Percolation

Lecture 2

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Last time, we defined $A \leftrightarrow B$ as the event:

$$U = \{\exists a \in A, b \in B, \text{ and an open path from } a \text{ to } b\}.$$

That U is an event means, of course, that $U \subseteq \{0,1\}^{E^d}$.

Exercise 3: Show that U is open in the product toplogy.

Now the solution to one of the exercises from last time:

Proposition 1. *If $p_1 \leq p_2$ then $\theta(p_1) \leq \theta(p_2)$.*

We will prove this by superimposing the two objects on the same probability space. This proof method is called a "coupling" and it is a very common and powerful tool in probability theory. The coupling used here is very direct, but they can be more involved.

Proof. Let $(X_e, Y_e)_{e \in E^d}$ be independent random vectors with the following distribution:

(X_e, Y_e)	Probability
$(1,1)$	p_1
$(0,1)$	$p_2 - p_1$
$(0,0)$	$1 - p_2$

Note that, in particular, $\mathbf{P}(X_e) = p_1$ and $\mathbf{P}(Y_e) = p_2$. Also, because the value $(1,0)$ never occurs, it holds that:

$$X_e = 1 \Rightarrow Y_e = 1. \tag{1}$$

Define the following events:

$$U = \left\{ \begin{array}{l} \exists \text{ an infinite path in } \mathbb{Z}^d \text{ starting at } 0 \\ \text{where } \mathbf{X_e = 1} \ \forall e \text{ in the path.} \end{array} \right\}$$

$$V = \left\{ \begin{array}{l} \exists \text{ an infinite path in } \mathbb{Z}^d \text{ starting at } 0 \\ \text{where } \mathbf{Y_e = 1} \ \forall e \text{ in the path.} \end{array} \right\}$$

Because the distribution of open edges in each case is the same as if they had been defined separately, it holds that $\mathbf{P}(U) = \theta(p_1)$ and $\mathbf{P}(V) = \theta(p_2)$. From (1) we know that $U \subseteq V$, and hence

$$\theta(p_1) = \mathbf{P}(U) \leq \mathbf{P}(V) = \theta(p_2).$$

$\square$

Last time we showed that $\forall d \geq 2$ there is a critical value $p_c(d) \in (0,1)$ such that:

$$\theta(p) = \begin{cases} 0 & \text{for } p < p_c(d) \\ > 0 & \text{for } p > p_c(d) \end{cases}.$$

Until now, we have been dealing with $\theta(p)$ which gives the probability that 0 is part of an infinite cluster. One can define a similar function $\psi(p)$ by:

$$\psi(p) = P_p(\exists \text{ an infinite cluster somewhere in } L^d).$$

Proposition 2. *With the above definitions, it holds that:*

1. $\theta(p) = 0 \Rightarrow \psi(p) = 0.$
2. $\theta(p) > 0 \Rightarrow \psi(p) = 1.$

Proof. 1. This follows from countable additivity:

$$\begin{aligned}\psi(p) &= \mathbf{P}_p(\exists \text{ an infinite cluster}) \\ &= \mathbf{P}_p(\bigcup_{x \in \mathbb{Z}^d} \{C(x) = \infty\}) \\ &\leq \sum_{x \in \mathbb{Z}^d} \mathbf{P}_p(|C(x)| = \infty) = 0.\end{aligned}$$

2. In order to show this, one actually shows that $\psi(p)$ must be either 0 or 1, with the help of a so-called 0-1 law. Since $\psi(p) \geq \theta(p)$, it then follows that $\psi(p) = 1$ when $p > p_c(d)$.

