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Adaptive discretization of an integro-differential equation with a weakly singular convolution kernel

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Abstract

An integro-differential equation involving a convolution integral with a weakly singular kernel is considered. The kernel can be that of a fractional integral. The integro-differential equation is discretized using the discontinuous Galerkin method with piecewise constant basis functions. Sparse quadrature is introduced for the convolution term to overcome the problem with the growing amount of data that has to be stored and used in each time-step. A priori and a posteriori error estimates are proved. An adaptive strategy based on the a posteriori error estimate is developed. Finally, the precision and effectiveness of the algorithm are demonstrated in the case that the convolution is a fractional integral. This is done by comparing the numerical solutions with analytical solutions.

Keywords: Integro-differential equation; Weakly singular kernel; Sparse quadrature; Error estimate; Adaptivity

1. Introduction

Fractional order operators (integrals and derivatives) have proved to be very suitable for modeling memory effects of various materials and systems of technical interest. In particular, they are very useful when modeling viscoelastic materials, see, e.g., [3,4,10]. The drawback of these models is that, when the response is integrated numerically, the whole previous stress or strain history must be included in each time-step. Rather few algorithms for integrating viscoelastic responses (integral equations with singular kernels) are available. Most of them are based on the Lubich convolution quadrature for fractional order operators, see [11] and, e.g., [8]. The Lubich convolution quadrature requires uniformly distributed time-steps. This is a cumbersome restriction, in particular, when analyzing non-linear viscoelastic responses. Furthermore, it is not possible to use adaptivity and goal oriented error estimates. It also restricts the possibility to use sparse

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time history. In the present work we discuss these difficulties in the context of the following integro-differential equation

$$u_t(t) + \int_0^t \beta(t-s)Au(s)\,\mathrm{d}s = f(t), \quad t \in (0,T), \\ u(0) = u_0, \tag{1.1}$$

where $u_t = \mathrm{d}u/\mathrm{d}t$ and A is a self-adjoint, positive definite, linear operator on a separable Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. The operator A may be unbounded with domain of definition $D(A) \subset H$, but we assume that it has compact inverse, so that the spectral theorem applies. In applications H could typically be $L_2(\Omega)$ for some spatial domain Ω, and A an elliptic partial differential operator with respect to the spatial variables, or the finite element approximation of such an operator. The abstract Hilbert space framework makes it possible to discuss time discretization without going into details about the spatial approximation. We assume that the data $u_0 \in H$ and $f(t) \in H$ are such that the equation has a unique, appropriately regular solution.

The kernel function β may be weakly singular but integrable. More precisely, we assume that β is real-valued, belongs to $L_1(0,T)$, and *positive definite* in the sense that, for any $T \geqslant 0$,

$$\int_0^T \int_0^t \beta(t-s)\varphi(s)\varphi(t)\,\mathrm{d}s\,\mathrm{d}t \geqslant 0, \quad \forall \varphi \in L_2(0,T). \tag{1.2}$$

Our chief example is

$$\beta(t) = \frac{1}{\Gamma(\alpha)}\frac{1}{t^{1-\alpha}}, \quad 0 < \alpha \leqslant 1, \tag{1.3}$$

for which the convolution integral can be interpreted as an integral of fractional order α, see, e.g., the textbook [15], and (1.1) can be written as

$$u_t(t) + D^{-\alpha}[Au](t) = f(t), \quad t \in (0,T), \\ u(0) = u_0, \tag{1.4}$$

where the operator $D^{-\alpha}$ is the fractional order integral. In the limit $\alpha = 0$ the kernel β approaches the delta function and we obtain the parabolic equation

$$u_t + Au = f; \quad u(0) = u_0,$$

while specializing to $\alpha = 1$ results in

$$u_t + \int_0^t Au(s)\,\mathrm{d}s = f,$$

which is equivalent to the hyperbolic equation

$$u_{tt} + Au = f_t; \quad u(0) = u_0, \quad u_t(0) = f(0).$$

This means that by varying the fractional integral exponent we obtain a link between parabolic behavior, (e.g., heat conduction) and hyperbolic behavior, (e.g., wave propagation).

Here we develop an adaptive algorithm with a priori and a posteriori error estimates for solving (1.1). The a posteriori error estimate forms the basis for the adaptive strategy. For the numerical integration we adopt the discontinuous Galerkin method with piecewise constant basis functions. To overcome the problem with the growing amount of data, that has to be stored and used in each time-step, we introduce sparse quadrature for the convolution integral. Sparsely distributed time-steps are used in the distant part of the history while small steps are used in the most recent part. The idea is to break up the convolution structure by using piecewise linear interpolants between the large steps in the distant part of the history. This was first studied in [16,18].

The present work is a further development of the study by McLean et al. [13]. Indeed, they allowed variable time-steps and introduced sparse quadrature, but their study was limited to a priori error estimates.

Similar considerations apply to fractional order differential equations, $D^{\alpha}u + Au = f$, which can be rewritten as Volterra integral equations of the second kind by applying the integral operator $D^{-\alpha}$, see [6]. The kernel function in the resulting integral equation is then of the same kind as in the present study. Fractional differential equations are studied in this way in [5–7]. The numerical schemes are based on the Lubich convolution quadrature and therefore need equispaced grids, or alternatively logarithmically distributed time-steps, as in [9]. This prevents the use of adaptivity and sparse quadrature. These issues will be addressed in the forthcoming paper [2] using the methods of the present work.

