Discretization of integro-differential equations modelling dynamic fractional order viscoelasticity

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Outline

- Fractional order viscoelasticity
- Stability and regularity
- Spatial discretization
- A priori error estimates
- Temporal discretization
- A priori error estimates
- Earlier and ongoing work

Fractional order linear viscoelasticity

stress tensor: σ_{ij}, displacement vector: u_i

strain tensor: $\epsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)$

$\sigma_{ij} = s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij}, \quad \epsilon_{ij} = e_{ij} + \frac{1}{3}\epsilon_{kk}\delta_{ij}$ shear and bulk modes

constitutive equations (Bagley and Torvik 1983):

$$s_{ij}(t) + \tau_1^{\alpha_1} D_t^{\alpha_1} s_{ij}(t) = 2G_\infty e_{ij}(t) + 2G\tau_1^{\alpha_1} D_t^{\alpha_1} e_{ij}(t)$$
$$\sigma_{kk}(t) + \tau_2^{\alpha_2} D_t^{\alpha_2} \sigma_{kk}(t) = 3K_\infty \epsilon_{kk}(t) + 3K\tau_2^{\alpha_2} D_t^{\alpha_2} \epsilon_{kk}(t)$$

relaxation time: $\tau_i > 0$, differentiation order: $\alpha_i \in (0,1)$

$$D_t^\alpha f(t) = D_t D_t^{-1+\alpha} f(t) = D_t \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f(s)\, ds$$

Fractional order linear viscoelasticity

solve for σ by means of Laplace transformation:

$$s_{ij}(t) = 2G\Big(e_{ij}(t) - \frac{G - G_\infty}{G}\int_0^t f_1(t-s)e_{ij}(s)\,ds\Big)$$

$$\sigma_{kk}(t) = 3K\Big(\epsilon_{kk}(t) - \frac{K - K_\infty}{K}\int_0^t f_2(t-s)\epsilon_{kk}(s)\,ds\Big)$$

where

$$f_i(t) = -\frac{d}{dt}E_{\alpha_i}\Big(-\Big(\frac{t}{\tau_i}\Big)^{\alpha_i}\Big), \quad E_\alpha(t) = \sum_{n=0}^{\infty}\frac{t^n}{\Gamma(1+\alpha n)}$$ Mittag-Leffler func.

simplifying assumptions: $\alpha = \alpha_1 = \alpha_2, \quad \tau = \tau_1 = \tau_2, \quad f = f_1 = f_2$

$$\gamma = \frac{G - G_\infty}{G} = \frac{K - K_\infty}{K} \in (0,1), \quad \beta(t) = \gamma f(t), \quad \mu = G, \quad \lambda = K - \tfrac{2}{3}G$$

Fractional order linear viscoelasticity

$$\sigma_{ij}(t) = \Big(2\mu\epsilon_{ij}(t) + \lambda\epsilon_{kk}(t)\delta_{ij}\Big) - \int_0^t \beta(t-s)\Big(2\mu\epsilon_{ij}(s) + \lambda\epsilon_{kk}(s)\delta_{ij}\Big)\,ds$$

$$\gamma \in (0,1), \quad \alpha \in (0,1), \quad \tau > 0$$

$$\beta(t) = -\gamma\frac{d}{dt}E_\alpha\Big(-\Big(\frac{t}{\tau}\Big)^\alpha\Big) = \gamma\frac{\alpha}{\tau}\Big(\frac{t}{\tau}\Big)^{-1+\alpha}E'_\alpha\Big(-\Big(\frac{t}{\tau}\Big)^\alpha\Big) \approx Ct^{-1+\alpha}, \;\; t \to 0$$

