SUPERCONVERGENCE OF TIME-SPACE DISCONTINUOUS FINITE ELEMENTS FOR FIRST ORDER HYPERBOLIC SYSTEMS *

Chuanmiao Chen

Institute of Computation, Hunan Normal University, Changsha, Hunan, 410081

E-mail address: cmchen@hunnu.edu.cn

Contents

§1 Summary

Denote by $u = (u_1, u_2)^T$ the unknowns. We consider the first order hyperbolic systems (diagonalization) with initial-boundary values

$$\begin{cases} Lu \equiv u_t + Au_x + Pu = f(x,t), \quad (x,t) \in Q = \{0 < x < X, 0 < t < T\}, \\ u = g(x,t) = (g_1, g_2)^T, \quad on\ \Gamma^- = (\Gamma_1^-, \Gamma_2^-). \end{cases} \tag{1}$$

where

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}.$$

Assume that $p_{11},\ p_{22} \geq C_1 >> 1$. Denote its two characteristic directions $\beta_i = (1, a_i),\ i = 1, 2$ and outer normal direction $n = (n_1, n_2)$ on boundary. The signs of products $\beta_i \cdot n$ play an important role to determine the direction of boundary.

The boundary Γ of Q is divided into two parts $\Gamma = \Gamma_i^- + \Gamma_i^+$ for $u_i,\ i = 1, 2$:

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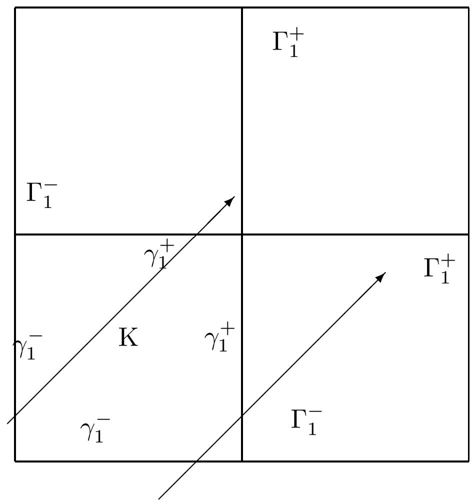
The inflow boundary Γ_i^- (i.e. down bottom and left side) if $\beta_i \cdot n = (\beta, n) < 0$ on which. The initial-boundary values g_i are given on Γ_i^-.

The outflow boundary Γ_i^- (i.e. super bottom and right side) if $\beta_i \cdot n > 0$ on which. The solution u_i is determined in $Q + \Gamma_i^+$.

The division of the boundaries in the case $a_1 > 0, a_2 < 0$ is pictured in Fig.1-2.



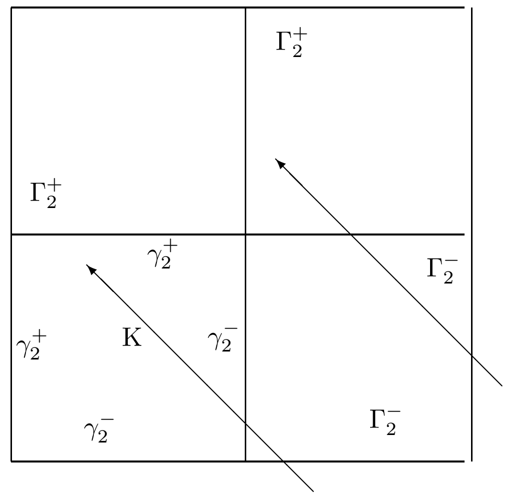


Fig. 1. Boundaries for $a_1 > 0$. Fig. 2. Boundaries for $a_2 < 0$.

We divide the domain $Q = \Omega \times J = (0, X) \times (0, T)$ into finite number of uniform rectangular elements K, whose semi-steplength of x, t-directions are h, k respectively(here taking $h = k$). Denote its node points by

$$(x_i, t_j) = (2ih, 2jk), \quad 0 \le i \le N, \quad 0 \le j \le M.$$

In an element $K = K_{ij} = \{x_i < x < x_{i+1}, t_j < t < t_{j+1}\}$, its inflow boundary γ_l^- and outflow boundary γ_l^+ can be defined similarly.

On the inflow boundary γ_i^- with an outer normal direction n, denote the right and left limit values of u_i on γ_i^- by

$$u_i^{\pm}(\gamma_i^-) = \lim_{s \to +0} u_i(\gamma_i^- \pm s\beta_i \cdot n), \quad \text{the jump } [u] = u^+ - u^-.$$

