On Dijkstra's proof of the Pythagorean Theorem

Recently I learned about E.W. Dijkstra's proof, from the journal of the Dutch mathematical society [Nieuw Archief voor Wiskunde, (5) 10 (2009), no. 2, 94–99], which published a copy of EWD975, with an introduction by K.P. Hart, who also mentions the proof of Guoping Zeng on the page
`http://www.cut-the-knot.org/pythagoras/Dijkstra.shtml`
using the Law of Cosines, without really being convinced of the simplicity of that proof.

Alex writes: "Intuitively clear as Dijkstra's proof may be, not everyone finds it simple enough." Actually, Dijkstra himself is not really satisfied: "No cheers at all for that stage of the argument in which lack of axiomatization forced us to resort to a picture. Pictures are almost unavoidably overspecific and thereby often force a case analysis upon you. Note that I carefully avoided the pictures for $\alpha+\beta>\gamma$; there are 9 of them: [...]". What he wants is a geometric proof of his formula

(D) $$\mathrm{sgn}(\alpha+\beta-\gamma) = \mathrm{sgn}(a^2+b^2-c^2)\ ,$$

preferably without pictures.

The problem with this is that (D) involves negative areas and angles. What one can do in Euclidean geometry, is to compare positive quantities in size. In this case this leads us to check

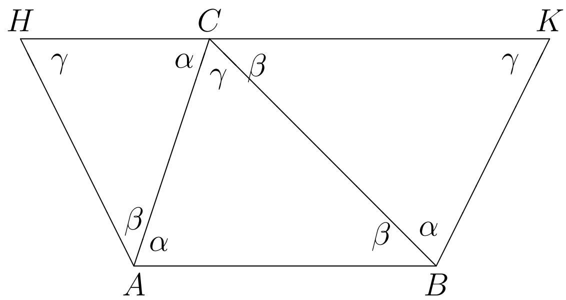
(1) $$\alpha+\beta>\gamma \quad\equiv\quad a^2+b^2>c^2\ ,$$

and

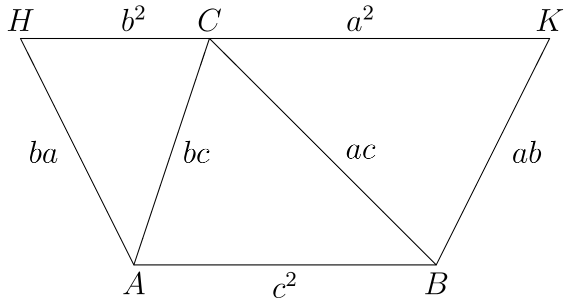
(2) $$\alpha+\beta<\gamma \quad\equiv\quad a^2+b^2<c^2\ .$$

But this are precisely the formulas Dijkstra started out from to arrive at his clever formula (D).

To avoid negative quantities one can construct, in stead of the difference $\alpha+\beta-\gamma$, the sum. For the figure this means placing the two smaller triangles on the outside. This gives the following figure, in the case $\alpha+\beta>\gamma$ (now only one case):



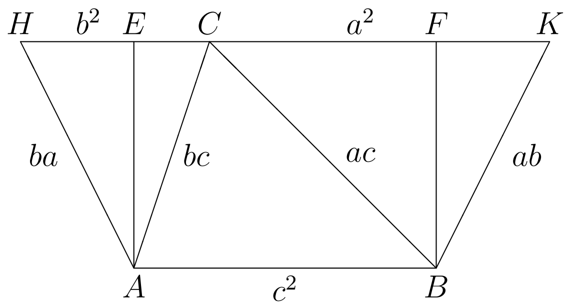
The quadrangle $KHAB$ is an isosceles trapezium. One can argue with areas, just as in Dijkstra's original proof, but as all triangles involved have the same height, one can as well work with the length of their bases. We compute the length of all sides, and multiply with c to avoid fractions (what amounts to taking a new unit of length); this could also be done in Dijkstra's picture. For a right angle we just get Proof #41.



A quantitative version of the qualitative statements (1) and (2) is, as I learned from Alex' website, already in Euclid (II.12 and II.13). Basically this is the cosine rule (Dijkstra gives a proof of this rule in EWD973), which we can state as quantitative version of (D):

$$\text{(D')} \qquad a^2 + b^2 - c^2 = 2ab \sin \frac{1}{2}(\alpha + \beta - \gamma) \ .$$

This formulation follows from the usual one, with the manipulations used by Guoping Zeng. However, we do not need these, as we easily can give a proof with our picture, which by the way also, and maybe more directly, proves the usual version. We have as always to distinguish the cases $\alpha+\beta \geq \gamma$ and $\alpha+\beta \leq \gamma$, to avoid problems with negative angles. All we have to do is draw perpendiculars from A and B onto the line HK.



As $\angle FKB = \gamma$, the length of the segment FK is $ab \cos \gamma$, but also $ab \sin \angle FBK$. If we reflect the segment BK in the line BF, we get the angle $\alpha+\beta-\gamma$ between BK and its reflection, so $\angle FBK = \frac{1}{2}(\alpha+\beta-\gamma)$.

The case $\alpha + \beta \leq \gamma$ works similarly. In this case one can also use Dijkstra's original figure, i.e., Thâbit ibn Qurra's figure in Proof #18. Then one only needs the altitude from C as extra line, which maybe even does not have to be drawn. With the notation of the figure on the Dijkstra page: let $|AB| = c^2$, $|AC| = bc$, $|BC| = ac$, then $|AL| = b^2$, $|BN| = a^2$ and $|CL| = |CN| = ab$. As $\angle LCN = \gamma - \alpha - \beta$ we have $|LN| = 2ab \sin \frac{1}{2}(\gamma - \alpha - \beta)$, which is also equal to $-2ab \cos \gamma$, as $\triangle LCN$ is isoceles with base angles $2R - \gamma$ (here I use Euclid's fundamental angle unit, which is the right angle — no need for transcendental quantities like π).

Actually, I have been a bit careless about the case $\alpha + \beta = \gamma$. In my proof of the case $\alpha + \beta \geq \gamma$ the angle $\angle FKB$ makes no sense, if $F = K$. But equally well as saying that $\angle FKB = \gamma$ we can observe that $\angle CKB = \gamma$. For the sine version one needs the angle $\angle FBK$, but according to most definitions (Euclid, Hilbert), an angle lies strictly between zero and two right angles. As I remarked earlier, the whole formulation with non-positive angles is problematical.

The easiest interpretation is maybe to view the statement (D') as a purely numerical one. The sine and cosine functions come in two sorts, one geometrical, defined as usual for angles occurring in right-angled triangles, while the other sort is just a function of a real variable (one may use radians, degrees or multiples of a right angle). An angle and the number, representing its size, have the same (co)sine, justifying the use of only one symbol. To give a geometrical proof of the cosine rule or the formula (D'), one has to treat, according to this view, the three cases, of the angle $\angle C$ being less than, equal to or more than a right angle, separately.

Such problems do not occur for the proofs of the formulas (1) and (2), which also work for the $\geq$-sign and the $\leq$-sign.