Class Lectures (for Chapter 6)

Probability Theory framework

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So, $(\Omega, \mathcal{M}, P)$ governs some "random experiment" where P tells us the "likelihood" that ω (chosen "randomly") falls in different sets.

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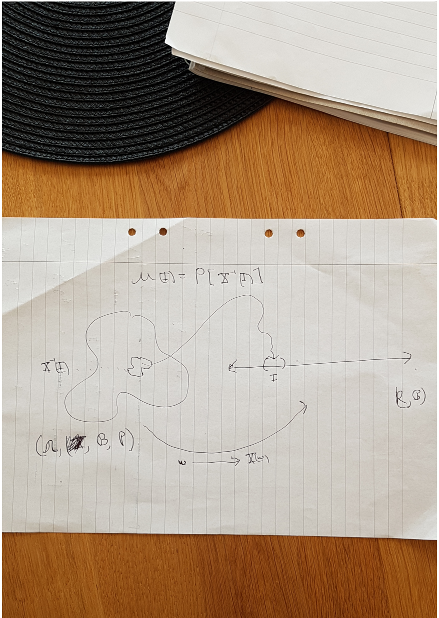
If X is a random variable on a probability space $(\Omega, \mathcal{M}, P)$, its **expectation**, denoted $E(X)$, is simply defined by

$$E(X) = \int X dP$$

provided this exists, meaning at least one of $\int X^+ dP$ and $\int X^- dP$ is finite.

Law or distribution of a random variable



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$$\mu_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}.$$

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An infinite collection of random variables on a probability space $(\Omega, \mathcal{M}, P)$ is called **independent** if each finite collection is independent as above.

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QED For the unit interval with Lebesgue measure, let $E_n = [0, 1/n]$, what is happening?

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Why? $P(|I_{E_n} - 0| \geq \epsilon) = P(E_n)$ which goes to 0. Hence convergence in measure. However, the second Borel-Cantelli Lemma says that $P(\limsup E_i) = 1$ and this means that it is not the case that I_{E_n} converges to 0 a.s.. In fact it says that I_{E_n} converges to 0 only on a set of probability 0.

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One should contemplate what is happening. For very large n, it is very unlikely that E_n occurs. But nonetheless, if you watch things in time (as n moves), I_{E_n} will a.s. pop up to be 1 infinitely often but more and more rarely.

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4. Which is more natural?

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Hence $P(S_n/n \to 0) = 1$.

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General SLLN

Theorem

(Strong Law of Large Numbers: General case) Let $X_1, X_2, \ldots$ *be independent random variables with the same distribution with* $E(|X|) < \infty$*. Then*

$$\frac{S_n}{n} := \frac{\sum_{i=1}^{n} X_i}{n} \text{ converges a.e. to } E(X).$$

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For very large n, $\frac{S_n}{n}$ is very likely to be close to 0, but if you watch the trajectory in time, there will be these very rare times at which $\frac{S_n}{n}$ is close to ∞ and times close to $-\infty$.

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One can show that $X_1, X_2, \ldots$ are independent and each has distribution $(\delta_1 + \delta_{-1})/2$.

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Explanation: The variance of $\sum_{k=1}^{n} \frac{X_k}{k^\alpha}$

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Explanation: The variance of $\sum_{k=1}^{n} \frac{X_k}{k^{\alpha}} = \sum_{k=1}^{n} \frac{1}{k^{2\alpha}}$ converges to ∞ if and only if $\theta \leq 1/2$.

A few words about the variance

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- Assuming X has finite expectation, $Var(X) < \infty$ if and only if $X \in L^2(\Omega, \mathcal{M}, P)$
- $Var(X) = E(X^2) - (E(X))^2$, which is something you might have seen, is actually the pythagorean theorem, viewed properly.

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Hence $E(X)$ and $X - E(X)$ are orthogonal. The pythagorean theorem tells us that $E(X^2) = E(X - E(X))^2 + (E(X))^2 = Var(X) + (E(X))^2$.

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Hence $E(X)$ and $X - E(X)$ are orthogonal. The pythagorean theorem tells us that $E(X^2) = E(X - E(X))^2 + (E(X))^2 = Var(X) + (E(X))^2$. So, the variance is the "squared distance from X to its projection onto the 1-dimensional space of constant functions".

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Since $\frac{S_n}{n}$ approaches 0 in probability (WLLN),
$P(\frac{S_n}{n} < -.001) \leq P(|\frac{S_n}{n} - 0| \geq .001)$ which goes to 0 as $n \to \infty$.

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False: the above limit is not zero and rather equals

$$\frac{2}{\pi} \arcsin(\sqrt{.1})$$

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What's happening? The WLLN is **not** applicable since the Y_i's are not independent. In fact, they are very correlated.