On the boundary $\gamma^- = \gamma_1^- \oplus \gamma_2^-$ (where $\beta_i \cdot n < 0,\ i = 1, 2$), we define the inner product and the norm respectively by

$$< u, v >_{\gamma^-} = \int_{\gamma_1^-} u_1 v_1^+ |\beta_1 \cdot n| dl + \int_{\gamma_2^-} u_2 v_2^+ |\beta_2 \cdot n| dl, \tag{2}$$

$$|u|_{\gamma^-} = \sqrt{< u, u >_{\gamma^-}}.$$

and bilinear form by

$$(Lu, v)_K = (L_1 u_1, v_1)_K + (L_2 u_2, v_2)_K.$$

Denote the piecewise discontinuous bi-n degree tensor product polynomial space by

$$S_g^h = \{U: \ U \in S^h, \ U_0^- = g_h \ on \ \Gamma^-\}, \quad S^h = S_0^h.$$

Which has $(n+1)^2 - 1$ freedoms in each K. Define the discrete initial-boundary values $g_h = P_h g(t)$ by n-degree (right or left) Radau's interplant, which has error estimate

$$|g - g_h|_{\Gamma_0} \le C h^{n+1} |g|_{n+1,\Gamma_0}.$$

We define the discontinuous finite element $U \in S_g^h$ in K such that

$$B_K(U, v) \equiv (LU, v)_K + < [U], v^+ >_{\gamma^-} = (f, v)_K, v \in S^h. \tag{3}$$

It is also satisfied by the solution u of (1) with jump $[u] = 0$. So the error $e = u - U$ satisfies the orthogonal relation

$$B_K(e, v) = 0, \ v \in S^h. \tag{4}$$

Assume that an interplant $u_I \in S^h$ of u is constructed, denote

$$e = u - U = u - u_I - (U - u_I) = R - \theta,$$

then $\theta = U - u_I \in S_0^h$ satisfies another equality

$$B_K(\theta, v) = B_K(R, v), \ v \in S^h. \tag{5}$$

Taking $v = \theta$ and noting

$$-(v_j^+)^2 + 2(v_j^+ - v_j^-)v_j^+ = (v_j^+ - v_j^-)^2 - (v_j^-)^2, \quad v_0 = 0,$$

In section 5, we shall derive a basic error inequality

$$\frac{1}{2}(|\theta^-|_{\Gamma^+}^2 + \sum_K |[\theta]|_{\gamma^-}^2) + ||\theta||_Q^2 \le B_Q(\theta, \theta) \le |B_Q(R, \theta)|. \tag{6}$$

However, as the bilinear form $B(R, v)$ is **not symmetrical, how to estimate is an essential difficulty, which greatly depends on choice of comparison function** u_I; The optimal order convergence have not been obtained. The discontinuous finite elements are studied as follows:

1. L.Saint-P. Raviart(1974) solved the neutron transport equation, only convergence order $O(h^n)$.

2. C.Johnson-J.Pitkaranta(1984) solved a scalar hyperbolic equation by time-space discontinuous finite elements, convergence order is less one half order,

$$|e^-|_{\Gamma^+} + ||e||_Q \le C h^{n+1/2} ||u||_{n+1,Q}.$$

They pointed out that this is optimal if $u \in H^{n+1}(Q)$.

3. B.Cockburn and C.W.Shu (1989) proposed TVB-LDG to solve first order nonlinear hyperbolic system, i.e. use the discontinuous finite elements in space and the implicit Runge-Kutta scheme in time. A series of the important results are obtained by them.

4. For a scalar equation, we have used a special Radau orthogonal expansion to construct a comparison function u_I, and proved that $R = u - u_I$ has a high order weak estimate

$$|B(R,v)| \le Ch^{n+p}||u||_{n+p+1}||v||, \quad p \ge 1. \tag{7}$$

It leads to following error estimate

$$|\theta^-|_\Gamma + ||\theta||_Q \le Ch^{n+p}, \quad p \ge 1.$$

So the optimal convergence ($p = 1$ for $n \ge 0$) and superconvegrnce ($p = 2$ for $n \ge 1$) are derived.(My talk in First China-Germany Conference on Computational and Applied Mathematics. Berlin. Sept.5-10,2005).

In this talk, we discuss the hyperbolic system and got the main results as follows.

Theorem 1. Assume that rectangular subdivision is quasiuniform , then time-space bi-n degree discontinuous finite element U has full order error estimate

$$||u - U||_{\Gamma^+} + ||(u - U)(t)||_Q \le C(T,u)h^{n+1}, \ n \ge 0. \tag{8}$$

where the constant

$$C(T,u) = C(T)(|g|_{n+1,\Gamma^-} + ||u||_{n+2,Q}).$$

Take a $n+1$ order Radau point z with relatively fixed place in each element K and define a discrete norm

$$||e||_{l^2(Q)} = \{\sum_{z \in Q} |e(z)|^2 h^2\}^{1/2}.$$

Theorem 2. Assume that rectangular mesh is uniform and $a_1 > 0, a_2 > 0$, then bi-n degree discontinuous finite element U has superconvergence on $n+1$ order right Radau's product point z_K in each element K,

$$||e||_{l^2(Q)} \le Ch^{n+2}||u||_{n+3,Q}, \ n = 1, 2.$$

If $a_1 > 0, a_2 < 0$ are opposite, then on a common side of left-right (or upper-lower) adjacent elements K and K', both U_1 and U_2 have superconvergence at $n+1$ order right Radau product points z, as left (or lower) limit value for U_1 in this element, and as right (or upper) limit value for U_2 in an adjacent element. In particular, they always have superconvergence at right-upper angular points.
(Superconvergence also holds at Radau's points on one-dimensional lines).