2. The discontinuous Galerkin method

Let $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n < \cdots < t_N = T$ be a temporal mesh with time intervals $I_n = (t_{n-1}, t_n)$ and steps $k_n = t_n - t_{n-1}$. We define the finite element space (with subscript D for "discrete")

$$\mathscr{W}_D = \{w : w(t) = w_n \quad \text{for } t \in I_n, w_n \in D(A), n = 1, \ldots, N\}. \tag{2.1}$$

Note that $w \in \mathscr{W}_D$ may be discontinuous at $t = t_n$; we write $w_n = w|_{I_n} = w_n^- = w_{n-1}^+$ and we let $[w]_n = w_n^+ - w_n^- = w_{n+1} - w_n$ denote the jump.

The approximation $U \in \mathscr{W}_D$ of the solution u of (1.1) is given by

$$\begin{aligned} &U \in \mathscr{W}_D, \quad \text{with} \quad U_0^- = u_0, \quad \text{and} \quad \text{for } n = 1, \ldots, N, \\ &\int_{I_n} \left(U_t(t) + \int_0^t \beta(t-s) AU(s) \mathrm{d}s - f(t), v(t) \right) \mathrm{d}t + ([U]_{n-1}, v_{n-1}^+) = 0 \quad \forall v \in \mathscr{W}_D. \end{aligned} \tag{2.2}$$

Here $(\cdot,\cdot)$ denotes the inner product in H. Since the functions in $\mathscr{W}_D$ are piecewise constant with respect to t, we get

$$\frac{U_n - U_{n-1}}{k_n} + q_n(AU) - \bar{f}_n = 0, \tag{2.3}$$

where

$$\begin{aligned} \bar{f}_n &= \frac{1}{k_n} \int_{t_{n-1}}^{t_n} f(t) \mathrm{d}t, \\ q_n(AU) &= \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \int_0^t \beta(t-s) AU(s) \mathrm{d}s \, \mathrm{d}t \\ &= \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j \wedge t} \beta(t-s) AU_j \, \mathrm{d}s \, \mathrm{d}t = \sum_{j=1}^{n} k_j \omega_{nj} AU_j, \\ \omega_{nj} &= \frac{1}{k_n k_j} \int_{t_{n-1}}^{t_n} \int_{t_{j-1}}^{t_j \wedge t} \beta(t-s) \mathrm{d}s \, \mathrm{d}t, \quad t_j \wedge t = \min(t_j, t). \end{aligned} \tag{2.4}$$

This is a variant of the backward Euler method, where the source term f and the convolution integral enter in the form of averages instead of point values. In each time-step we have to solve a linear equation for U_n, namely,

$$(I + k_n^2 \omega_{nn} A) U_n = U_{n-1} - k_n \sum_{j=1}^{n-1} k_j \omega_{nj} AU_j + k_n \bar{f}_n. \tag{2.5}$$

Since $\omega_{nn} > 0$ and A is positive definite, it is clear that (2.5) has a unique solution. In the case of the weakly singular kernel (1.3) we have

$$k_n^2\omega_{nn} = \frac{k_n^{1+\alpha}}{\alpha(1+\alpha)\Gamma(\alpha)}. \tag{2.6}$$

In addition to the finite element space $\mathscr{W}_D$ we introduce the space $\mathscr{W}$ of functions with values in $D(A)$ that are piecewise smooth with respect to the temporal mesh. Note the inclusion $\mathscr{W}_D \subset \mathscr{W}$ and that the error $e = U - u \in \mathscr{W}$.

We define the bilinear form $B : \mathscr{W} \times \mathscr{W} \to \mathbf{R}$ by

$$\begin{aligned} B(w,v) &= \sum_{n=1}^{N} \int_{I_n} \left(w_t(t) + \int_0^t \beta(t-s)Aw(s)\mathrm{d}s, v(t) \right) \mathrm{d}t + \sum_{n=1}^{N-1} ([w]_n, v_n^+) + (w_0^+, v_0^+) \\ &= \sum_{n=1}^{N} \int_{I_n} \left(w(t), -v_t(t) + \int_t^T \beta(s-t)Av(s)\mathrm{d}s \right) \mathrm{d}t + \sum_{n=1}^{N-1} (w_n^-, -[v]_n) + (w_N^-, v_N^-), \end{aligned} \tag{2.7}$$

where the second variant is obtained by integration by parts:

$$\int_{I_n} (w_t, v)\mathrm{d}t = \int_{I_n} (w, -v_t)\mathrm{d}t + (w_n^-, v_n^-) - (w_{n-1}^+, v_{n-1}^+)$$

and rearrangement of the jump terms and the convolution integral.

By adding Eqs. (2.2) and the identity $(U_0^-, v_0^+) = (u_0, v_0^+)$, that determine the finite element solution U up to time $t_N = T$, we obtain

$$U \in \mathscr{W}_D : B(U,v) - \int_0^T (f,v)\mathrm{d}t - (u_0, v_0^+) = 0 \quad \forall v \in \mathscr{W}_D. \tag{2.8}$$

We recall from (1.1) that the exact solution satisfies

$$u \in \mathscr{W} : B(u,v) - \int_0^T (f,v)\mathrm{d}t - (u_0, v_0^+) = 0 \quad \forall v \in \mathscr{W}. \tag{2.9}$$

The a posteriori error estimate is based on the residual of the computed solution, which is the linear functional $r : \mathscr{W} \to \mathbf{R}$ defined by

$$\langle r, v \rangle = B(U,v) - \int_0^T (f,v)\mathrm{d}t - (u_0, v_0^+) \quad \forall v \in \mathscr{W}. \tag{2.10}$$

With this notation (2.8) becomes

$$\langle r, v \rangle = 0 \quad \forall v \in \mathscr{W}_D, \tag{2.11}$$

reflecting that this is a Galerkin method.