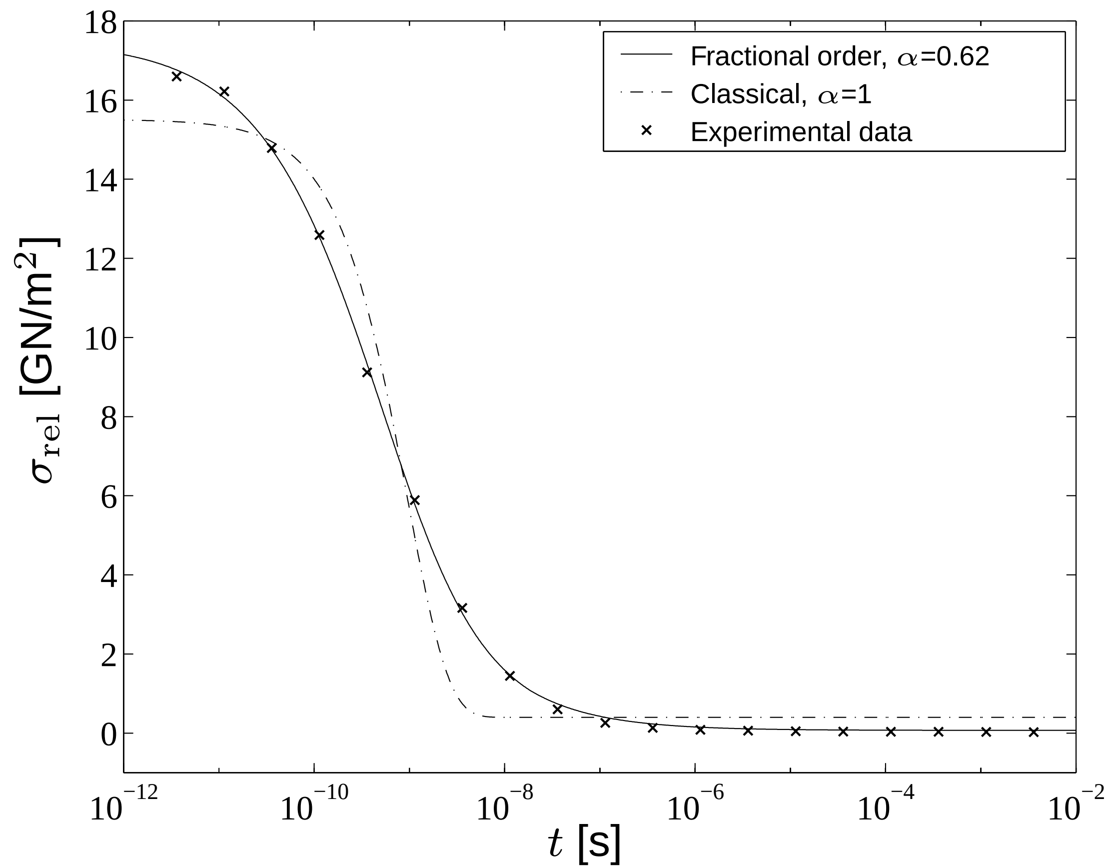
$$\beta(t) \geq 0, \quad \|\beta\|_{L_1(\mathbf{R}^+)} = \int_0^\infty \beta(t)\,dt = \gamma\Big(E_\alpha(0) - E_\alpha(\infty)\Big) = \gamma < 1$$

$$(-1)^n\frac{d^n}{dt^n}\beta(t) \geq 0 \quad \Rightarrow \quad (\beta * \phi, \phi) = \int_0^T \int_0^t \beta(t-s)\phi(s)\,ds\,\phi(t)\,dt \geq 0$$

viscoelastic model: time domain, few parameters

Stress response to unit strain – parameter fit

Uniaxial data at small strains for the rubber material: GR-S gum vulcanizates. Data from Tobolsky. Classical: $\alpha = 1,\ \beta(t) = \frac{\gamma}{\tau}\exp\Big(-\frac{t}{\tau}\Big)$.



Equations of motion

equations of motion:

$$\begin{cases} \rho \ddot{u}_i - \sigma_{ij,j} = f_i, & \text{in } \Omega \\ u_i = 0, & \text{on } \Gamma \end{cases}$$

(also: traction boundary condition $\sigma \cdot n = g$ on Γ_{N})

constitutive equations:

$$\begin{cases} \epsilon_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \\ \sigma_{ij}(t) = \Big(2\mu\epsilon_{ij}(u(t)) + \lambda\epsilon_{kk}(u(t))\delta_{ij}\Big) \\ \qquad\qquad - \int_0^t \beta(t-s)\Big(2\mu\epsilon_{ij}(u(s)) + \lambda\epsilon_{kk}(u(s))\delta_{ij}\Big)\, ds \end{cases}$$