§2 Element Orthogonal Expansion and Error Decomposition

Make transform $x = sh$, $s \in E = \{-1 < s < 1\}$ in a standard element $\tau = (-h, h)$. For smooth function $u(x) = u(hs) = u(s)$ and its parameter variable s, we have differentiating law, $\partial_s^i u(s) = D_i^x u(x) h^i = O(h^i)$.

Using Legendre orthogonal polynomial in E

$$l_0 = 1, l_1(s) = s, l_2(s) = (3s^2 - 1)/3, ..., l_j(s) = \partial_s^j (s^2 - 1)^j / (2^j j!),$$

$$l_j(s) \perp P_{j-1}(s),\ j > 0, \quad l_j(\pm 1) = (-1)^j,$$

we expand any function $u(s)$ in the form

$$u(s) = \sum_{j=0}^{\infty} a_j l_j(s),\ a_j = (j + 1/2)(u, l_j), \tag{9}$$

its Fourier coefficient has estimate, by integration by parts in $i \leq j$ times,

$$a_j = (j + 1/2)\gamma_j (-1)^i (\partial_s^i u, \partial^{j-i}(s^2 - 1)^j) = O(h^i),\ i \leq j. \tag{10}$$

Define n-degree part sum $u_n = \sum_{j=0}^n a_j l_j(s)$, its remainder R^* is orthogonal to any n-degree polynomials

$$R^* = u - u_n = \sum_{j=n+1}^{\infty} a_j l_j(s) = a_{n+1} l_j(s) + ... \perp P_n(s).$$

and can be estimated by Bramble-Hilbert lemma,

$$||R^*||_{0,p,\tau} \leq C h^{n+1+(1/p-1/2)} ||u||_{n+1,\tau}, \quad 1 \leq p \leq \infty.$$

Now we define the new bases

$$\phi_0 = 1,\ \phi_1 = l_1 - l_0, ..., \phi_j = l_j(s) - l_{j-1}(s) \perp P_{j-2}(s). \tag{11}$$

Obviously $\phi_j(1) = 0,\ \phi_j(-1) = 2(-1)^j, \quad j > 0$. Assume that

$$u(s) = \sum_{j=0}^{\infty} b_j \phi_j(s), \quad u_I(s) = P^s u = \sum_{j=0}^{n} b_j \phi_i(s),$$

where P^s is a projection operator from $L^2(\tau)$ to P_n. Using the expression of ϕ_j we rewrite

$$u(t) = (b_0 - b_1) l_0 + (b_1 - b_2) l_1 + ...$$

Comparing its Legendre's expansion, b_j can be expressed as a series (to be estimated by Bramble-Hilbert lemma)

$$b_j = a_j + b_{j+1} = a_j + a_{j+1} + a_{j+2} + ... = O(h^j)|D^j u|.$$

The roots s_j of $\phi_{n+1}(s) = 0$ called $n+1$-order (right) Radau's points Z. Denote an operator $G^s = I - P^s$, the remainder $R = G^s u$ has four important properties:

1. $R = G^s u = \sum_{j=n+1}^{\infty} b_j \phi_j(t) \perp P_{n-1}$,
2. $||R||_\tau \leq Ch^{n+1}||u||_{n+1,\tau}, \quad |R(s)| \leq Ch^{n+1-1/2}||u||_{n+1,\tau}$,
3. $R(1) = 0$,
   $R(-1) = 2(-1)^{n+1}(b_{n+1} - b_{n+2} + ...) = O(h^{n+1})$,
4. $|R(s_j)| \leq |b_{n+2}\phi_{n+2}(s_j) + ...| \leq Ch^{n+2-1/2}||u||_{n+2,\tau}$.

On $n+1$-order Radau's point-set Z, there is superconvergence in discrete norm

$$||R(Z)||_K^* = \{\frac{1}{n+1}\sum_{j=0}^{n}|R(s_j)|^2 h\}^{1/2} \leq Ch^{n+2}||u||_{n+2,K}.$$

Table 1. Roots s_j of basis $\phi_{n+1}(s) = l_{n+1}(s) - l_n(s)$.

n	s_j
1	1 , -1/3
2	1, 0.28989 79485 5664, -0.68989 79485 5663
3	1 , 0.57531 89235 2169, -0.18106 62711 1853, -0.82282 40809 7459
4	1 , 0.72048 02713 12438, 0.16718 08647 3783,
	-0.44631 39727 2375, -0.88579 16077 7097

