Probability theory: Homework 4: Due February 20 (if you feel you need/want more time, feel free to talk to me)

You are free to work together as this sometimes maximizes the amount of mathematics that you learn. However, you should first spend a fair amount of time trying to solve the problems yourself before discussions with others. If you do discuss with others, you should write up the final solutions yourselves and also write down with whom you have discussed the problems. Solutions should always be clear and concise (and hopefully of course correct).

Recall that normalizing a random variable X means to consider $aX + b$ for some $a > 0$ and some b. Recall a measure is nontrivial means it is not 1 point mass.

1. Let X be an exponential distribution. How does one normalize the conditional distribution of X given $X \geq n$ to get a nontrivial limit (as $n \to \infty$) and what is this limiting distribution?

2. Let Z be a standard normal distribution. How does one normalize the conditional distribution of Z given $Z \geq n$ to get a nontrivial limit (as $n \to \infty$) and what is this limiting distribution?

3. a. Show that for any random variable X, there exist random variables $\{X_n\}$ whose distributions are continuous so that X_n converges to X in distribution.

b. Show that for any random variable X with a continuous distribution, there exist random variables $\{X_n\}$ each of which takes on only finitely many values so that X_n converges to X in distribution. (This is true for any random variable X but doing it in general might be a little "meckig".)

4. Find a sequence of probability measures which has both (a) subsequences which converge vaguely to a probability measure and (b) subsequences which converge vaguely to a subprobability measure which is not a probability measure.

5. Does there exist a sequence of subprobability measures, each which has total mass at most 1/2 which converges vaguely to a probability measure?

6. (a). Is the collection of probability measures on the real line compact under vague convergence?

(b). Is the collection of probability measures on the real line which vanish on $(-\infty,-2)\cup(2,\infty)$ compact under vague convergence?

(c). Is the collection of probability measures on the real line which vanish on $(-\infty,-2]\cup[2,\infty)$ compact under vague convergence?

7. (Optional challenge problem.)

Find a sequence of random variables $\{X_n\}$ such that for every subsequence, one cannot normalize it and obtain a nontrivial limit. More precisely, find a sequence of random variables $\{X_n\}$ such that if we take any subsequence $\{X_{n_k}\}$, if we have sequences a_k and b_k with $a_k > 0$ for all k and such that $a_k X_{n_k} + b_k$ converges in distribution to some random variable U, then U

has a degenerate distribution.

You are not allowed to do trivial things. For example, if the $\{X_n\}$'s are degenerate, this would be true. Or if $\{X_n\}$ had a point mass whose weight approached 1. To keep things nontrivial, let's say that no $\{X_n\}$ can have a point mass of weight larger than 1/2. I think (hope?) this will eliminate all "trivial" examples.