Probability theory: Homework 2: Due February 6

You are free to work together as this sometimes maximizes the amount of mathematics that you learn. However, you should first spend a fair amount of time trying to solve the problems yourself before discussions with others. If you do discuss with others, you should write up the final solutions yourselves and also write down with whom you have discussed the problems. Solutions should always be clear and concise (and hopefully of course correct).

1. Fix $\gamma > 0$. Let $X_1, X_2, X_3, ...$ be independent random variables such that for each i

$$P(X_i = i^\gamma) = P(X_i = -i^\gamma) = 1/2.$$

a. Show that for $\gamma < 1/2$,

$$(\sum_{i=1}^{n} X_i)/n$$

converges to 0 in probability.

b. Show that for $\gamma \geq 1$,

$$(\sum_{i=1}^{n} X_i)/n$$

does not converge to 0 in probability.

c. (Optional Challenge Problem)

Show that for $\gamma \in [1/2, 1)$,

$$(\sum_{i=1}^{n} X_i)/n$$

does not converge to 0 in probability.

(One way to do this, which I don't want you to do since we haven't gotten to these things yet, is using the Lindeberg-Feller Theorem which is a generalization of the central limit theorem. So, the point is to get an elementary argument from first principles. I haven't thought about it myself and so I don't know how elementary one can make it.)

2. In the proof of the weak law of large numbers that we did in class, discuss whether pairwise independence (as opposed to full independence) suffices for the proof.

3. In class, we have proved that if S_n is the sum of your winnings after n plays of the St. Petersburg game, then

$$S_n/(n \log n)$$

approaches 1 in probability.

(a). Show that nonetheless

$$\limsup_n S_n/(n \log n) = \infty \ \ a.s.$$

Explain in words how it is possible that these two things can both hold.

(b). Show that

$$\liminf_n S_n/(n \log n) \leq 1 \ \ a.s.$$

It is in fact the case that

$$\liminf_n S_n/(n \log n) = 1 \ \ a.s.$$

but you do not need to prove that.

4. (a). Let $X_1, X_2, X_3, ...$ be i.i.d. with

$$P(X_1 = (-1)^k k) = C/(k^2 \log k)$$

for each $k \geq 2$ and where C is chosen so the probabilities add to 1. Show that the expected value of X_1 is infinite but nonetheless there exists a finite μ so that

$$S_n/n \to \mu$$

in probability.

(b). Find a distribution with the property that if $X_1, X_2, X_3, ...$ are i.i.d. with this distribution, then $xP(|X_1| \geq x)$ goes to 0 as $x \to \infty$ (which implies that there is a type of weak law of large numbers as proved in class) but such that

$$S_n/n \to \infty$$

in probability.