Class Lectures (for Chapter 8)

Theory of Differentiation in R^n: Overview

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Weierstrass then shocked the community when he constructed a continuous nowhere differentiable function on $[0, 1]$. (Almost all continuous functions are nowhere differentiable).

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Since $|g| \leq 1$ and is continuous, f is continuous by the Weierstrass M-Test (or some other test from advanced calculus). But, not worrying about being rigorous, we have that

$$f'(x) := \sum_n \frac{4^n}{3^n} g'(4^n(x))$$

whose terms go to ∞.

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$$P(f \text{ is nowhere differentiable}) = 1$$

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2. For the Cantor ternary function, we have seen that the derivative is 0 a.e.

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Most of the rest of the course are generalizations of the fundamental theorems of calculus.

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have, by "First", the same derivative ($f'(x)$) and agree at a. Hence equal.

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This holds if and only if the measure μ_f, associated to f, is absolutely continuous with respect to m.

For the Cantor Ternary function, we have seen the measure is singular w.r.t. m.

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2. Let ν be a regular Borel measure on R^n. If E is a Borel set, then

$$\nu(E) = \inf\{\nu(O) : O \supseteq E, O \text{ open }\}.$$

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However, one cannot do it the other way.

The irrationals $[0,1]\backslash Q$ cannot be approximated "from the inside" by open sets, since it contains no nonempty open sets. Similarly the rationals $[0,1] \cap Q$ cannot be approximated "from the outside" by closed sets, since $[0,1]$ is the only closed set containing the rationals.

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$$\int_{R^n} |f(x) - g(x)| dm(x) < \epsilon.$$

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$$\sum_{i=1}^{k} m(B_i) \geq \frac{c}{3^n}.$$

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Claim: If A_i is not in the final list $B_1, B_2, \ldots, B_k$,

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(1) implies that $K \subseteq \bigcup_{j=1}^{k} B_j^*$ which in turn yields

$$c < m(K) \leq m(\bigcup_{j=1}^{k} B_j^*) \leq \sum_{j=1}^{k} m(B_j^*) = 3^n \sum_{j=1}^{k} m(B_j).$$

QED

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where the last inequality follows from the disjointness of the balls. Since this is true for all $c < m(E_\alpha)$, we are done.
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Since this is true for every $\epsilon > 0$, we can conclude $m(E_\alpha) = 0$.
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(The set of x where this holds is called the **Lebesgue set** *of f and is denoted by L_f.)*

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Since this inequality holds for all $q \in Q$, the LHS is 0. QED

Nicely shrinking sets

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Example: In R^2, $E_r := [0, \frac{r}{2}] \times [0, \frac{r}{200}]$ shrinks nicely to $(0,0)$ but $E_r := [0, \frac{r}{2}] \times [0, \frac{r^2}{2}]$ does not shrink nicely to $(0,0)$.

Corollary

Let f be locally integrable.

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$$\frac{1}{m(E_r))}\int_{E_r}|f(y)-f(x)|dm(y)\leq\frac{1}{\alpha m(B(x,r))}\int_{B(x,r)}|f(y)-f(x)|dm(y).$$

QED

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However, this still doesn't give us Lebesgue's Theorem that a monotone function is differentiable a.e.

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We leave this to read on your own as it gets quite technical.

A corollary of Lebesgue's Differentiation Theorem for measures (after checking a number of the previous things)

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Theorem

(Lebesgue) If $f : [0,1] \to R$ is monotone ($x \leq y$ implies that $f(x) \leq f(y)$), then for a.e. x, f is differentiable with a finite derivative.

A proof of Lebesgue's Theorem

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Examples: (1). f is the Cantor ternary function.

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A proof of Lebesgue's Theorem

Proof: Let μ_f be the measure on $[0,1]$ corresponding to f. Then

$$\frac{f(x+h)-f(x)}{h} = \frac{\mu_f((x,x+h])}{m((x,x+h])}$$

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