Using (2.9) and (2.10) we obtain, with $e = U - u$,

$$B(e,v) = B(U,v) - B(u,v) = B(U,v) - \int_0^T (f,v)\mathrm{d}t - (u_0, v_0^+) = \langle r, v \rangle \quad \forall v \in \mathscr{W}.$$

This means that the error satisfies the equation

$$e \in \mathscr{W} : B(e,v) = \langle r, v \rangle \quad \forall v \in \mathscr{W}. \tag{2.12}$$

Note that this is of the same form as (2.9) but with the data terms $(u_0, v_0^+) + \int_0^T (f,v)\mathrm{d}t$ replaced by the residual.

3. A priori error estimate

The following a priori error estimate is Theorem 6.1 in [13]. We repeat the proof, expressed in our present notation. We recall that $\|\cdot\|$ denotes the norm in H.

Theorem 3.1. *Let u and U be the solutions of* (2.9) *and* (2.8), *respectively. Let $\tilde{u} \in \mathscr{W}_D$ denote the piecewise constant interpolant determined by $\tilde{u}(t) = u(t_n)$ for $t \in I_n$. Then, for all $t_N \geqslant 0$,*

$$\|U_N - u(t_N)\| \leqslant 2\|\beta\|_{L_1(0,t_N)} \int_0^{t_N} \|A(u(t) - \tilde{u}(t))\| \mathrm{d}t \leqslant 2\|\beta\|_{L_1(0,t_N)} \sum_{n=1}^{N} k_n \int_{I_n} \|Au_t(t)\| \mathrm{d}t.$$

Proof. We begin by showing that the bilinear form B is positive. By adding the first and second variants of $B(v,v)$ in (2.7) we get

$$B(v,v) = \frac{1}{2}\|v_N^-\|^2 + \frac{1}{2}\|v_0^+\|^2 + \frac{1}{2}\sum_{n=1}^{N-1}\|[v]_n\|^2 + \int_0^{t_N}\int_0^t \beta(t-s)(Av(s), v(t))\mathrm{d}s\,\mathrm{d}t.$$

Let $\lambda_j > 0$ be the eigenvalues of A and φ_j a corresponding ON-basis of eigenvectors. Writing $v(t) = \sum_{j=1}^{\infty} \hat{v}_j(t)\varphi_j$, $\hat{v}_j = (v, \varphi_j)$ and using (1.2), we get

$$\int_0^{t_N}\int_0^t \beta(t-s)(Av(s), v(t))\mathrm{d}s\,\mathrm{d}t = \sum_{j=1}^{\infty} \lambda_j \int_0^{t_N}\int_0^t \beta(t-s)\hat{v}_j(s)\hat{v}_j(t)\mathrm{d}s\,\mathrm{d}t \geqslant 0.$$

Therefore,

$$B(v,v) \geqslant \frac{1}{2}\|v_N^-\|^2 \quad \forall v \in \mathscr{W}.$$

Let $e = (U - \tilde{u}) + (\tilde{u} - u) = \theta + \rho$. Then

$$B(\theta, v) = B(e,v) - B(\rho, v) = -B(\rho, v) \quad \forall v \in \mathscr{W}_D,$$

since $B(e,v) = 0$ in view of (2.12) and (2.11). We choose $v = \theta$ here, to get

$$\|\theta_N^-\|^2 \leqslant 2|B(\rho,\theta)|.$$

Moreover, using the second variant of $B(w,v)$ in (2.7), and the fact that $\rho_n^- = 0$, we get

$$\begin{aligned} B(\rho,\theta) &= \sum_{n=1}^{N}\int_{I_n}\left(\rho(t), -\theta_t(t) + \int_t^{t_N}\beta(s-t)A\theta(s)\mathrm{d}s\right)\mathrm{d}t + \sum_{n=1}^{N-1}(\rho_n^-, -[\theta]_n) + (\rho_N^-, \theta_N^-) \\ &= \int_0^{t_N}\int_t^{t_N}\beta(s-t)(\rho(t), A\theta(s))\mathrm{d}s\,\mathrm{d}t = \int_0^{t_N}\int_0^t \beta(t-s)(A\rho(s), \theta(t))\mathrm{d}s\,\mathrm{d}t. \end{aligned}$$

Hence,

$$|B(\rho,\theta)| \leqslant \int_0^{t_N}\int_0^t \beta(t-s)\|A\rho(s)\|\|\theta(t)\|\mathrm{d}s\,\mathrm{d}t \leqslant \|\beta\|_{L_1(0,t_N)}\int_0^{t_N}\|A\rho(s)\|\mathrm{d}s \max_{0\leqslant t\leqslant t_N}\|\theta(t)\|. \tag{3.1}$$

Putting these things together, recalling that $\theta|_{I_n} = \theta_n^- = \theta_n$, we arrive at

$$\|\theta_N\|^2 \leqslant 2\|\beta\|_{L_1(0,t_N)}\int_0^{t_N}\|A\rho(t)\|\mathrm{d}t \max_{1\leqslant n\leqslant N}\|\theta_n\|,$$