In a rectangular element $K = \{-h < x, t < h\}$, we can construct similarly bi-n degree tensor product Radau orthogonal projection $u_I = P^t \otimes P^x u$, whose error has the important tensor product decomposition (see Douglas-Dupont-Wheeler(1974) in an elliptic finite element case)

$$R(sh, yh) = R(x,t) = (I - P_k^t \otimes P_h^x)u = G^t u + G^x u - G^t \otimes G^x u,$$

where

$$G^t u = (I - P^t)u \perp P_{n-1}(t), \quad G^x u = (I - P^x)u \perp P_{n-1}(x), \tag{12}$$
$$G^t G^x u = G^t \otimes G^x u \perp P_{n-1}(t) \otimes P_{n-1}(x).$$

Note that $G^t G^x u$ is high order small quantity(dependent on smoothness of u)

$$||G^t G^x u||_{0,\tau} \leq Ch^m ||u||_{m,\tau}, \quad n+1 \leq m \leq 2n+2.$$

Therefore the main part of the tensor product errors is a sum of the error in t and the error in x, this is very important property. Especially $R(1,1) = 0$ and

$$G^s u(1,y) = 0, \; R(1,y) = G^y u(1,y) \; at \; s = 1;$$

$$G^y u(s,1) = 0, \; R(s,1) = G^s u(s,1) \; at \; y = 1. \tag{13}$$

Note that $R = O(h^{n+1})$ on sides does'nt disappear, this is a new difficulty in multiple case. $G^t u, G^x u$ can be differentiated with respect to another variable and same orthogonality is reserved with the following laws (very important !),

$$D_x G^t u = G^s u_x \perp P_{n-1}(t), \quad D_t G^x u = G^x u_t \perp P_{n-1}(x). \tag{14}$$

Finally, on $n+1$-order Radau point-set $Z: \ (s_i, y_j), \ i, j = 0, 1, ..., n$, the remainder has high order accuracy

$$R(Z) = O(h^{n+2-2/p})||u||_{n+2,p,K}, \ ||R(Z)||_K^* = O(h^{n+2})||u||_{n+2,K}. \tag{15}$$

Later we denote the norm by $M_n = ||u||_{n,2,K}$ in element K. Noting an element area $mes(K) = O(h^2)$ and two dimensional embedding theorem $W^{1,2} \hookrightarrow L^p, \ p >> 1$, we can always understand approximately $M_n = Ch||u||_{n,\infty,K} \sim Ch||u||_{n+1,2,Q}$ if $u \in W^{n+1,2}(Q)$.

§3 Convergence Analysis for A Scalar Equation

Denote a conjugate operator of L by $L^* v = -v_t - (av)_x + bv, \ a > 0, \ b >> 0$. We get conjugate equality by integration by parts,

$$B_K(R, v) =< R^-, v^- >_{\gamma^+} + < R^-, v^+ >_{\gamma^-} + (R, L^* v)_K. \tag{16}$$

Proof of Theorem 1. Taking tensor product $u_I = P_h \otimes P_k u \in S_g^h$ as a comparison function, its error has the following tensor product decomposition

$$R = u - u_I = G^x u + G^t u - G^x G^t u,$$

Note that $G^t u = G^t G^x u = 0$ on line $t = t_j - 0$, and $G^x u$ is continuous with respect to t. Using the orthogonality $G^t u \perp v_t$ and integration by parts twice, we have

$$(R_t, v)_K - < R^+ - R^-, v^+ >_{\gamma^-}$$

$$=< R^-, v^- >_{\gamma^+} + < R^+, v^+ >_{\gamma^-} - < R^+ - R^-, v^+ >_{\gamma^-} - (R, v_t)_K$$

$$=< R^-, v^- >_{\gamma^+} + < R^-, v^+ >_{\gamma^-} - (G^x u, v_t)_K$$

$$=< G^x u, v^- >_{\gamma^+} + < G^x u, v^+ >_{\gamma^-} - (G^x u, v_t)_K = (G^x u_t, v)_K.$$

In general, we can derive the following**basic equality**

$$B_K(R, v) = (G^x u_t + aG^t u_x + qR, v)_K = l_K(v), \tag{17}$$

where the linear functional $l_k(v)$ in L^2 has an important estimate

$$|l_K(v)| \le Ch^{n+1}||u||_{n+2,K}||v||_{,K} \le \epsilon||v||_K^2 + Ch^{2n+2}||u||_{n+2,K}^2,$$

where $\epsilon > 0$ is suitably small. By (7), cancelling a term $||v||_K^2$ on the right side and summing over K, we get an estimate of $\eta = U - u_I$

$$|\eta^-|_\Gamma^2 + \sum_K |[\eta]|_{\gamma_-}^2 + ||\eta||_Q^2 \leq Ch^{2n+2}||u||_{n+2}^2. \tag{18}$$

Finally using a given estimate of $R = u - u_I$, the theorem 1 follows.

