Probability theory: Homework 1: Due January 24

You are free to work together as this sometimes maximizes the amount of mathematics that you learn. However, you should first spend a fair amount of time trying to solve the problems yourself before discussions with others. If you do discuss with others, you should write up the final solutions yourselves and also write down with whom you have discussed the problems. Solutions should always be clear and concise (and hopefully of course correct).

1. Consider the measurable space $[0, \infty)$ with the Borel sets. Consider the measurable function on this given by $X(\omega) = \omega^2$. Does there exist a probability measure on this measurable space for which the resulting law of X has an exponential distribution with parameter 1? Find one or prove it does not exist. Is it unique?

If X remains the same but $[0, \infty)$ were replaced by $(-\infty, \infty)$, would your answers change?

2(i). Give an example of a sequence $\{X_n\}$ of independent symmetric random variables such that

$$\sum_n Var(X_n) = \infty$$

but

$$\sum_n X_n \text{ converges a.s.}$$

Hint: Borel-Cantelli.

(ii). Give an example of a sequence $\{X_n\}$ of independent random variables with mean 0 such that

$$\sum_n X_n = \infty \text{ a.s.}$$

Hint: Borel-Cantelli.

3. Is it possible to have two distinct probability measures on a measurable space $(\Omega, \mathcal{F})$ and a subset $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{F}$ is the smallest σ-algebra containing $\mathcal{G}$ and the two probability measures agree on $\mathcal{G}$?

4. (i). Do there exist random variables X_1, X_2, X_3 which are not independent but which are pairwise independent?

(ii). If the events E_1, E_2, E_3 satisfy

$$P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2)P(E_3),$$

are they necessarily independent?

5. We have seen an example where convergence in probability does not imply convergence a.s. The following gives a very different construction.

Let $A_1, A_2, ...$ be independent events with $P(A_i) = p_i$. Determine necessary and sufficient conditions in terms of the p_i's such that the sequence I_{A_i} converges to 0 in probability but not a.s.

Hint: Borel-Cantelli.

6. (Optional). We know that if X_n are random variables, then $\sup_n X_n$ is a random variable. If we have random variables X_t, for $t \in [0, 1]$, is $\sup_t X_t$ necessarily a random variable? Prove or give a counterexample.