This holds for all N and the right side is increasing with respect to N. With $\widetilde{N}$ chosen so that $\|\theta_{\tilde{N}}\| = \max_{1\leqslant n\leqslant N}\|\theta_n\|$, we therefore have

$$\|\theta_{\tilde{N}}\|^2 \leqslant 2\|\beta\|_{L_1(0,t_N)} \int_0^{t_N} \|A\rho(t)\|\mathrm{d}t\|\theta_{\tilde{N}}\|,$$

which proves the first error estimate, because $U_N - u(t_N) = \theta_N$. The second one follows by noting that

$$\int_{I_n} \|A\rho(t)\|\mathrm{d}t = \int_{I_n} \left\| \int_t^{t_n} Au_t(s)\mathrm{d}s \right\| \mathrm{d}t \leqslant k_n \int_{I_n} \|Au_t(s)\|\mathrm{d}s. \qquad \square$$

4. A posteriori error estimate

In order to obtain a representation of the error we introduce the adjoint problem with arbitrary data $\phi \in H$ and $T = t_N$:

$$\begin{aligned} &-\psi_t(t) + \int_t^T \beta(s-t)A\psi(s)\mathrm{d}s = 0, \quad t \in (0,T),\\ &\psi(T) = \phi, \end{aligned} \tag{4.1}$$

By the transformation $t \mapsto T - t$, Eq. (4.1) takes the same form as (1.1) with $f = 0$, whose solution we denote by $u(t) = E(t)u_0$ following [12]. The solution of (4.1) is thus $\psi(t) = E(T-t)\phi$.

Eq. (4.1) is the adjoint of (2.12). Using the second variant of (2.7) it can be written in weak form as

$$\psi \in \mathscr{W} : B(w,\psi) = (w_N^-, \phi) \quad \forall w \in \mathscr{W}. \tag{4.2}$$

Using $w = e$ in (4.2) and $v = \psi$ in (2.12) we obtain

$$(e_N^-, \phi) = \langle r, \psi\rangle = \langle r, E(t_N - \cdot)\phi\rangle. \tag{4.3}$$

This equation is the basis for our a posteriori error estimates. It expresses the error in terms of the residual r, which tells how well the approximate solution satisfies the original equation, and the adjoint solution ψ, which captures the stability properties of the error equation (2.12).

In the case of the weakly singular kernel (1.3) we recall from ([12, Theorems 5.1, 5.5]) the stability estimates

$$\|\psi(t)\| \leqslant \|\phi\|, \quad t \in [0,T], \tag{4.4}$$

$$\|\psi_t(t)\| \leqslant C_\alpha (T-t)^{-1}\|\phi\|, \quad t \in [0,T). \tag{4.5}$$

In the following theorem we present a sequence of increasingly larger a posteriori error estimates.

Theorem 4.1. *Let u and U be the solutions of* (2.9) *and* (2.8), *respectively, and let $\psi(t) = E(t_N - t)\phi$ be the solution of* (4.1). *Let*

$$R(t) = \int_0^t \beta(t-s)AU(s)\mathrm{d}s - f(t) \tag{4.6}$$

and let $\bar{\psi} \in \mathscr{W}_D$ denote the orthogonal projection of ψ onto the space of piecewise constant functions, determined by

$$\bar{\psi}(t) = k_n^{-1} \int_{I_n} \psi(s)\mathrm{d}s, \quad t \in I_n. \tag{4.7}$$

Then, for all $t_N \geqslant 0$,

$$\|U_N - u(t_N)\| \leqslant \sup_{\|\phi\|=1} \sum_{n=1}^{N} \left(\int_{I_n} \|R - \chi\| \mathrm{d}t + \|[U]_{n-1}\| \right) \max_{I_n} \|\psi - \bar{\psi}\| \tag{4.8}$$

$$\leqslant \sup_{\|\phi\|=1} \sum_{n=1}^{N} \left\{ \left(\int_{I_n} \|R - \chi\| \mathrm{d}t + \|[U]_{n-1}\| \right) \min\left(\int_{I_n} \|\psi_t\| \mathrm{d}t, 2 \max_{I_n} \|\psi\| \right) \right\}, \tag{4.9}$$

where $\chi \in \mathscr{W}_D$ *is arbitrary. If* ψ *satisfies stability estimates of the form* (4.4), (4.5), *then*

$$\|U_N - u(t_N)\| \leqslant C_{\alpha,N} \max_{1 \leqslant n \leqslant N} \left(k_n \max_{I_n} \|R - \chi\| + \|[U]_{n-1}\| \right), \tag{4.10}$$

where $C_{\alpha,N} = 2 + C_\alpha \log(t_N/k_N)$.

Here $\int_{I_n} \|R - \chi\| \mathrm{d}t + \|[U]_{n-1}\|$ is an estimate of the residual, while the quantity $\min(\int_{I_n} \|\psi_t\| \mathrm{d}t, 2 \max_{I_n} \|\psi\|)$ is a stability factor.