Abstract formulation

$$V = \left[H_0^1(\Omega)\right]^3, \quad \|v\| = \|v\|_{L_2} = \left(\int_\Omega v_i v_i \, dx\right)^{1/2}, \quad (f,v) = \int_\Omega f_i v_i \, dx$$

$$a(u,v) = \int_\Omega \Big(2\mu\epsilon_{ij}(u)\epsilon_{ij}(v) + \lambda\epsilon_{ii}(u)\epsilon_{jj}(v)\Big)\, dx$$

weak formulation:

$$\begin{cases} u(t) \in V, \quad u(0) = u_0, \quad \dot{u}(0) = v_0 \\ \rho(\ddot{u}(t),\psi) + a(u(t),\psi) - \displaystyle\int_0^t \beta(t-s) a(u(s),\psi)\, ds = (f(t),\psi) \quad \forall \psi \in V \end{cases}$$

strong formulation:

$$\rho\ddot{u}(t) + Au(t) - \int_0^t \beta(t-s) Au(s)\, ds = f(t), \quad (Au)_i = -(2\mu\epsilon_{ij}(u) + \lambda\epsilon_{kk}(u)\delta_{ij})_{,j}$$

Stability and regularity

$$\xi(t) = 1 - \int_0^t \beta(s)\,ds, \quad \xi(0) = 1, \quad \lim_{t\to\infty} \xi(t) = 1-\gamma, \quad \xi(t) \geq 1-\gamma > 0$$

$w(t,s) = u(t) - u(t-s),\ s \in [0,t],$ "the history"

$$\begin{aligned} Au(t) &- \int_0^t \beta(t-s)\big(Au(s) \pm Au(t)\big)\,ds \\ &= Au(t) - \int_0^t \beta(t-s)\,ds\,Au(t) + \int_0^t \beta(t-s)\big(Au(t) - Au(s)\big)\,ds \\ &= \Big(1 - \int_0^t \beta(s)\,ds\Big)Au(t) + \int_0^t \beta(s)\big(Au(t) - Au(t-s)\big)\,ds \\ &= \xi(t)Au(t) + \int_0^t \beta(s)Aw(t,s)\,ds \end{aligned}$$

$$\rho(\ddot{u}(t),\psi) + \xi(t)a(u(t),\psi) + \int_0^t \beta(s)a(w(t,s),\psi)\,ds = (f(t),\psi) \quad \forall \psi \in V$$

Theorem 1: stability and regularity

$$\|v\|_l = \|A^{l/2}v\| = \sqrt{(v, A^l v)}, \quad l \in \mathbf{R}$$

$$\rho(\ddot{u}(t), \psi) + \xi(t)a(u(t), \psi) + \int_0^t \beta(s)a(w(t,s), \psi)\, ds = (f(t), \psi) \quad \forall \psi \in V$$

take $\quad \psi = A^l \dot{u}(t)$

$$\rho\|\dot{u}(T)\|_l^2 + \underbrace{\xi(T)}_{\geq 1-\gamma} \|u(T)\|_{l+1}^2 + \int_0^T \beta(t)\|u(t)\|_{l+1}^2\, dt$$

$$+ \int_0^T \beta(s)\|w(T,s)\|_{l+1}^2\, ds + \underbrace{\int_0^T \int_0^t [\beta(s) - \beta(t)]\, D_s\|w(t,s)\|_{l+1}^2\, ds\, dt}_{\geq 0}$$

$$= \rho\|v_0\|_l^2 + \|u_0\|_{l+1}^2 + 2\int_0^T (f, A^l \dot{u})\, dt$$

Theorem 1: stability and regularity

$$\rho^{\frac{1}{2}}\|\dot{u}(T)\|_l + (1-\gamma)^{\frac{1}{2}}\|u(T)\|_{l+1} \leq C\Big[\rho^{\frac{1}{2}}\|v_0\|_l + \|u_0\|_{l+1} + \rho^{-\frac{1}{2}}\int_0^T \|f\|_l \, dt\Big]$$

$l \in \mathbf{R}$

Assume elliptic regularity: $\|v\|_{H^2} \leq C\|Av\| = C\|v\|_2 \quad \forall v \in D(A)$

$$\|\dot{u}(T)\|_{H^1} + \|u(T)\|_{H^2} \leq C\Big[\|v_0\|_{H^1} + \|u_0\|_{H^2} + \rho^{-\frac{1}{2}}\int_0^T \|f\|_{H^1} \, dt\Big]$$

$$\begin{aligned}\|\ddot{u}(T)\|_{H^2} &\leq C\Big\|Au(T) - \int_0^T \beta(T-t)Au(t)\,dt + f(T)\Big\|_2 \\ &\leq C \max_{0\leq t\leq T}\|u(t)\|_4 + \|f(T)\|_2\end{aligned}$$

