Class Lectures (for Chapter 4)

Riemann Integral

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Theorem

If f is a bounded function, then f is RI if and only if the set $\{x : f \text{ is not continuous at } x\}$ has Lebesgue measure 0.

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If f takes values in $[0, 1]$, we partition $[0, 1]$ in the y-axis into $0 = a_0 < a_1 < a_2 < \ldots < a_n = 1$ and approximate "the integral" by

$$\sum_{i=0}^{n-1} a_i m(\{x : f(x) \in [a_i, a_{i+1})\})$$

where m is Lebesgue measure.

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What happens with I_Q? Only is the first term and the last term giving

$$0m([0, 1]\backslash Q) + a_{n-1}m(Q) = 0.$$

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The structure of the domain is irrelevant which allows us to do this on a general measure space.

Measurable functions

Definition

If $(X, \mathcal{M})$ is a measurable space, a mapping $f : X \to R$ is called **measurable** if for all $B \in \mathcal{B}$ (recall that $\mathcal{B}$ is the collection of Borel sets in R), we have that (see picture)

$$f^{-1}(B) := \{x \in X : f(x) \in B\} \in \mathcal{M}.$$

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$f : (X, \mathcal{M}) \to \overline{R} := R \cup \{-\infty, \infty\}$ is measurability if for all $B \in \mathcal{B}$,

$$\{x \in X : f(x) \in B\} \in \mathcal{M}$$

and

$$\{x \in X : f(x) = \infty\} \in \mathcal{M}, \ \{x \in X : f(x) = -\infty\} \in \mathcal{M}.$$

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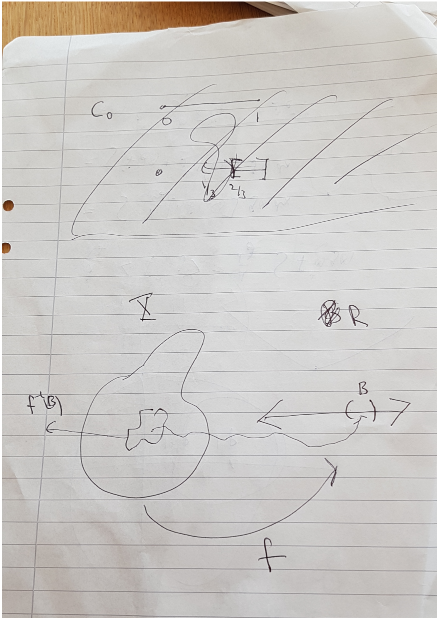

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1. $X, \emptyset \in \mathcal{F}$.

2.

$$E \in \mathcal{F} \to f^{-1}(E) \in \mathcal{M} \to (f^{-1}(E))^c \in \mathcal{M} \to f^{-1}(E^c) \in \mathcal{M} \to E^c \in \mathcal{F}$$

noting that $(f^{-1}(E))^c = f^{-1}(E^c)$ (Check this!).

3.

$$E_1, E_2, \ldots \in \mathcal{F} \to f^{-1}(E_1), f^{-1}(E_2), \ldots \in \mathcal{M} \to \bigcup_i (f^{-1}(E_i)) \in \mathcal{M}$$

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The exact same proof shows that to show that f is measurable, it is enough to check that for all c

$$f^{-1}(c, \infty) = \{x : f(x) > c\} \in \mathcal{M}.$$

Measurable functions are closed under addition

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For all $a \in R$, we have

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Now, f, g being measurable implies each of the terms in the union are in $\mathcal{M}$ and since we have a *countable* union, the RHS and hence the LHS belongs to $\mathcal{M}$. QED

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$$\{x : h^2(x) \geq c\} = \{x : h(x) \geq c^{1/2}\} \cup \{x : h(x) \leq -c^{1/2}\} \text{ if } c > 0.$$

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$$\{x \in X : (\sup_j f_j)(x) > a\} = \bigcup_j \{x \in X : f_j(x) > a\}.$$

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Apply the previous proposition twice.
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Theorem

(Folland Theorem 2.10) If $(X, \mathcal{M})$ is a measurable space and $f : X \to [0, \infty]$ is measurable, then there exists a sequence (ϕ_n) of simple functions such that $0 \leq \phi_1 \leq \phi_2 \leq \ldots$ so that ϕ_n approaches f pointwise.

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Let ϕ and ψ be simple nonnegative functions. Then the following hold.

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a. $\int c\phi(x)\ dm(x) = c \int \phi(x)\ dm(x) \quad \forall c \geq 0.$

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b. $\int (\phi(x) + \psi(x))\ dm(x) = \int \phi(x)\ dm(x) + \int \psi(x)\ dm(x).$

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b. $\int (\phi(x) + \psi(x))\ dm(x) = \int \phi(x)\ dm(x) + \int \psi(x)\ dm(x)$.

c. $\phi \leq \psi$ implies that $\int \phi(x)\ dm(x) \leq \int \psi(x)\ dm(x)$.