Proof. Recalling the definition of the residual (2.10) and the bilinear form (2.7), adding and subtracting $U_0^- = u_0$, we obtain

$$\begin{aligned} \langle r, \psi \rangle &= B(U, \psi) - \int_0^{t_N} (f, \psi) \mathrm{d}t - (u_0, \psi_0^+) \\ &= \sum_{n=1}^{N} \int_{I_n} \left(U_t(t) + \int_0^t \beta(t-s) A U(s) \mathrm{d}s - f(t), \psi(t) \right) \mathrm{d}t + \sum_{n=1}^{N} ([U]_{n-1}, \psi_{n-1}^+). \end{aligned}$$

We also recall the orthogonality property (2.11), which implies that

$$\langle r, \psi \rangle = \langle r, \psi - \bar{\psi} \rangle. \tag{4.11}$$

Therefore, in view of (4.3),

$$\|U_N - u(t_N)\| = \|e_N^-\| = \sup_{\|\phi\|=1} |(e_N^-, \phi)| = \sup_{\|\phi\|=1} |\langle r, \psi - \bar{\psi} \rangle|.$$

Since U is piecewise constant with respect to t, recalling the definition (4.6) of $R(t)$, and noting that $\bar{\psi}$ is the orthogonal projection of ψ onto $\mathscr{W}_D$, we get for any $\chi \in \mathscr{W}_D$,

$$\langle r, \psi - \bar{\psi} \rangle = \sum_{n=1}^{N} \int_{I_n} ((R - \chi)(t), (\psi - \bar{\psi})(t)) \mathrm{d}t + \sum_{n=1}^{N} ([U]_{n-1}, (\psi - \bar{\psi})_{n-1}^+). \tag{4.12}$$

Hence,

$$\begin{aligned} |\langle r, \psi - \bar{\psi} \rangle| &\leqslant \sum_{n=1}^{N} \left(\int_{I_n} \|(R - \chi)(t)\| \|(\psi - \bar{\psi})(t)\| \mathrm{d}t + \|[U]_{n-1}\| \|(\psi - \bar{\psi})_{n-1}^+\| \right) \\ &\leqslant \sum_{n=1}^{N} \left(\int_{I_n} \|(R - \chi)(t)\| \mathrm{d}t + \|[U]_{n-1}\| \right) \max_{t \in I_n} \|(\psi - \bar{\psi})(t)\|. \end{aligned}$$

This proves (4.8). Here,

$$\max_{t \in I_n} \|(\psi - \bar{\psi})(t)\| = \max_{t \in I_n} \left\| k_n^{-1} \int_{I_n} (\psi(t) - \psi(s)) \mathrm{d}s \right\| \leqslant \int_{I_n} \|\psi_t\| \mathrm{d}s,$$

and also

$$\max_{t\in I_n}\|(\psi-\bar{\psi})(t)\|\leqslant 2\max_{I_n}\|\psi\|.$$

This proves (4.9). In order to prove (4.10), we use (4.9) to get

$$\|e_N^-\|\leqslant \sup_{\|\phi\|=1}\left\{\sum_{n=1}^{N-1}\left(k_n\max_{I_n}\|R-\chi\|+\|[U]_{n-1}\|\right)\int_{I_n}\|\psi_t\|\mathrm{d}s+2\left(k_N\max_{I_N}\|R-\chi\|+\|[U]_{N-1}\|\right)\max_{I_N}\|\psi\|\right\}$$
$$\leqslant \max_{1\leqslant n\leqslant N}\left(k_n\max_{I_n}\|R-\chi\|+\|[U]_{n-1}\|\right)\sup_{\|\phi\|=1}\left(\int_0^{t_{N-1}}\|\psi_t\|\mathrm{d}t+2\max_{I_N}\|\psi\|\right).$$

Using (4.4) and (4.5) we get

$$\int_0^{t_{N-1}}\|\psi_t\|\mathrm{d}t+2\max_{I_N}\|\psi\|\leqslant(2+C_\alpha\log(t_N/k_N))\|\phi\|,$$

and (4.10) follows. □

5. Sparse quadrature

Sparse quadrature was introduced in [13], but only for the case of a kernel without singularity. Here we use the same procedure, but extend it to the case of the singular kernel with an emphasis on adaptivity.

We introduce time levels $0=M_0<M_1<M_2<\cdots$ and replace the function $s\mapsto\beta(t-s)$ in the integral $q_n(AU)$ in (2.4) by a piecewise linear interpolant,

$$\tilde{\beta}(t,s)=\begin{cases}\beta(t-t_{M_{l-1}})\phi_{1,l}(s)+\beta(t-t_{M_l})\phi_{2,l}(s), & s\in[t_{M_{l-1}},t_{M_l}],\quad l=1,\ldots,L,\\ \beta(t-s), & s\in[t_{M_L},t],\end{cases}\tag{5.1}$$

where

$$\phi_{1,l}(s)=\frac{t_{M_l}-s}{K_l},\quad \phi_{2,l}(s)=\frac{s-t_{M_{l-1}}}{K_l},\quad K_l=t_{M_l}-t_{M_{l-1}},\tag{5.2}$$

and L is the largest integer such that $t-t_{M_L}\geqslant 1$. This gives a margin from the singularity at $s=t$, see Fig. 1.

The discrete problem (2.8) is then replaced by

$$U\in\mathscr{W}_D:\widetilde{B}(U,v)-\int_0^T(f,v)\mathrm{d}t-(u_0,v_0^+)=0\quad\forall v\in\mathscr{W}_D,\tag{5.3}$$

where instead of (2.7)

$$\widetilde{B}(w,v)=\sum_{n=1}^{N}\int_{I_n}\left(w_t(t)+\int_0^t\tilde{\beta}(t,s)Aw(s)\mathrm{d}s,v(t)\right)\mathrm{d}t+\sum_{n=1}^{N-1}([w]_n,v_n^+)+(w_0^+,v_0^+).\tag{5.4}$$

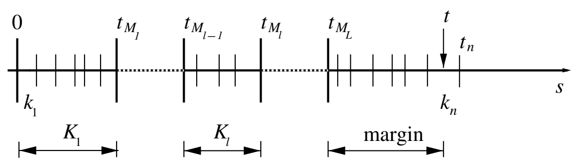

Fig. 1. Time mesh showing original time-steps k_n and sparse time-steps K_l. Note the margin.