Functional analytic framework

Fabiano and Ito (1990)

$$\ddot{u}(t) + Au(t) - \int_{-\infty}^{t} \beta(t-s)Au(s)\,ds = f(t)$$

$$\ddot{u}(t) + Au(t) - \int_{0}^{t} \beta(t-s)Au(s)\,ds = f(t) + \int_{-\infty}^{0} \beta(t-s)Au(s)\,ds$$

known pre-history

Functional analytic framework

Fabiano and Ito (1990)

$$\ddot{u}(t) + Au(t) - \int_{-\infty}^{t} \beta(t-s)Au(s)\,ds = f(t)$$

$$\ddot{u}(t) + Au(t) - \int_{0}^{\infty} \beta(s)Au(t-s)\,ds = f(t)$$

$$\ddot{u}(t) + \Big(1 - \int_{0}^{\infty} \beta(s)\Big)Au(t) + \int_{0}^{\infty} \beta(s)A\big(u(t) - u(t-s)\big)\,ds = f(t)$$

$$\ddot{u}(t) + (1-\gamma)Au(t) + \int_{0}^{\infty} \beta(s)A\big(u(t) - u(t-s)\big)\,ds = f(t)$$

$$v = \dot{u}, \quad \dot{v}(t) + (1-\gamma)Au(t) + \int_{0}^{\infty} \beta(s)Aw(t,s)\,ds = f(t)$$

Functional analytic framework

Fabiano and Ito (1990)

$$\dot{u} = v, \quad \dot{v}(t) + (1-\gamma)Au(t) + \int_0^\infty \beta(s)Aw(t,s)\,ds = f(t), \quad \dot{w} = v - w'$$

$$U(t) = \begin{bmatrix} u(t) \\ v(t) \\ w(t,\cdot) \end{bmatrix} \in H^1(\Omega) \times L_2(\Omega) \times W^1_{2,\beta}((0,\infty); H^1(\Omega)) = X$$

$$\dot{U}(t) + \mathcal{A}U(t) = F(t)$$

$\mathcal{A}$ is dissipative on X: $\quad \langle \mathcal{A}V, V \rangle \geq 0, \quad V \in D(\mathcal{A})$

Lumer-Phillips theorem, strongly continuous semigroup

existence and uniqueness

Spatial discretization

standard finite element space: $\quad V_h \subset V$

$$\begin{cases} u_h(t) \in V_h, \quad u_h(0) = u_{0,h}, \quad \dot{u}_h(0) = v_{0,h} \\ \rho(\ddot{u}_h(t), \psi) + a(u_h(t), \psi) - \displaystyle\int_0^t \beta(t-s) a(u_h(s), \psi)\, ds = (f(t), \psi) \; \forall \psi \in V_h \end{cases}$$

Theorem 2: stability

$$\begin{cases} A_h : V_h \longrightarrow V_h \\ a(v_h, w_h) = (A_h v_h, w_h) \quad \forall v_h, w_h \in V_h \end{cases}$$

$$\|v_h\|_{h,l} = \|A_h^{l/2} v_h\| = \sqrt{(v_h, A_h^l v_h)}, \quad v_h \in V_h,\ l \in \mathbf{R}$$

$$\|\dot{u}_h(T)\|_{h,l} + \|u_h(T)\|_{h,l+1} \leq C\Big[\|v_{0,h}\|_{h,l} + \|u_{0,h}\|_{h,l+1} + \int_0^T \|P_h f\|_{h,l}\,dt\Big]$$

$l \in \mathbf{R}$

we use only $\quad l = 0,\ -1$

Theorem 3: A priori error estimates

$e = u_h - u$

$$
\begin{aligned}
\|\dot e(T)\| &\le C\Big(\|v_{h,0} - v_0\| + \|u_{h,0} - R_h u_0\|_{H^1}\Big) \\
&\qquad + Ch^2\Big(\|v_0\|_{H^2} + \|\dot u(T)\|_{H^2} + \int_0^T \|\ddot u\|_{H^2}\,dt\Big), \\
\|e(T)\|_{H^1} &\le C\Big(\|v_{h,0} - v_0\| + \|u_{h,0} - u_0\|_{H^1}\Big) \\
&\qquad + Ch\Big(\|v_0\|_{H^1} + \|u_0\|_{H^2} + \|u(T)\|_{H^2} + \int_0^T \|\ddot u\|_{H^1}\,dt\Big), \\
\|e(T)\| &\le C\Big(\|v_{h,0} - v_0\| + \|u_{h,0} - u_0\|\Big) \\
&\qquad + Ch^2\Big(\|v_0\|_{H^2} + \|u_0\|_{H^2} + \|u(T)\|_{H^2} + \int_0^T \|\ddot u\|_{H^2}\,dt\Big).
\end{aligned}
$$

Proof: same as for the wave equation.

