Paper we review the use of shadows in the study of chaotic iterations such as (1.1) (Section 4).

Shadows provide a means for the analysis of such situations. In this exposition

one can see how the use of shadows is often used to find such solutions. This

method provides a means for studying the behaviour of numerical methods in long-time

integrations. We refer to [1] for a detailed discussion.

Numerical methods are often used to investigate the long-time behavior of solutions of

ordinary or partial differential equations. Similarly, mappings such as the well-known logistic

expression

absence

Keywords. Numerical methods; chaotic dynamics; shadows; error estimation.

and in the analysis of numerical integrators in the neighbourhood of hyperbolic equilibrium points, such as saddles (Section 5). Sections 2 and 3 are respectively devoted to the presentation of the idea of shadowing and to the shadowing lemma. The final Section 6 contains some concluding remarks.

2. Shadowing: the basic idea

Let us begin with a very simple example. Consider the scalar initial value problem

$$\frac{dx}{dt} = -x, \quad 0 \leq t \leq T_{\text{max}}, \quad x(0) = 1, \quad x(0) = 1,$$

that we integrate by Euler's method

$$x_{n+1} = \psi x_n := (1 - \Delta t)x_n, \quad 0 \leq n \leq \left[\frac{T_{\text{max}}}{\Delta t}\right], \quad x_0 = 1.$$  

Figure 1 depicts the exact trajectory (solid line) and the computed points (stars) when $T_{\text{max}} = 5$ and $\Delta t = 0.1$. The largest absolute value of the errors $e_n := x(n\Delta t) - x_n$ is 0.019 and occurs at $n = 10$ (i.e. at $t = 1$). Note that $|e_n|$ does not grow larger and larger as $n$ increases. The reason for this is quite simple, see e.g. the venerable textbook by Isaacson and Keller [16, Chapter 8, Section 1, Corollary 2]. The error $e_n$ can be written as

$$e_n = x(n\Delta t) - x_n = [x(n\Delta t) - \psi x((n-1)\Delta t)] + \psi x((n-1)\Delta t) - \psi x_{n-1},$$

![Graph](image-url)
conditioning a solution of the correct differential equation with a wrong initial approximation, which we integrated numerically; however they show no poor approximation to the solution of the system of the form: 

\[ \begin{aligned}
&\phi' = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \\
&\chi' = \frac{\partial \chi}{\partial x} + \frac{\partial \chi}{\partial y}
\end{aligned} \]

This shows that the maximum error occurs where the derivative of the function is zero. We have seen that this maximum occurs at the maximum value of \( \phi \), which is \( y_0 \). With this in mind, we can now state that the maximum error occurs at \( y_0 \). 

The solution to (2.7) is given by the expression:

\[ \begin{aligned}
I & = I_0 + \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \\
& = I_0 + \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}
\end{aligned} \]

Let us now move to the practical problem of finding solutions for smooth solutions in a bounded domain. The method of characteristics is a powerful tool for solving such problems, especially in the case of non-linear systems. 

For the case in (2.5) the bound is 0.05 and the maximum actual error 0.019.
In the scenario studied in numerical analysis textbooks it is assumed that when integrating problems such as (2.4) the goal is to accurately compute the value of $y(t)$, $0 \leq t \leq T_{\text{max}}$. Due to the exponential growth of the errors, this goal is not realistic when $T_{\text{max}}$ is not small. The idea of shadowing does not help in this scenario. On the other hand, a more realistic scenario may be envisaged. We could consider that the interest lies in finding the behaviour of the trajectories of the differential system being integrated. In this alternative scenario shadowing is certainly of use: in Fig. 2 we know that the computed points represent, except for a small error of 2.1, a suitable trajectory of the differential equation.

Before proceeding any further, let us examine why in Fig. 2 the errors $\tilde{e}_n := \tilde{y}(n\Delta t) - y_n$ turn out to be small. Instead of (2.2) we now have the error recursion

$$\tilde{e}_n = \psi \tilde{e}_{n-1} + \tilde{I}_n,$$

where $\tilde{I}_n := \tilde{y}(n\Delta t) - \psi \tilde{y}((n-1)\Delta t)$ and $\psi = 1 + \Delta t$. This formula is now used iteratively for $n$ increasing up to $N = [T_{\text{max}}/\Delta t]$. The result is

$$|\tilde{e}_n| \leq |\psi^{-1}| (|\tilde{e}_{n+1}| + \tilde{I}) \leq |\psi^{-1}|^2 (|\tilde{e}_{n+2}| + \tilde{I}) + |\psi^{-1}| \tilde{I}$$

$$\leq \cdots \leq \left( \sum_{i=1}^{N-n} |\psi^{-1}|^i \right) \tilde{I} \leq (1 + \Delta t) \tilde{I}/\Delta t.$$

After estimating $\tilde{I}$ we conclude

$$|\tilde{e}_n| \leq \frac{1}{2} \Delta t (1 + \Delta t) |\tilde{y}(T_{\text{max}})|, \quad n \leq N.$$  \hspace{1cm} (2.5)
We now look at processes of the form

\[ \mathcal{F} \text{ The Shadrono Lemma} \]
We then say that \( \{ p_n \}_{n=\nu}^{n=N} \) is a \( \delta \)-pseudo-orbit. Finally we say that a true orbit \( \{ \tilde{u}_n \}_{n=\nu}^{n=N} \) \( \varepsilon \)-shadows the pseudo-orbit \( \{ p_n \}_{n=\nu}^{n=N} \) if \( |\tilde{u}_n - p_n| \leq \varepsilon \), \( \nu \leq n \leq N \).