   To do this, we need the concept of a *tail event.* Formally, tail events are defined in terms of σ-algebras and alike, but intuitively it is a clear concept. If we have a infinite sequence of random variables, then a tail event is an event for which we can determine if it occurs even when any finite set of variables is excluded from knowledge. The simplest example is whether an independent sequence converges - not knowing the values of a finite number of variables cannot change whether we know this is true. In light of this, we have:

   Theorem 1. (Kolmogorov's 0-1 Law) *Consider a sequence of independent random variables $(X_i)_{i>0}$. Any tail event of this sequence has probability either 0 or 1.*

   That $\psi(p)$ is 0 or 1 follows by application of this law: Take any ordering of the E^d, and let X_n be an indicator that the n-th edge is on. This is a sequence of independent random variables, and one can see that the existence of an infinite cluster is a tail event.

□

What vital questions can we ask about the critical value?

- What is the value of $p_c(d)$?

  It is known that $p_c(2) = 1/2$, but the value is not known for any $d > 2$, nor believed to be easily expressible. $p_c(d)$ is known to have asymptotic order of about $1/2d$ (the lower bound from the last lecture) as $d \to \infty$.

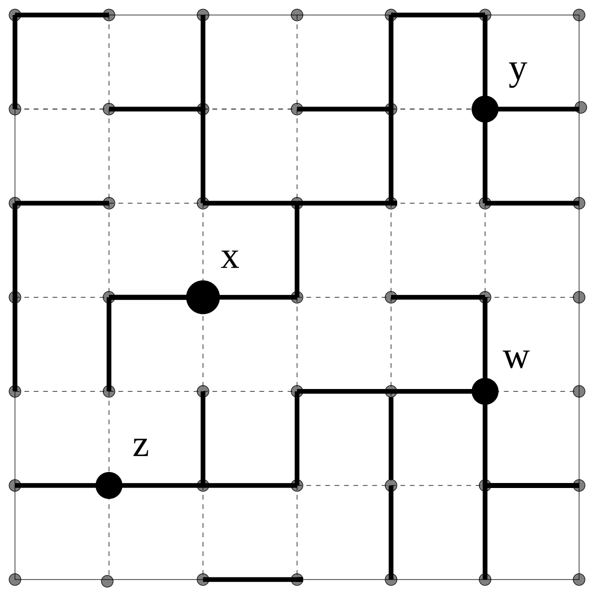


Figure 1: Finite percolation used in motivating the FKG inequality.

- What is $\theta(p_c)$?

  $\theta(p_c)$ is known to be 0 when $d = 2$ and $d \geq 19$. All other cases are still open, though they are very strongly believed to be 0 as well. The proof of this is the big open question in the field (it turns out to be the same question as whether $\theta(p)$ is continuous everywhere).

The FKG Inequality

Motivation: Consider percolation in a finite grid, and let x, y, z, and w be points in the grid. We can consider two events:

$$\begin{aligned} A &= x \leftrightarrow y \\ B &= z \leftrightarrow w. \end{aligned}$$

See Figure 1. Note that in this case $\leftrightarrow$ means that the points are connected by an open path within the finite grid.

The events A and B are **not** independent. They are, however, positively correlated in the sense that:

$$\mathbf{P}(A \cap B) \geq \mathbf{P}(A)\mathbf{P}(B)$$

or (trivially) equivalently:

$$\mathbf{P}(A|B) \geq \mathbf{P}(A).$$

This fact is a special case of the FKG inequality.

For a countable set S let $\Omega = \{0,1\}^S$, and use the usual product measure:

$$\mathbf{P}_p = \prod_{s \in S} (p\delta_1 + (1-p)\delta_0).$$

We define a partial order on the configurations by $\omega \leq \omega'$ if $\omega(s) \leq \omega'(s)$ for all $x \in S$. This ordering is only partial – many configurations are not comparable – but there is a smallest configuration $\omega \equiv 0$ and a largest $\omega \equiv 1$.