Time discretization – discontinuous Galerkin method

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n < \cdots < t_N = T$$

$$I_n = (t_{n-1}, t_n), \quad k_n = t_n - t_{n-1}$$

$$\mathcal{W} = \Big\{ w : w(t) = w_n \text{ for } t \in I_n,\ w_n \in V_h,\ n = 1, \dots, N \Big\}$$

$$\begin{cases} U_1, U_2 \in \mathcal{W}, \quad U_{1,0}^- = u_{0,h},\ U_{2,0}^- = v_{0,h}, \quad \text{and for } n = 1, \dots, N, \\ \displaystyle\int_{I_n} \Big((\dot{U}_1(t), V_1(t)) - (U_2(t), V_1(t)) \Big)\, dt + ([U_1]_{n-1}, V_{1,n-1}^+) = 0, \\ \displaystyle\int_{I_n} \Big(\rho(\dot{U}_2(t), V_2(t)) + a(U_1(t), V_2(t)) \\ \displaystyle\qquad - \int_0^t \beta(t-s) a(U_1(s), V_2(t))\, ds - (f(t), V_2(t)) \Big)\, dt + \rho([U_2]_{n-1}, V_{2,n-1}^+) = 0 \\ \hfill \forall V_1, V_2 \in \mathcal{W} \end{cases}$$

$$U_i|_{I_n} = U_{i,n} \in V_h,\ V_i|_{I_n} = \chi_i \in V_h \ , \quad i = 1, 2, \quad \text{implicit Euler}$$

Temporal discretization – discontinuous Galerkin

$$\begin{cases} k_n^{-1}\big(U_{1,n} - U_{1,n-1}\big) - U_{2,n} = 0 \\ k_n^{-1}\big(U_{2,n} - U_{2,n-1}\big) + A_h U_{1,n} - q_n(A_h U_1) - P_h \bar{f}_n = 0 \end{cases}$$

where

$$\begin{aligned} \bar{f}_n &= \frac{1}{k_n}\int_{t_{n-1}}^{t_n} f(t)\,dt \\ q_n(A_h U_1) &= \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\int_0^t \beta(t-s)A_h U_1(s)\,ds\,dt \\ &= \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j\wedge t}\beta(t-s)A_h U_{1,j}\,ds\,dt = \sum_{j=1}^{n} k_j \omega_{nj} A_h U_{1,j} \\ \omega_{nj} &= \frac{1}{k_n k_j}\int_{t_{n-1}}^{t_n}\int_{t_{j-1}}^{t_j\wedge t}\beta(t-s)\,ds\,dt, \quad t_j \wedge t = \min(t_j, t) \end{aligned}$$

Temporal discretization – discontinuous Galerkin

$$k_n^{-1}\big(U_{2,n} - U_{2,n-1}\big) + A_h U_{1,n} - \sum_{j=1}^{n} k_j \omega_{nj} A_h U_{1,j} - P_h \bar{f}_n = 0$$

$$k_n^{-1}\big(U_{2,n} - U_{2,n-1}\big) + (1 - k_n \omega_{nn}) A_h U_{1,n} = \sum_{j=1}^{n-1} k_j \omega_{nj} A_h U_{1,j} + P_h \bar{f}_n$$

$$\begin{aligned} k_n \omega_{nn} &= \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} \beta(t-s)\, ds\, dt \\ &\approx \frac{\gamma}{(1+\alpha)\Gamma(1+\alpha)} \Big(\frac{k_n}{\tau}\Big)^{\alpha} < 1 \quad \text{if } k_n \text{ small} \end{aligned}$$