§4 Element Orthogonality Correction for a Scalar Equation

Proof of Theorem 2. Use orthogonality correction technique proposed by author (1997) to treat $B(R, v)$, i.e. decompose

$$S^h = S^0 + S'^h,\ S^0 = span\{\phi_{00}\},$$

$$S'^h = span\{\phi_{ij} = \phi_i(t)\phi_j(x), (i,j) \in I_n^*\},$$

$$I_n^* = \{(i,j) : 0 \leq i, j \leq n, i + j > 0\},$$

and hope to construct a correction function in element K

$$w = \sum_{(p,q)\in I_m^*} b_{pq}^* \phi_{pq}(t,x) \in S'^h, \quad w(1,1) = 0,$$

such that there is approximately orthogonal estimate

$$|B_K(R - w, v)| \leq Ch^{n+p}||u||_{n+1+p,K}||v||_K, \quad p \geq 2, \quad v \in S'^h. \tag{19}$$

By orthogonal relation $B(u - u_h, v) = 0$, so $\theta = u_h - u_I - w \in S^h$ will satisfies a new equality

$$B_K(\theta, v) = B_K(R - w, v) = l_K(v) - B_K(w, v) = r_K(v), \quad v \in S^h. \tag{20}$$

Taking $v = \theta \in S'^h$, by (20) we have **a general estimate**

$$|\theta^-|_{\gamma^+}^2 - |\theta^-|_{\gamma^-}^2 + |[\theta]|_{\gamma^-}^2 + ||\theta||_K^2 \leq 2|r_K(\theta)|. \tag{21}$$

If we define w such that $B(w, v') = l(v')$, which cannot be solved independently in K, because $B(w, v)$ includes a limit value w^- in former elements. Now we change its definition, i.e. replace w^- on former element K' by $w^-(\gamma^+)$ in this element K,

$$B'_K(w, v) = (Lw, v)_K - < w^+ - w^-(\gamma^+), v^+ >_{\gamma^-}$$

$$= (w, L^*v)_K + < w^-, v^- - v^+(\gamma^-) >_{\gamma^+},$$

which is only dependent in K. So

$$B_K(u, v) = B'_K(u, v) + B_K^0(u, v), \quad B_K^0(w, v) =< w^-(\gamma^+) - w^-, v^+ >_{\gamma^-} .$$

1. Now we decompose $v = v^0 + v'$ and construct $w \in S'$ in K satisfying **a new equation**

$$B'_K(w, v') = B_K(R, v) = l_K(v'),\ v' \in S'^h,\ w_0^- = 0. \tag{22}$$

Taking $v' = w$, then the correction function w has the following energy estimate

$$|w^-(\gamma^+) - w^+|^2_{\gamma^-} + 2||w||^2_K \le 2|l_K(w)|. \tag{23}$$

If w is already defined by (22), and noting $l(v^0) = 0$, then right side in (20) is simplified to the form

$$r_K(v) = l(v) - B(w, v) = -(w, pv^0)_K - < w^-(\gamma^+) - w^-, v^+ >_{\gamma^-}. \tag{24}$$

For w constructed above, we shall prove that $r_K(v)$ is of high order small

$$|r_K(v)| \le Ch^{n+2} M_{n+3} ||v'||_K,\quad n \ge 1. \tag{25}$$

Here the most difficulty is to estimate the second term in (24), the measure of γ has only $O(h)$, but one requires the translation quantity

$$|w^-(\gamma^+) - w^-|_{\gamma^-} \le Ch^{n+2+1/2} M_{n+3}.$$

2. To estimate w in K is still difficult. Take new trial functions

$$\phi_{ij} = \phi_i(t)\phi_j(x),\ \phi(1,1) = 0,$$

$$\phi_0(t) = 1,\ \phi_1(t) = t - 1,\ \phi_j(t) = M_j(t) = \partial_t^{j-2}(t^2 - 1)^{j-1},\quad j \ge 2.$$

Their derivatives have better orthogonality. Taking the correction function

$$w = \sum_{i,j \le n, i+j>0} b_{ij}\phi_{ij},\ w(1,1) = 0,\quad b' = \{b_{ij}, i + j > 0\}.$$

where b contains $(n + 1)^2 - 1 = n^2 + 2n$ parameters. Obviously,

$$w^- = w(1, x) = \sum_{j=1}^{n} b_{0j}\phi_j(x) = g(x),$$

$$w^+ = w(-1, x) = g(x) + 2\sum_{j=0}^{n} b_{1j}\phi_j(x),$$

So their difference can be simply expressed by b_{1j},

$$\eta = w^- - w^+ = w(1, x) - w(-1, x) = 2\sum_{j=0}^{n} b_{1j}\phi_j(x),$$

i.e. the boundary estimate (22) contains $2n + 1$ coefficients

$$b^* = \{b_{10}, b_{01}, b_{11}, b_{12}, b_{21}, ..., b_{1n}, b_{n1}\} \subset b'.$$