In each time-step we then have to solve

$$\frac{U_n - U_{n-1}}{k_n} + \tilde{q}_n(AU) - \bar{f}_n = 0, \tag{5.5}$$

where the quadrature formula, $\tilde{q}_n(\varphi) \approx q_n(\varphi)$ is defined for $\varphi \in \mathscr{W}_{\mathrm{R}}$, the space of all real-valued functions that are piecewise constant with respect to the temporal mesh, by

$$\begin{aligned}\tilde{q}_n(\varphi) &= \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\int_0^t \tilde{\beta}(t,s)\varphi(s)\,\mathrm{d}s\,\mathrm{d}t = \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\left(\int_0^{t_{M_L}} \tilde{\beta}(t,s)\varphi(s)\,\mathrm{d}s + \int_{t_{M_L}}^{t} \beta(t-s)\varphi(s)\,\mathrm{d}s\right)\mathrm{d}t \\ &= \sum_{j=1}^{M_L}\left(\frac{1}{k_n k_j}\int_{t_{n-1}}^{t_n}\int_{t_{j-1}}^{t_j}\tilde{\beta}(t,s)\,\mathrm{d}s\,\mathrm{d}t\,k_j\varphi_j\right) + \sum_{j=M_L+1}^{n}\left(\frac{1}{k_n k_j}\int_{t_{n-1}}^{t_n}\int_{t_{j-1}}^{t_j\wedge t}\beta(t-s)\,\mathrm{d}s\,\mathrm{d}t\,k_j\varphi_j\right) \\ &= \sum_{j=1}^{M_L}\tilde{\omega}_{nj}k_j\varphi_j + \sum_{j=M_L+1}^{n}\omega_{nj}k_j\varphi_j,\end{aligned} \tag{5.6}$$

where

$$\tilde{\omega}_{nj} = \frac{1}{k_n k_j}\int_{t_{n-1}}^{t_n}\int_{t_{j-1}}^{t_j}\tilde{\beta}(t,s)\,\mathrm{d}s\,\mathrm{d}t, \tag{5.7}$$

and ω_{nj} is defined in (2.4). The first sum can be computed as

$$\sum_{j=1}^{M_L}\tilde{\omega}_{nj}k_j\varphi_j = \sum_{l=1}^{L}\left(\tilde{\beta}_{nl,1}\tilde{\varphi}_{l,1} + \tilde{\beta}_{nl,2}\tilde{\varphi}_{l,2}\right), \tag{5.8}$$

where

$$\tilde{\beta}_{nl,1} = \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\beta(t - t_{M_{l-1}})\,\mathrm{d}t, \qquad \tilde{\beta}_{nl,2} = \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\beta(t - t_{M_l})\,\mathrm{d}t, \tag{5.9}$$

and

$$\tilde{\varphi}_{l,i} = \sum_{j=M_{l-1}+1}^{M_l}\int_{t_{j-1}}^{t_j}\phi_{i,l}(s)\,\mathrm{d}s\,\varphi_j, \quad i = 1,2. \tag{5.10}$$

We now estimate the quadrature error. Recall that $\mathscr{W}_{\mathrm{R}}$ denotes the real-valued piecewise constant functions.

Theorem 5.1. *The local quadrature error is bounded by*

$$|\tilde{q}_n(\varphi) - q_n(\varphi)| \leqslant \sum_{j=1}^{M_L}\epsilon_{nj}k_j|\varphi_j|, \quad \forall \varphi \in \mathscr{W}_{\mathrm{R}}, \tag{5.11}$$

where

$$\begin{aligned}\epsilon_{nj} &= \frac{1}{8}\max_{s\in I_{nl}}|\beta''(s)|K_l^2, \quad \text{if } j = M_{l-1}+1,\ldots,M_l, \\ I_{nl} &= [t_{n-1} - t_{M_l}, t_n - t_{M_{l-1}}].\end{aligned} \tag{5.12}$$

The global quadrature error is bounded by

$$|\widetilde{B}(w,v) - B(w,v)| \leqslant \sum_{n=1}^{N}k_n\sum_{j=1}^{M_L}\epsilon_{nj}k_j\|Aw_j\|\max_{I_n}\|v\|, \quad \forall w \in \mathscr{W}_D, \; v \in \mathscr{W}. \tag{5.13}$$

Proof. The standard interpolation error formula gives, with $\tilde{s} \in [t_{M_{l-1}}, t_{M_l}]$,

$$\begin{aligned}|\tilde{q}_n(\varphi) - q_n(\varphi)| &\leqslant \sum_{l=1}^{L} \sum_{j=M_{l-1}+1}^{M_l} \frac{1}{k_n k_j} \int_{I_n} \int_{I_j} |\tilde{\beta}(t,s) - \beta(t-s)|\,\mathrm{d}s\,\mathrm{d}t\, k_j |\varphi_j| \\ &\leqslant \frac{1}{2} \sum_{l=1}^{L} \sum_{j=M_{l-1}+1}^{M_l} \frac{1}{k_n k_j} \int_{I_n} \int_{I_j} |\beta''(t-\tilde{s})|(s - t_{M_{l-1}})(t_{M_l} - s)\mathrm{d}s\,\mathrm{d}t\, k_j |\varphi_j| \\ &\leqslant \frac{1}{8} \sum_{l=1}^{L} \sum_{j=M_{l-1}+1}^{M_l} \max_{s \in I_{nl}} |\beta''(s)| K_l^2 k_j |\varphi_j|.\end{aligned}$$

This is (5.11). The estimate (5.13) is proved in the same way as (5.11) using the definitions of B and $\widetilde{B}$. □

Note that ϵ_{nj} is piecewise constant with respect to j. This implies that the sums in (5.11), (5.13) can be computed without storing the whole history.