Definition 1. *A function $f : \Omega \to \mathbb{R}$ is increasing if $\omega \leq \omega'$ implies that $f(\omega) \leq f(\omega')$. A subset A of Ω is said to be increasing if I_A is (i.e. $\omega \in A, \omega \leq \omega' \Rightarrow \omega' \in A$).*

Theorem 2. (FKG)

1. *If X, Y are increasing functions in $L^2(\Omega)$ (that is, they have finite second moments) then $E_p(XY) \geq E_p(X)E_p(Y)$.*

   In particular, if A, B are increasing events, then $\mathbf{P}(A \cap B) \geq \mathbf{P}(A)\mathbf{P}(B)$.

Corollary 1. *If $A_1, \ldots, A_n$ are increasing, then:*

$$\mathbf{P}\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} \mathbf{P}(A_i).$$

An application: Consider an arbitrary connected graph G, and let x be a vertex. Then $p_c(x) = \sup\{p : \mathbf{P}_p(|C(x)| = \infty) = 0\}$ does not depend on our choice of x.

Proof. Let x and y be arbitrary vertices, we will show that $p_c(x) \geq p_c(y)$, whence the result follows by symmetry. Assume that $p_c(x) < 1$ (if $p_c(x) = 1$ there is nothing to prove), and choose $p > p_c(x)$. We then have:

$$\begin{aligned} \mathbf{P}_p(|C(y)| = \infty) &\geq \mathbf{P}_p(y \leftrightarrow x, \text{ and } |C(x)| = \infty) \\ &\geq \mathbf{P}_p(y \leftrightarrow x)\mathbf{P}_p(|C(x)| = \infty) \end{aligned}$$

where the last step is an application of the FKG inequality, possible because both events are indeed increasing. $\mathbf{P}_p(y \leftrightarrow x) > 0$ whenever $p > 0$, and $\mathbf{P}_p(|C(x)| = \infty) > 0$ by our choice of p. This shows that $p \geq p_c(y)$, and since this holds for any $p < p_c(x)$, the inequality is obtained. □

We will now prove FKG for variables X, Y which depend only on the variables at finitely many elements of S. The generalization to the full theorem uses Martingale convergence theorem and can be found in Grimmet's book.

Lemma 1. *Let $f, g \in L^2(\mathbb{R})$ be increasing functions (in the usual sense), and let Z be a random variable. Then*

$$E[f(Z)g(Z)] = E[f(Z)]E[g(Z)].$$

Proof. Let Z_1, Z_2 be independent random variables with the same distribution as Z. Then

$$(f(Z_1) - f(Z_2))(g(Z_1) - g(Z_2)) \geq 0$$

Since this statement is necessarily non-negative, so is its expectation, whence it follows that

$$E[f(Z_1)g(Z_1)] + E[f(Z_2)g(Z_2)] - E[f(Z_2)]E[g(Z_1)] - E[f(Z_1)]E[g(Z_2)] \geq 0.$$

Each term is unchanged if Z_1 and Z_2 are replaced with Z, so the inequality holds for Z alone, and the result follows. □

Proof. (Of finite case FKG) Assume (WLG) that X, Y depend on the same set of variables at $s_1, \ldots, s_n$. The proof works by induction on the number variables.

$n = 1$ is a special case of the previous lemma. Now assume that the statement holds if X, and Y depend on ω's value at $s_1, \ldots, s_{n-1}$.

$$E[XY] = E[E[XY|\omega(s_1), \ldots, \omega(s_{n-1})]]$$

Applying the case for $n = 1$, and that X and Y are still increasing after conditioning, gives

$$E[XY|\omega(s_1), \ldots, \omega(s_{n-1})] \geq E[X|\omega(s_1), \ldots, \omega(s_{n-1})]E[Y|\omega(s_1), \ldots, \omega(s_{n-1})].$$

Therefore:

$$\begin{aligned} E[XY] &\geq E\left[E[X|\omega(s_1), \ldots, \omega(s_{n-1})]E[Y|\omega(s_1), \ldots, \omega(s_{n-1})]\right] \\ &\geq E\left[E[X|\omega(s_1), \ldots, \omega(s_{n-1})]\right] E\left[E[Y|\omega(s_1), \ldots, \omega(s_{n-1})]\right] \\ &= E[X]E[Y] \end{aligned}$$

where the last inequality is the application of the induction hypothesis, possible since the conditional expectations are increasing functions of $n - 1$ variables. □

