Probability theory: Homework 5: Due March 7

You are free to work together as this sometimes maximizes the amount of mathematics that you learn. However, you should first spend a fair amount of time trying to solve the problems yourself before discussions with others. If you do discuss with others, you should write up the final solutions yourselves and also write down with whom you have discussed the problems. Solutions should always be clear and concise (and hopefully of course correct).

1. One of our definitions of weak convergence is that μ_n goes to μ_∞ if for all continuous f vanishing at ∞

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu_\infty.$$

Note there is no uniformity over f. Consider the stronger property that

$$\lim_{n\to\infty} \sup_{f\in A} \left| \int f d\mu_n - \int f d\mu_\infty \right| = 0$$

where A is the set of continuous functions vanishing at ∞ which are bounded in absolute value by 1. Give examples of convergence in distribution where this stronger property fails and also give (nontrivial) examples where this stronger property holds.

2. In elementary probability books, it is often stated, concerning the Central Limit Theorem, that it is a good approximation if $n \geq 30$. The

only precise mathematical interpretation of this that I see is the following statement.

For every fixed interval I,

$$\lim_{n\to\infty} \sup_{X_1 \in D} |Prob(\frac{S_n}{\sqrt{n}} \in I) - Prob(Z \in I)| = 0$$

where D is the set of random variables with mean 0 and variance 1 and Z is a standard normal. This corresponds to having the CLT hold uniformly over the distributions used for the summands.

a. Show that such a statement is false.

b. If $D(\epsilon, C)$ is the set of random variables with mean 0 and variance 1 and with

$$E(|X_1|^{2+\epsilon}) \leq C$$

show, using the Lindeberg-Feller Theorem that the above statement is true if D is replaced by $D(\epsilon, C)$. This would say that the CLT holds uniformly over distributions used for the summands if we restrict to having a uniformly bounded $2+\epsilon$ moment.

3. (a). Let X_1, X_2 be two independent exponential random variables with parameter 1. Show that the pdf of $X_1 - X_2$ is continuous without computing it.

(b). Use what you did in Part (a) to obtain the characteristic function of the Cauchy distribution.

4. (a). Show that with probability 1, Brownian motion is not 0 throughout a time interval of positive length.

(b). Show that with probability 1, Brownian motion is not constant on a time interval of positive length.

5. Show using the Kolmogorov 0-1 Law that with probability 1, Brownian motion takes both positive and negative values in every interval around the time point 0.

6. Show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$P(B_t \text{ hits location } -\delta \text{ before location } 1) \geq 1 - \epsilon$$