Note that the boundary norm $|w|_{\gamma^-} \approx |b^*|h^{1/2}$(Equivalence), and the element norm $||w||_K \approx |b'|h$. By (23), we have

$$|b^*|^2 h + |b'|^2 h^2 \le Ch^{n+1} M_{n+2} ||v'||_K \le Ch^{n+1} M_{n+2} |b'| h^{1/2}.$$

If $n = 1$, three parameters $b^* = \{b_{01}, b_{10}, b_{11}\} = b'$ are all of coefficients, and it leads to

$$|b^*| \le Ch^2 M_3 = O(h^3), \quad ||w||_K \le |b^*|h \le Ch^3 M_3, \quad |w_{j+1}^-|_\gamma \le Ch^{2+1/2} M_3.$$

To improve the estimate of the difference $\delta = w(K') - w(K)$, we use a translation techniques in two adjacent elements K and K'. By (22), δ satisfies

$$B_K(\delta, v') = l_{K'}(v') - l_K(v') = O(h^{n+2}) M_{n+3,K+K'} ||v'||_K,$$

So we get a high order estimate

$$|\delta|_{\gamma^-}^2 + ||\delta||_K^2 \le Ch^{n+2} M_{n+3}^2 ||\delta||_K,$$

When $n = 1$, we get

$$| < w^-(\gamma^+) - w^-, v' >_{\gamma^-} | \le C|\delta|_{\gamma^-} |v'|_{\gamma^-} \le Ch^3 M_4 ||v'||_K.$$

Therefore theorem 2 for $n = 1$ is proved.

3. But when $n \ge 2$, the techniques is not enough. At this time, b^* is only a part of b'. To estimate other coefficients $\bar{b}$, we have to choose other ways. Assume that, for simplicity of analysis, $p = 0$, so the equation (22) becomes a linear system of equations

$$Mb' = r, \quad r = O(h^{n+1}) M_{n+2},$$

where the matrix M is independent of h. Below we shall prove that its homogenous system $Mb' = 0$ has only zero solution. By the theory of linear system, the inverse M^{-1} exists and then $b' = M^{-1} r = O(h^{n+1}) M_{n+2}$. By a translation we also get $b'(K) - b'(K') = O(h^{n+2}) M_{n+3}$. The theorem 2 will be proved.

In fact, by energy estimate (23), it is known that $b^* = 0$. It remains to prove $\bar{b} = 0$. We take new test functions

$$\psi_{ij} = l_i(t) l_j(x) - 1, \ \psi_{ij}(1, 1) = 0, \ i + j > 0.$$

Due to $B(R, 1) = 0$, we can take off the constant 1 in ψ_{ij}. Obviously, $\psi_{ij} = l_i(t) l_j(t)$ have the best orthogonality.

When $n = 2$, it remains to analyze $w = a_{02}\phi_{02} + a_{20}\phi_{20} + a_{22}\phi_{22}$. Taking $v = t, x, tl_2(x)$ (or $l_2(t)x$) in (22), we get three equations $b_{22} = 3b_{20}, b_{22} = 3b_{02}, b_{22} = 0$ respectively, i.e. $b_{02} = b_{20} = b_{22} = 0$. So the theorem 2 holds for $n = 2$. Similarly the case $n = 3$ is also studied concretely.

Finally, if $a > 0, p > 0$ are variable coefficients, taking $a^0 = a(z)$ at a point $z \in K$ and $a - a^0 = O(h)$, the corresponding linear system is written in the form $Mb' = hM'b' + r$, where the matrix M is independent of h and M' is bounded. By above arguments, we get a new linear system $b' = hM^{-1}M'b' + M^{-1}r$. For h suitably small, using a contract mapping leads to $|b'| \le C|r| \le Ch^{n+1}M_{n+2}$. Other arguments above are still valid.

We hope that more general argument can be proposed.

§5 The Linear System with Variable Coefficients

First consider the case of $a_1 > 0, a_2 > 0$, its proof is almost same with single equation. Make bi-n degree right Radau projections for $u = (u_1, u_2)^T$ and its remainder $R = u - u_I = (R_1, R_2)^T$. We have the expression (without linear integrals)

$$B_K(R, v) = \int_K \{(G^x u_t + AG^t u_x + (G^x u - G^x G^t u)(A_x v - (A - A^0)v_x) + PRv\} dxdt,$$

and then

$$B_K(\theta, v) = B_K(R, v) \le Ch^{m+1}||u||_{m+2,K}||v||_K.$$

On the other hand, taking $v = \theta$, the energy on the left has

$$B_K(v, v) = \frac{1}{2}(|v^-|^2_{\gamma^+} + |[v]|^2_{\gamma^-} - |v^-|^2_{\gamma^-}) + ((P - A_x/2)v, v)_K,$$

Summing over all elements K and noting $v^- = \theta^- = 0$ on Γ^-, we have

$$B_Q(v, v) = \frac{1}{2}(|v|^2_{\Gamma^+} + \sum_{K \subset Q} |[v]|^2_{\gamma^-}) + ((P - A_x/2)v, v)_Q,$$