In the case of the weakly singular kernel (1.3) we have

$$\max_{s \in I_{nl}} |\beta''(s)| = |\beta''(t_{n-1} - t_{M_l})| = \frac{(1-\alpha)(2-\alpha)}{\Gamma(\alpha)} (t_{n-1} - t_{M_l})^{-3+\alpha}. \tag{5.14}$$

The quadrature formula $\tilde{q}_n$ is not necessarily positive definite, i.e., the modified kernel $\tilde{\beta}$ does not satisfy an analog of (1.2). This is needed in the a priori analysis. We therefore add a positive term on the diagonal:

$$\hat{q}_n(\varphi) = \tilde{q}_n(\varphi) + \delta_n \varphi_n. \tag{5.15}$$

So instead of (2.3) we now use

$$\frac{U_n - U_{n-1}}{k_n} + \hat{q}_n(AU) - \bar{f}_n = 0. \tag{5.16}$$

The discrete problem (2.8) is then replaced by

$$U \in \mathscr{W}_D : \widehat{B}(U, v) - \int_0^T (f, v)\mathrm{d}t - (u_0, v_0^+) = 0 \quad \forall v \in \mathscr{W}_D, \tag{5.17}$$

where instead of (2.7), and with $\delta(t) = \delta_n$ for $t \in I_n$,

$$\widehat{B}(w, v) = \sum_{n=1}^{N} \int_{I_n} \left(w_t(t) + \int_0^t \tilde{\beta}(t,s) A w(s)\,\mathrm{d}s + \delta(t) A w(t), v(t) \right) \mathrm{d}t + \sum_{n=1}^{N-1} ([w]_n, v_n^+) + (w_0^+, v_0^+). \tag{5.18}$$

The following lemma is identical to ([13, Lemma 5.2]).

Lemma 5.2. *Assume that the numbers δ_j are positive and increasing with $\delta_j \geqslant \epsilon_{Nj} t_N/2$, where ϵ_{Nj} is defined in* (5.12). *Then we have the following analog of* (1.2):

$$\int_0^{t_N} \left(\int_0^t \tilde{\beta}(t,s)\varphi(s)\varphi(t)\mathrm{d}s + \delta(t)\varphi(t)^2 \right) \mathrm{d}t \geqslant 0, \quad \forall \varphi \in \mathscr{W}_{\mathrm{R}}. \tag{5.19}$$

This guarantees stability and we can prove an a priori error estimate.

Theorem 5.3. *Let u and U be the solutions of* (2.9) *and* (5.17), *respectively, with δ_j as in Lemma* 5.2. *Let $\tilde{u} \in \mathscr{W}_D$ denote the piecewise constant interpolant determined by $\tilde{u}(t) = u(t_n)$ for $t \in I_n$. Then, for all $t_N \geqslant 0$,*

$$\|U_N - u(t_N)\| \leqslant 2\|\beta\|_{L_1(0,t_N)} \int_0^{t_N} \|A(u(t) - \tilde{u}(t))\|\mathrm{d}t + 2\hat{\epsilon}_N \leqslant 2\|\beta\|_{L_1(0,t_N)} \sum_{n=1}^{N} k_n \int_{I_n} \|Au_t(t)\|\mathrm{d}t + 2\hat{\epsilon}_N. \tag{5.20}$$

Here $\hat{\epsilon}_N$ is a bound for the quadrature error:

$$\hat{\epsilon}_N = \sum_{n=1}^{N} k_n \sum_{j=1}^{M_L} \epsilon_{nj} k_j \|Au_j\| + \sum_{n=1}^{N} k_n \delta_n \|Au_n\|. \tag{5.21}$$

Proof. We modify the proof of Theorem 3.1. Let $e = (U - \tilde{u}) + (\tilde{u} - u) = \theta + \rho$. Then

$$\widehat{B}(\theta, v) = -B(\rho, v) - (\widehat{B}(\tilde{u}, v) - B(\tilde{u}, v)) \quad \forall v \in \mathscr{W}_D.$$

We choose $v = \theta$. By Lemma 5.2 we have $\widehat{B}(\theta,\theta) \geqslant \frac{1}{2}\|\theta_N\|^2$ and hence

$$\|\theta_N\|^2 \leqslant 2(|B(\rho,\theta)| + |\widehat{B}(\tilde{u},\theta) - \widetilde{B}(\tilde{u},\theta)| + |\widetilde{B}(\tilde{u},\theta) - B(\tilde{u},\theta)|),$$

which, in view of (3.1) and (5.13), proves the desired result. □

We next prove a posteriori estimates. These are based on the stability of the adjoint problem (4.1) and we need not assume that δ_j are strictly positive.