If the dynamics of \( f \) includes exponential divergence of orbits, for instance if it is chaotic, then very small perturbations \( \xi_n \) can have a big impact on the computed points. In (1.1) with \( \mu = 3.8 \) the orbit that starts at \( u_0 = 0.4 \) has \( u_{30} = 0.1936 \); when the initial datum is perturbed to 0.400001, the value of \( u \) at \( n = 30 \) changes to 0.9335, i.e. when the initial data are known with an uncertainty of one part in a million, the result at \( n = 30 \) is completely undetermined, because the size of the underlying errors is comparable to the size of the dynamic variable \( u \). In such situations shadowing would be of much use; at least we could ensure that the computed points closely represent some true orbit. We have the following result due to Anosov [2] and Bowen [5,6] and usually known as the shadowing lemma.

**Theorem 3.1.** If \( f \) is a hyperbolic diffeomorphism, then for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that every \( \delta \)-pseudo-orbit can be \( \varepsilon \)-shadowed.

Note that the length \( N - \nu \) of the pseudo-orbit may be arbitrarily large. Hyperbolicity means that at each point \( u \) the Jacobian matrix of \( f \) has eigenvalues that are in modulus either \( < 1 \) (damping) or \( > 1 \) (growth). Furthermore the angle between the stable and unstable directions should be bounded away from zero uniformly in \( u \) and the rates of damping and growth should be bounded away from 1 uniformly in \( u \). Unfortunately, hyperbolicity is a very restrictive condition; for instance the logistic map (1.1) does not satisfy it. Shadowing results for nonhyperbolic situations can be seen in [7,19].

4. Checking shadowing with the computer

The results quoted at the end of the previous section are mainly of theoretical interest. Hammel, Yorke and Grebogi [14] devised a method whereby practical shadowing results can be obtained. In their approach once a numerical orbit has been generated by the computer, the same computer is used to calculate rigorously how long a true trajectory exists near the numerical orbit. In [14] it is assumed that \( f \) in (3.1) is one-dimensional, \( d = 1 \). Once a numerical orbit \( \{ p_n \}_{n=\nu}^{n=N} \) has been found, the computer is used to successively find for \( n = N, \ldots, \nu \) intervals \( I_n \) in such a way that \( I_N = [p_N, p_N] \) and \( f(I_{n-1}) \supseteq I_n \), \( \nu \leq n \leq N \). The last condition guarantees that an exact orbit \( \{ \tilde{u}_n \}_{n=\nu}^{n=N} \) exists such that \( \tilde{u}_n \in I_n \), \( \nu \leq n \leq N \). Finally by measuring the distance between \( p_n \) and the endpoints of the interval \( I_n \) a bound for the distances \( |\tilde{u}_n - p_n| \) is obtained. Note that the \( I_n \) are found in a recursion with \( n \) decreasing, in agreement with our treatment of the unstable problem (2.4). When working in single precision with a Cray X-MP computer, Hammel, Yorke and Grebogi find that for (1.1) with \( \mu = 3.8 \), \( p_0 = 0.4 \), \( \nu = 0 \), \( N = 10^7 \), the computed pseudo-orbit is \( \varepsilon \)-shadowed by a true orbit with \( \varepsilon \approx 10^{-5} \). It is conjectured that with a noise amplitude \( \delta \), \( \varepsilon \)-shadowing with \( \varepsilon \approx \sqrt{\delta} \) holds for orbits of length \( N - \nu \approx 1/\sqrt{\delta} \).

In [9,15] similar results are derived for two-dimensional problems. Further developments may be seen in [23]. Chow and Palmer have considered a similar approach; this work is reported in [13].
Approximate individual trajectories very well over long time intervals and the idea of shadowing with 1, even if they are initially very close. Therefore numerical methods cannot be expected to

\[ p_{\Delta} = \frac{\Delta y}{\Delta x} \]

Figure 3.

\[ n_{\Delta} = \frac{\Delta p}{\Delta y} \]

Hence the modes of the linearized system

\[ \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = J(0) \begin{pmatrix} x \\ y \end{pmatrix} \]

hyperbolic means that the Jacobian matrix of the linearized system is at the equilibrium is non-degenerate. Therefore the equilibrium is

\[ (n)_{\Delta} = \frac{\Delta p}{\Delta y} \]

is similar. Another situation where the idea of shadowing is useful concerns the numerical solution of