Denote $c_0 = \max_Q(|p_{12}|, |p_{21}|)$. Always assume $p_{11}, p_{22} \ge C_1 > 1 + c_0$. So

$$B_Q(v, v) \ge \frac{1}{2}(||v||^2_{\Gamma^+} + \sum_{\gamma^- \subset K} ||[v]||^2_{\gamma^-} + 2C_1||v||^2_Q - 2c_0||v_1||_Q||v_2||_Q).$$

or

$$||v||^2_{\Gamma^+} + \sum_K ||[v]||^2_{\gamma^-} + 2||v||^2_Q \le 2B(v, v) = 2B(R, v) \le Ch^{n+1}||u||_{n+2,Q}||v||_Q.$$

By Young's inequality, cancel $||v||_Q$ on the right side and get the error estimate

$$||\theta||^2_{\Gamma^+} + \sum_K ||[\theta]||^2_{\gamma^-} + ||\theta||^2_Q \le Ch^{2n+2}||u||^2_{n+2,Q}. \tag{26}$$

Finally the theorem 1 directly follows from estimate $||R||$.

Besides, we consider $a_1 > 0, a_2 < 0$. At this time, the characteristic $\beta_2 = (1, a_2)$ and the inflow boundary γ_2^- for u_2 are changed (see Fig. 2). In the analysis, a left Radau approximation $u_{2,I}$ for u_2 should be constructed. In the computation, the corresponding linear system of equations for U_1, U_2 should be solved simultaneously in a row elements. But its theoretical analysis is not essentially changed, the theorem 1 still holds.

Superconvergence. Assume that rectangular mesh is uniform and $a_1 > 0, a_2 > 0$. The key of proof is that for each equation, its main part $L_i u = D_t u_i + a_i D_x u_i$ is same with single equation. So we can construct the correction function $W = (W_1, W_2)^T$ and get the linear system in the form

$$\left(\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} + h \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \right) \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}.$$

where the matrices M_1, M_2 are independent of h, whose inverse M_1^{-1}, M_2^{-1} exist and M_{ij} are bounded. The corresponding linear system can be rewritten in the form

$$W_1 = A_1^{-1}(-A_{11} h W_1 - A_{12} h W_2 + r_1),$$

$$W_2 = A_2^{-1}(-A_{21h} W_1 - A_{22h} W_2 + r_2).$$

By a contract map and translation techniques, it leads to

$$W = O(h^{n+1})||u||_{n+2,K}, \quad W(K) - W(K') = O(h^{n+2})||u||_{n+3,K+K'}.$$

So superconvergence can be still proved.

The proof for $a_1 > 0, a_2 < 0$ is similar. But now, superconvergence structures change. U_1 has still $n+1$ order right Radau product points, but U_2 has the product points of right Radau points (for t) and left Radau points (for x). At this case, in a same element, U_1 and U_2 have not a common superconvergence point. But in fact, on the common side in two left-right (or upper-lower) adjacent elements, at $n+1$ order right Radau points, both U_1 as a left (or lower) limit and U_2 as a right (or upper) limit have superconvergence. In particular, two angular points on the upper side of K are always common superconvergence points.

Remark. We find in surprise in numerical experiments that bi-$n \geq 2$ degree discontinuous finite elements have ultraconvergence $O(h^{n+2+1/2})$ at the right-upper angular points in each element. This wonderful phenomenon will be discussed in another paper.

§6 Numerical Experiments

Subdivide a square $Q = [0,1] \times [0,1]$ into the $N \times N$ uniform rectangular elements and use bi-quadratic elements. Take third order Radau point z with relatively fixed place in each element K and define a discrete norm in $l^2(Q)$ by

$$||e||_{l^2(Q)} = \{\sum_{z \in Q} |e(z)|^2 h^2\}^{1/2}.$$

1. Two positive characteristic directions.

$$\begin{cases} u_t + (1 + x + 2t)u_x + v = f_1, & u(x,0) = 0, \; u(0,t) = 0, \\ v_t + (1 + 2x + 3t)v_x + 2u = f_2, & v(x,0) = 0, \; v(0,t) = 0. \end{cases} \tag{27}$$