Time discretization — stability

$$
\begin{aligned}
&Au(t) - \int_0^t \beta(t-s) Au(s)\, ds \\
&\quad = Au(t) - \int_0^t \beta(t-s)\, ds Au(t) + \int_0^t \beta(t-s)\big(Au(t) - Au(s)\big)\, ds \\
&\quad = \left(1 - \int_0^t \beta(s)\, ds\right) Au(t) + \int_0^t \beta(s)\big(Au(t) - Au(t-s)\big)\, ds
\end{aligned}
$$

discrete time, variable steps: $U_1(t-s) \notin \mathcal{W}$

Time discretization — stability

$$
\begin{aligned}
Au(t) - \int_0^t \beta(t-s)Au(s)\,ds \\
&= Au(t) - \int_0^t \beta(t-s)\,ds\,Au(t) + \int_0^t \beta(t-s)\big(Au(t) - Au(s)\big)\,ds \\
&= \Big(1 - \int_0^t \beta(s)\,ds\Big)Au(t) + \int_0^t \beta(t-s)A\big(u(t) - u(s)\big)\,ds \\
&= \xi(t)Au(t) + \int_0^t \beta(t-s)Aw(t,s)\,ds
\end{aligned}
$$

$$
\rho(\ddot{u}(t),\psi) + \xi(t)a(u(t),\psi) + \int_0^t \beta(t-s)a(w(t,s),\psi)\,ds = (f(t),\psi) \quad \forall \psi \in V
$$

Time discretization — stability

the crucial term:

$$\int_0^T \int_0^t \beta(t-s) a(w(t,s), \dot{u}(t))\, ds\, dt$$

$$\Big\{ w(t,s) = u(t) - u(s), \quad \dot{u}(t) = D_t w(t,s) \Big\}$$

$$\begin{aligned} &= \int_0^T \int_0^t \beta(t-s) a(w(t,s), D_t w(t,s))\, ds\, dt \\ &= \frac{1}{2} \int_0^T \int_0^t \beta(t-s) D_t \|w(t,s)\|_1^2\, ds\, dt \\ &= \frac{1}{2} \int_0^T \int_s^T \beta(t-s) D_t \|w(t,s)\|_1^2\, dt\, ds \end{aligned}$$

Time discretization — stability

the crucial term:

$$\int_0^T \int_0^t \beta(t-s)a(w(t,s),\dot{u}(t))\,ds\,dt$$

$$= \frac{1}{2}\int_0^T \int_s^T \beta(t-s)D_t\|w(t,s)\|_1^2\,dt\,ds$$

$$= \frac{1}{2}\int_0^T \beta(T-s)\|w(T,s)\|_1^2\,ds - \frac{1}{2}\int_0^T \beta(0)\|w(s,s)\|_1^2\,ds$$

$$- \frac{1}{2}\int_0^T \int_s^T \beta'(t-s)\|w(t,s)\|_1^2\,dt\,ds \quad \geq 0 \quad ??$$

Time discretization — stability

the crucial term:

$$
\begin{aligned}
&\int_0^T \int_0^{t-\epsilon} \beta(t-s)a(w(t,s),\dot{u}(t))\,ds\,dt \\
&\quad = \frac{1}{2}\int_0^T \int_{s+\epsilon}^T \beta(t-s)D_t\|w(t,s)\|_1^2\,dt\,ds \\
&\quad = \frac{1}{2}\int_0^T \beta(T-s)\|w(T,s)\|_1^2\,ds - \frac{1}{2}\int_0^T \beta(\epsilon)\|w(s+\epsilon,s)\|_1^2\,ds \\
&\qquad - \frac{1}{2}\int_0^T \int_{s+\epsilon}^T \beta'(t-s)\|w(t,s)\|_1^2\,dt\,ds \quad \geq 0 \quad \text{as } \epsilon \to 0^+ \text{ ??}
\end{aligned}
$$

Time discretization — stability

the crucial term:

$$
\begin{aligned}
&\int_0^T \int_0^{t-\epsilon} \beta(t-s) a(w(t,s), \dot{u}(t))\, ds\, dt \\
&\quad = \frac{1}{2} \int_0^T \int_{s+\epsilon}^T \beta(t-s) D_t \|w(t,s)\|_1^2\, dt\, ds \\
&\quad = \frac{1}{2} \int_0^T \beta(T-s) \|w(T,s)\|_1^2\, ds - \frac{1}{2} \int_0^T \beta(\epsilon) \|w(s+\epsilon, s)\|_1^2\, ds \\
&\qquad - \frac{1}{2} \int_0^T \int_{s+\epsilon}^T \beta'(t-s) \|w(t,s)\|_1^2\, dt\, ds \quad \geq 0 \quad \text{as } \epsilon \to 0^+ \text{ ??}
\end{aligned}
$$

This works with $\epsilon = 0$ in the time-discrete case.