5. Equilibrium points in differential equations
becomes useful as shown by Beyn [4]. In [4] both one-step and multi-step methods are considered, but for simplicity we only consider here the one-step case

\[ u_{n+1} = u_n + \Delta t F_{\Delta t}(u_n). \]

(5.3)

The two following reasonable consistency hypotheses are made:

(H1) \( F_{\Delta t}(u) \) converges to \( F(u) \) as \( \Delta t \to 0 \), uniformly in \( u, u \in W \). Furthermore the Jacobian matrix \( F'_{\Delta t}(u) \) of \( F_{\Delta t}(u) \) with respect to \( u \) converges to the Jacobian matrix \( F'(u) \) as \( \Delta t \to 0 \), uniformly in \( u, u \in W \).

(H2) The truncation errors \( f_{\Delta t}(u) - [u + \Delta t F_{\Delta t}(u)] \) are \( O(\Delta t^{p+1}) \) uniformly in \( u, u \in W \).

Here \( f_{\Delta t} \) represents the \( \Delta t \) flow of (5.1) and \( p \) is the order of the numerical method.

Beyn shows that there are a neighbourhood \( W \) of the equilibrium and positive constants \( C \) and \( \eta \) such that, for \( \Delta t \leq \eta \), to each numerical orbit \( \{u_n\}_{n=1}^{N} \) in \( W \) there corresponds a solution \( \tilde{u}(t) \) of (5.1) such that

\[ |\tilde{u}(n\Delta t) - u_n| \leq C\Delta t^p, \quad \nu \leq n \leq N. \]

Note that \( N \) may be \( \gg 1 \) while \( \nu \) may be \( \ll -1 \). The cases \( \nu = -\infty \) and \( N = \infty \) are allowed and respectively correspond to numerical orbits in the unstable and stable manifold of the mapping (5.3). The conclusion is that the numerical method faithfully reproduces the phase portrait of (5.1) near the equilibrium point.

As in Section 2, the shadowing solution \( \tilde{u}(t) \) is found by imposing that the stable \( X \)-component of \( \tilde{u}(t) \) coincides with the \( X \)-component of \( u \), while the unstable \( Y \)-component of \( \tilde{u}(t) \) coincides with the \( Y \)-component of \( u_n \). To bound the errors \( \tilde{e}_n := \tilde{u}(n\Delta t) - u_n \), note that, as in Section 2, they satisfy

\[ \tilde{e}_{n+1} = \tilde{e}_n + \Delta t \left[ F_{\Delta t}(\tilde{u}(n\Delta t)) - F_{\Delta t}(u_n) \right] + \tilde{l}_n, \]

where \( \tilde{l}_n \) denote the \( O(\Delta t^{p+1}) \) truncation errors (see (H2) above). Hence

\[ \tilde{e}_{n+1} = \tilde{e}_n + \Delta t A \tilde{e}_n + \tilde{l}_n + \Delta t \left[ F_{\Delta t}(\tilde{u}(n\Delta t)) - F_{\Delta t}(u_n) - A(\tilde{u}(n\Delta t) - u_n) \right]. \]

Without the term in square brackets, this would be a linear recursion that is easily analyzed as in Section 2 by working forward in the \( X \)-component and backward in the \( Y \)-component. The term in brackets introduces a small perturbation because, due to (H1), it possesses a bound \( L |\tilde{e}_n| \) with a Lipschitz constant that may be rendered arbitrarily small by suitably reducing the size of the neighbourhood \( W \).

The results in [4] have been extended to partial differential equations by Alouges and Debussche [1] and by Larsson and Sanz-Serna [18]. The paper [1] works in an abstract framework and only considers discretization of the time variable. The analysis provided is a direct extension of that in [4] and the error bounds obtained are not of optimal order. The paper [18] considers the piecewise linear Galerkin space discretization of the reaction–diffusion problem

\[ \begin{align*}
    \partial_t u - \Delta u &= f(u), & x &\in \Omega, \\
    u &= 0, & x &\in \partial \Omega,
\end{align*} \]

where \( \Omega \) is a bounded convex polygonal domain in \( \mathbb{R}^d \), \( d = 1, 2, \) or \( 3 \) and \( f \in C^2(\mathbb{R}) \). If \( d = 2 \) or \( 3 \), then \( f \) is also supposed to satisfy a growth condition

\[ f^{(j)}(u) \leq C(1 + |u|^{\delta-j}), \quad j = 0, 1, 2, \]
References

Acknowledgments

6. Concluding Remarks

Note that the expressions of $\gamma$ are optimal if $\gamma > 0$, but $\gamma$ is not necessarily smooth.

A true solution of $\gamma$ exists in a way that shadowing of $\gamma$ can be achieved by

For a survey of the use of numerical methods in long-time integrations see [20].


