Probability theory: Homework 3: Due February 13

You are free to work together as this sometimes maximizes the amount of mathematics that you learn. However, you should first spend a fair amount of time trying to solve the problems yourself before discussions with others. If you do discuss with others, you should write up the final solutions yourselves and also write down with whom you have discussed the problems. Solutions should always be clear and concise (and hopefully of course correct).

1. Construct an example of independent random variables, $X_1, X_2, X_3, \ldots$, such that all the distributions are symmetric, all have infinite mean but

$$\sum_n X_n$$

converges a.s.

2. Let $X_1, X_2, X_3, \ldots$ be i.i.d. mean 0, variance 1 random variables.

a. Show that for each t,

$$\sum_n X_n \frac{\sin(nt)}{n}$$

converges a.s.

b. Show that it follows from part (a) that

$$P(\sum_n X_n \frac{\sin(nt)}{n} \text{ converges for every rational } t \in [0,1]) = 1.$$

c. Show that it follows from part (a) that

$$P(\sum_n X_n \frac{\sin(nt)}{n} \text{ converges for Lebesgue a.e. } t \in [0,1]) = 1.$$

Hint: Fubini's Theorem. You may assume, in the application of Fubini's Theorem, that the relevant sets are measurable without proving that.

d. Explain why you are not necessarily able to conclude that

$$P(\sum_n X_n \frac{\sin(nt)}{n} \text{ converges for every } t \in [0,1]) = 1.$$

The latter turns out to be true nonetheless but it is a nontrivial theorem. The general theme of this question (when are there exceptional values of t for something to occur even though it holds a.s. for each fixed t) is a question which permeates alot of probability theory and we will see some of this when studying Brownian motion.

3. Let $X_1, X_2, X_3, ...$ be independent and S_n be, as usual, the partial sums. Which of the following two statements are true? Prove or give a counterexample.

(i). If X_i converges to 0 a.s., then S_n/n converges to 0 a.s.

(ii). If X_i converges to 0 in probability, then S_n/n converges to 0 in probability.

4. Let $g(x) = x^2$ for $|x| \leq 1$ and $= |x|$ for $|x| \geq 1$. Show that if $X_1, X_2, X_3, ...$ are independent with $E(X_i) = 0$ for each i and

$$\sum_n E(g(X_n)) < \infty$$

then

$$\sum_n X_n$$

converges a.s.

5. (a). Suppose that $X_1, X_2, X_3, \ldots$ are independent and all bounded by 1, and assume that for any sequence a_n of 1's and -1's,

$$\sum_n a_n X_n$$

converges a.s. Show that

$$\sum_n X_n^2 < \infty \ a.s.$$

Hints: (i). You may (and should) use the fact, stated in class, that for an independent sequence of mean 0 random variables which are uniformly bounded, then finiteness of the sum of the variances is also necessary for a.s. convergence of the corresponding random series.

(ii). Fubini's Theorem. You may assume, in the application of Fubini's Theorem, that the relevant sets are measurable without proving that.

(b). Switch the order of quantifiers and assume (the stronger fact) that

$$P(\sum_n a_n X_n \text{ converges for every sequence } (a_n)) = 1.$$

Show then that

$$\sum_n |X_n| < \infty \ a.s.$$

(c). Let $X_1, X_2, X_3, \ldots$ be independent with X_n being $\pm 1/n^{3/4}$ each with probability 1/2. Show that for every sequence a_n of 1's and -1's, $\sum_n a_n X_n$ converges a.s. but that nontheless there a.s. exists a sequence of a_n's for which $\sum_n a_n X_n$ does not converge.

Remark: This sequence of a_n's for which $\sum_n a_n X_n$ does not converge will depend on the particular outcome of ω. This contrasts in an interesting way from the stated behavior in problem 1 where you almost surely obtain convergence for all values of t. (The order of the quantifiers is crucial in all of this stuff.)