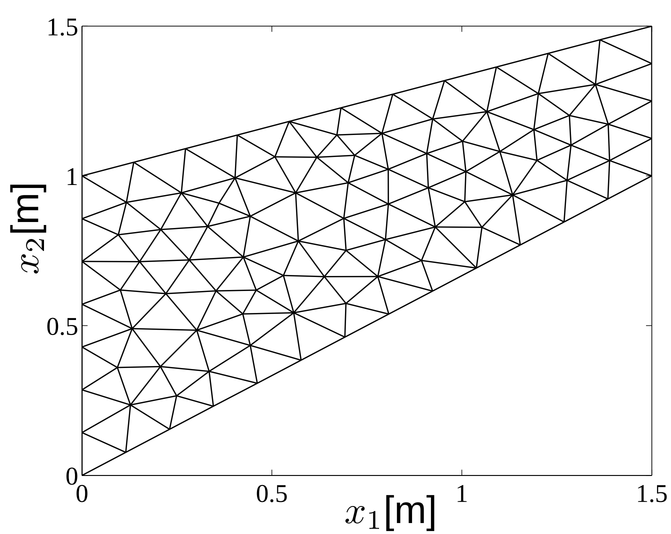
Time discretization

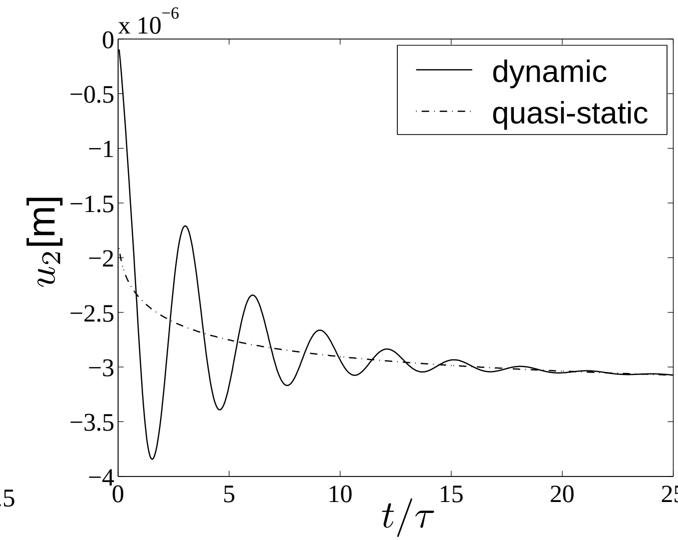
We can now prove stability and error estimates in a similar way as in the semidiscrete case.

$$\|U_{2,n}\|_{h,l} + \|U_{1,n}\|_{h,l+1} \le C \Big[\|v_{0,h}\|_{h,l} + \|u_{0,h}\|_{h,l+1} + \int_0^T \|P_h f\|_{h,l} \, dt \Big]$$

$0 \le t_n \le t_N = T, \quad l \in \mathbf{R}$

Numerical experiment





$$u(x,0) = 0\,\text{m}, \quad \dot{u}(x,0) = 0\,\text{m/s}, \quad f(x,t) = 0\,\text{N/m}^3,$$
$$u(x,t) = 0\,\text{m at } x_1 = 0\,\text{m}, \quad g(x,t) = (0,-1)\Theta(t)\,\text{Pa at } x_1 = 1.5\,\text{m},$$
$$\gamma = 0.5, \quad E_0 = 10\,\text{MPa}, \quad \alpha = 0.5, \quad \nu = 0.3, \quad \rho = 40\,\text{kg/m}^3,$$

The left figure shows the spatial discretization. The right figure shows the computed vertical displacement at the point (1.5,1.5) m.

Earlier work

- smooth kernel: Lin, Thomée, and Wahlbin (a priori error estimates)
- quasi-static motion $\rho\ddot{u} \approx 0$:

  Shaw and Whiteman (a posteriori, adaptivity)

  Adolfsson, Enelund, Larsson (a posteriori, adaptivity, sparse history)

Ongoing work

- a posteriori error estimates
- adaptivity
- sparse history
- other time-stepping methods
- two kernels: β_1 (shear), β_2 (bulk)