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d. The mapping from $\mathcal{M}$ to $[0, \infty]$ given by $A \to \int_A \phi(x)\, dm(x)$ is a measure on $\mathcal{M}$. (We call this measure ϕm.).

The Lebesgue Integral

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See the lecture notes for the proof.

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See the lecture notes for the proof. Part d takes one measure m and gives us a new measure ϕm. Note that $m(A) = 0$ implies that $\phi m(A) = 0$. IMPORTANT!

Step 2. Definition of the integral for nonnegative measurable functions

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- $\int (f+g)dm = \int fdm + \int gdm$ (requires some work and we will return to)

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Since this inequality holds for every $\alpha < 1$, we obtain (2).
QED

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where the MCT was used in the outer most equalities.
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Let $N \to \infty$ using MCT on LHS.
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- Strict inequality possible: Recall our example of functions which converge to 0 for all x but the integrals are all 1.

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Since this is true for all $j \geq k$, we have

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We have what we want on the RHS and now we take $k \to \infty$. Note that $\inf_{n \geq k} f_n$ is an increasing sequence in k and converges to $\liminf f_n$. Hence by the MCT, the LHS converges, as $k \to \infty$, to $\int \liminf_{n \to \infty} f_n dm$.
QED

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and note that both f^+, f^- are nonnegative, $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

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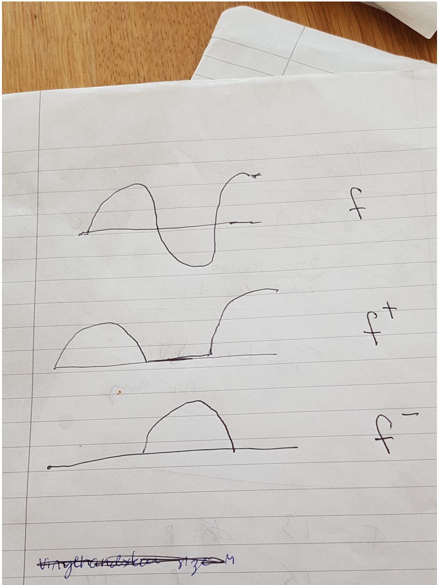

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(L^p are Banach spaces and L^2 is a Hilbert space.)

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This requires an order of the domain.

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Then $f \in L^1((X, \mathcal{M}, m))$ *and*

$$\int f dm = \lim_{n \to \infty} \int f_n dm.$$

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Since $|f_n| \leq g$ for all n and $f_n \to f$, we also have $|f| \leq g$ and hence $f \in L^1((X, \mathcal{M}, m))$.

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Since $|f_n| \leq g$ for all n and $f_n \to f$, we also have $|f| \leq g$ and hence $f \in L^1((X, \mathcal{M}, m))$. Observe that for all n

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Similarly, we have

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So we have

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Hence the limit of $\int f_n dm$ exists and is $\int fdm$ as claimed.
QED

An example on how one shows a set is measurable

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Lemma

If the sequence (f_n) and f are measurable functions on $(X, \mathcal{M}, m)$, then

$$\{x : f_n(x) \to f(x)\} \in \mathcal{M}.$$

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Untangling what the definition of a limit is (and thinking a bit), it is not hard to see that the set above is the same as

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This belongs to $\mathcal{M}$ since the events on the RHS do and then we are applying countable set operations.
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3. This is best described by a picture. See the (admittedly terrible) picture.

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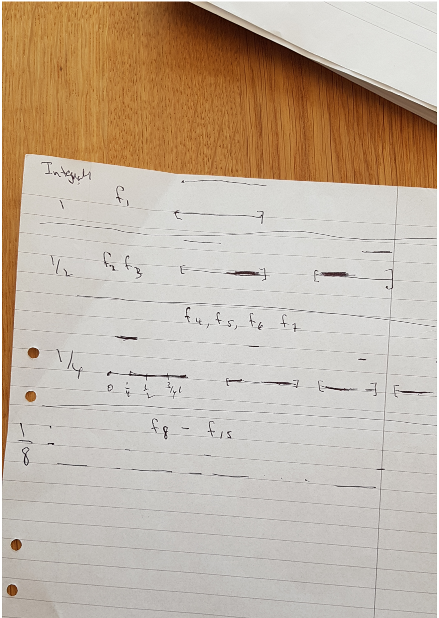

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Proof:
Apply Markov's inequality to the nonnegative function $(f(x) - \int fdm)^2$.
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