Russo's Formula

Let $\Omega = \{0,1\}^S$ for a finite set S, and $A \subseteq \Omega$ be increasing. Then $P_p(A)$ is an increasing function in p, which tells us that:

$$\frac{dP_p(A)}{dp} \geq 0.$$

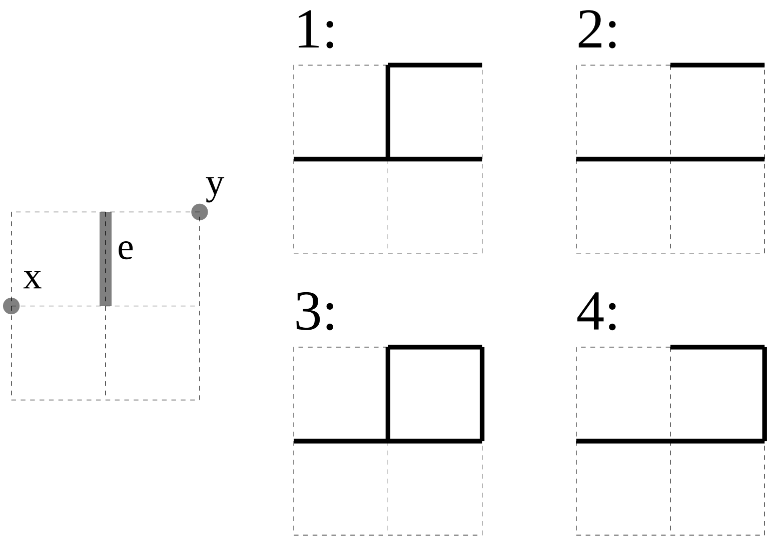
We are interested in finding an explicit formula for this derivative: the rate at which the probability of an event increases with p.

Definition 2. *For $s \in S$ and $A \subseteq \Omega$, the event "s is pivotal for A" is the subset of all configurations for which changing the state at s changes whether the event occurs.*

That is:

$$\{\omega \in \Omega : I_A(\omega) \neq I_A(\omega_s) \text{ where } \omega_s(j) = \omega(j)\ \forall s \neq j,\ \omega_s(s) = 1 - \omega(s)\}$$

Example: Let S be edges in a three-by-three lattice, and let the points x, y and edge e be as in the picture.



Now let $A = x \leftrightarrow y$. Of the four configurations shown, e is pivotal for A in 1 and 2 (which become one another when s is changed), but not in 3 and 4, where x and y remain connected regardless of edge e.

Theorem 3. (Margulis, Russo (indep.)) *Let $A \subseteq \{0,1\}^S$ and $|S| < \infty$. Let N_A be the number of elements that are pivotal for A. If A is increasing, then*

$$\begin{aligned}\frac{dP_p(A)}{dp} &= \sum_{s\in S} P_p(s \textit{ is pivotal for } A) \\ &= E[N_A]\end{aligned}$$

Two examples follow, the first silly and the second trivial. Consider first:

$$A = \{e \text{ is open}\}$$

for some edge e in finite percolation. The $P_p(A) = p$ and the number of pivotal edges is of course always 1.

If we instead consider two edges e_1 and e_2 and let

$$A = \{\text{Both } e_1 \text{ and } e_2 \text{ are open}\}$$

then $P_p(A) = p^2$, and

$$\frac{dP_p(A)}{dp} = 2p.$$

Now consider the conditions under which e_1 and e_2 are pivotal. Both edges are pivotal exactly when the other is open, because that is when changing their state changes the truth value of A. Since this occurs with probability p for each edge, $E[N_A] = 2p$.