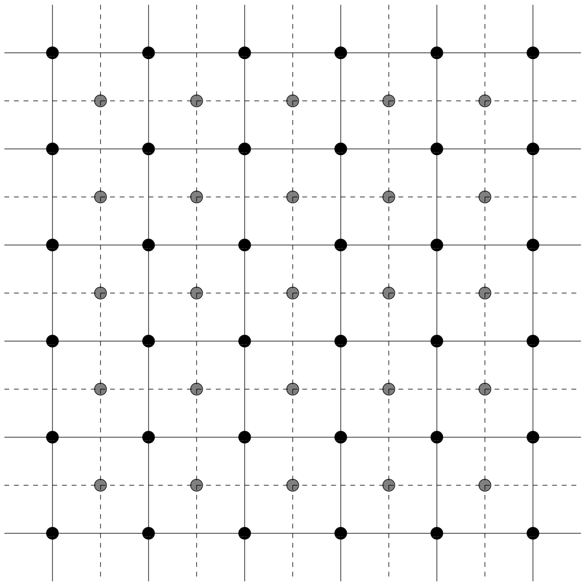

Figure 2: A section of L^2 and its dual $(L^2)^*$.

We do each in turn:

1. In particular, we will show that if $p < 1/2d$ then $\theta(p) = 0$.

   Let $\sigma(n)$ be the number of paths in L^d of length n starting from 0. This value is very hard to calculate, but we can easily bound it by looking at the number of choices in each step:
   $$\sigma(n) \leq (2d)^n.$$
   Let the random variable N_n be the number of paths which are open. Since each bond is open or closed independently with probability p, it follows that:
   $$E_p[N_n] = \sigma(n)p^n \leq (2dp)^n$$
   Since we have chosen $p < 1/2d$, the right hand side $\to 0$ as $n \to \infty$. Now for all natural numbers n it holds that:
   $$\begin{aligned} \theta(p) &= \mathbf{P}_p(|C(0)| = \infty) \\ &\leq \mathbf{P}(N_n \geq 1) \\ &\leq E[N_n] \end{aligned}$$
   whence it follows that $\theta(p)$ is smaller than any positive number, and thus $\theta(p) = 0$.

2. In view of Exercise 2, it is enough to show that $p_c(2) < 1$ for the same thing to hold in all dimensions.

   To do this, we define the dual graph of L^2, $(L^2)^*$, as follows. In each quadrant of L^2, we place a vertex and then we connect each vertex with the ones above, below, and to the left and right. In other words, the dual graph is L^2 shifted by [1/2, 1/2], and each edge in $(L^2)^*$ crosses one edge in L^2. See Figure 2.

   There is a one-to-one correspondence between E^2 and $(E^2)^*$ defined by letting each edge correspond to the one it crosses. We can define a coupled percolation on $(L^2)^*$ as follows: each edge in $e^* \in (E^2)^*$ is open iff the corresponding edge in E^2 is closed.

Lemma 1. (Due to Whitney) *For any configuration of open and closed edges:*

$$|C(0)| < \infty \Leftrightarrow \exists \text{ an open circuit in } (L^2)^* \text{ surrounding 0.}$$

Now, let $\rho(n)$ be the number of circuits of length n in $(L^2)^*$ surrounding 0. It is not difficult to see that $\rho(n) \leq n4^{n-1}$. Thus:

$$\begin{aligned}
\mathbf{P}_p(|C(0)| < \infty) &= \mathbf{P}_p\left(\exists \text{ an open circuit in } (L^2)^* \text{ surrounding } 0.\right) && (1)\\
&\leq E_p\left(\text{Number of open circuits around } 0.\right) && (2)\\
&= \sum_{n=1}^{\infty} \rho(n)(1-p)^n && (3)\\
&= \sum_{n=1}^{\infty} n4^{n-1}(1-p)^n < \infty && (4)
\end{aligned}$$

where the last inequality holds if $p > 3/4$. That a probability is less than infinity is perhaps not surprising, but as $p \to 1$, the final, bounded, sum $\to 0$. Hence there $\exists p < 1$ such that $\mathbf{P}_P(|C(0)| < \infty) < 1$, which implies:

$$\theta(p) = P_p(|C(0)| = \infty) > 0.$$

□