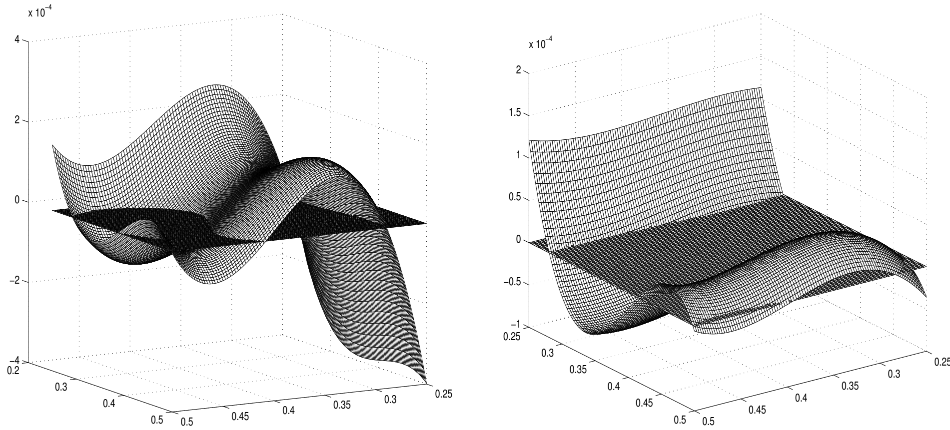


Figure 1. The error surfaces in an element(left for u, right for v) on 4×4 meshes.

Take an exact solution $(x,t) = sin(x)(e^t - 1), v(x,t) = (e^x - 1)(cos(t) - 1)$. The boundary values given on the left boundary and the computation can be completed in each element from the left to the right. The average quadratic root error at the right-upper or left-lower angular points in all elements are listed in table 2. We see that these errors at the left-lower angular points(inflow points) have convergence $O(h^3)$, whereas they have superconveregcne about $O(h^{4.5})$ at the right-upper angular points(outflow points).

Table 2. The errors and their ratio for two positive characteristics

N	u(right-upper)	v(right-upper)	u(left-lower)	v(left-lower)
4	2.939e-6	7.819e-6(30.834)	1.458e-4	3.737e-5
8	1.052e-7(28.0)	2.310e-7(33.8)	1.945e-5(7.50)	5.377e-6(6.95)
16	4.155e-9(25.3)	7.186e-9(32.2)	2.476e-6(7.86)	7.182e-7(7.49)
32	1.739e-10 (23.9)	2.374e-10(30.3)	3.112e-7(7.95)	9.321e-8(7.71)
64	7.494e-12(23.2)	8.390e-12(28.3)	3.898e-8(7.98)	1.189e-8(7.84)

2. Two characteristics have opposite signs

$$\begin{cases} u_t + (1+x+2t)u_x + v = f_1, \quad u(x,0) = 0, \ u(0,t) = 0, \\ v_t - (2+x+t)v_x + 3u = f_2, \quad v(x,0) = 0, \ v(1,t) = 0. \end{cases} \tag{28}$$

take the exact solution $u(x,t) = sin(x)(e^t - 1), v(x,t) = sin(t)(e^x - e)$. From the table 3, we see that U at left-lower angular points and V at right-upper angular points have convergence order $O(h^3)$, whereas U at right-upper angular points and V at left-lower angular points have superconvergence about $O(h^{4.5})$.

Table 3. The errors and their ratio for two opposite characteristics

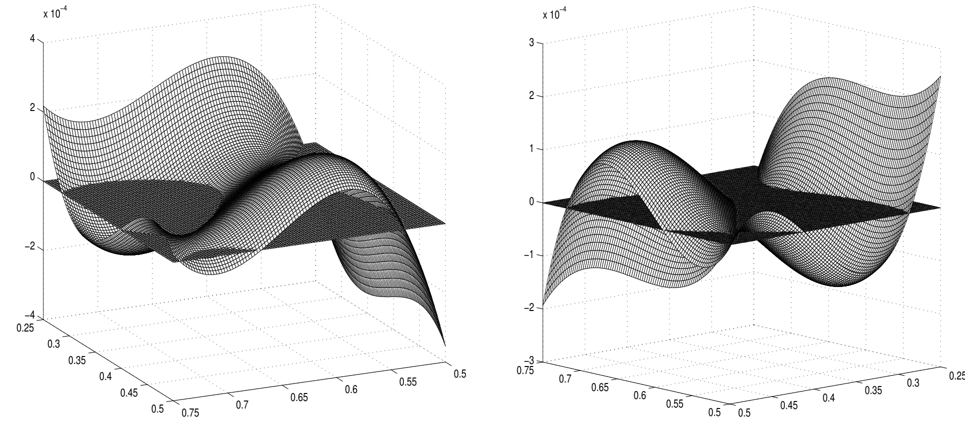


Figure 2. The error surfaces in an element(left for u,right for v) on 4×4 meshes.

N	u(right-upper)	v(left-upper)	u(left-lower)	v(right-lower)
4	3.479e-6	8.876e-6	1.456e-4	2.142e-4
8	1.303e-7 (26.7)	5.905e-7(15.0)	1.952e-5 (7.46)	2.779e-5(7.71)
16	5.403e-9(24.1)	3.843e-8(15.4)	2.483e-6(7.86)	3.491e-6(7.96)
32	2.401e-10(22.5)	2.456e-9(15.7)	3.118e-7(7.96)	4.3591e-7(8.01)
64	1.129e-11(21.3)	1.553e-10(15.8)	3.902e-8(7.99)	5.441e-8(8.01)

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