Theorem 5.4. *Let u and U be the solutions of* (2.9) *and* (5.17), *respectively, with* $\delta_j \geqslant 0$. *Let* $\psi(t) = E(t_N - t)\phi$ *be the solution of* (4.1). *Let*

$$\widehat{R}(t) = \int_0^t \tilde{\beta}(t,s) AU(s)\mathrm{d}s - f(t) + \delta(t) AU(t), \tag{5.22}$$

and let $\bar{\psi} \in \mathscr{W}_D$ *denote the orthogonal projection of* ψ *onto the space of piecewise constant functions, defined in* (4.7). *Then, for all* $t_N \geqslant 0$,

$$\|U_N - u(t_N)\| \leqslant E_\mathrm{G} + E_\mathrm{Q}. \tag{5.23}$$

Here E_G *is the error due to the Galerkin approximation, which is estimated by*

$$E_\mathrm{G} = \sup_{\|\phi\|=1} \left| \sum_{n=1}^{N} \int_{I_n} (\widehat{R}, \psi)\mathrm{d}t + ([U]_{n-1}, \psi_{n-1}^+) \right| \tag{5.24}$$

$$\leqslant \sup_{\|\phi\|=1} \sum_{n=1}^{N} \left(\int_{I_n} \|\widehat{R} - \chi\|\mathrm{d}t + \|[U]_{n-1}\| \right) \max_{I_n} \|\psi - \bar{\psi}\| \tag{5.25}$$

$$\leqslant \sup_{\|\phi\|=1} \sum_{n=1}^{N} \left\{ \left(\int_{I_n} \|\widehat{R} - \chi\|\mathrm{d}t + \|[U]_{n-1}\| \right) \min\left(\int_{I_n} \|\psi_t\|\mathrm{d}t, 2\max_{I_n} \|\psi\| \right) \right\}, \tag{5.26}$$

where $\chi \in \mathscr{W}_D$ *is arbitrary. Finally,* E_Q *is the global quadrature error, which is estimated by*

$$E_\mathrm{Q} = \sup_{\|\phi\|=1} |B(U,\psi) - \widehat{B}(U,\psi)| \leqslant \sup_{\|\phi\|=1} \sum_{n=1}^{N} k_n \left(\sum_{j=1}^{M_L} \epsilon_{nj} k_j \|AU_j\| + \delta_n \|AU_n\| \right) \max_{I_n} \|\psi\|. \tag{5.27}$$

If ψ *satisfies stability estimates of the form* (4.4), (4.5), *then*

$$\|U_N - u(t_N)\| \leqslant C_{\alpha,N} \max_{1\leqslant n\leqslant N} \left(k_n \max_{I_n} \|\widehat{R} - \chi\| + \|[U]_{n-1}\| \right) + \sum_{n=1}^{N} k_n \left(\sum_{j=1}^{M_L} \epsilon_{nj} k_j \|AU_j\| + \delta_n \|AU_n\| \right), \tag{5.28}$$

where $C_{\alpha,N} = 2 + C_\alpha \log(t_N/k_N)$.

Proof. Using (2.9) we obtain for all ϕ with $\|\phi\| = 1$

$$\begin{aligned}(e_N^-, \phi) &= B(e, \psi) = B(U, \psi) - B(u, \psi) \\ &= \Big(B(U, \psi) - \widehat{B}(U, \psi)\Big) + \Big(\widehat{B}(U, \psi) - \int_0^T (f, \psi)\mathrm{d}t - (u_0, \psi_0^+)\Big) \leqslant E_\mathrm{Q} + E_\mathrm{G}.\end{aligned}$$

The estimate (5.27) of E_Q follows immediately from (5.13) with the addition of the terms involving δ_n. For E_G we argue as in the proof of Theorem 4.1. We first use (5.17) with $v = \bar{\psi} \in \mathscr{W}_D$ to get

$$\begin{aligned}\widehat{B}(U, \psi) - \int_0^T (f, \psi)\mathrm{d}t - (u_0, \psi_0^+) &= \widehat{B}(U, \psi - \bar{\psi}) - \int_0^T (f, \psi - \bar{\psi})\mathrm{d}t - (u_0, \psi_0^+ - \bar{\psi}_0^+) \\ &= \sum_{n=1}^N \int_{I_n} \Big((\widehat{R} - \chi)(t), (\psi - \bar{\psi})(t)\Big)\mathrm{d}t + \sum_{n=1}^N ([U]_{n-1}, (\psi - \bar{\psi})_{n-1}^+),\end{aligned}$$

cf. (4.12). The proof is now completed as the proof of Theorem 4.1 with R replaced by $\widehat{R}$. □

Note that the estimates of the Galerkin error E_G can be computed using the sparse quadrature and therefore does not require storage of the whole history U_n, $1 \leqslant n \leqslant N$. The sums in the quadrature error estimate (5.27) can also be computed without storing the whole history. The optimal choice of χ is the orthogonal projection of $\widehat{R} = \widetilde{R} + \delta A U$,

$$\chi = \overline{\widehat{R}} = \overline{\widetilde{R}} + \overline{\delta A U} = \overline{\widetilde{R}} + \delta A U,$$

where in the last step we used that $\delta A U$ is piecewise constant. Other choices are possible, for example, we may choose $\chi = \delta A U$ in order to make the latter term disappear. In our numerical experiments in Section 7 we use $\chi = 0$.

The a priori error estimate in Theorem 5.3 guarantees that, in general, the error converges to zero if the mesh is refined. We are not able to prove this without adding the positive terms δ_j. On the other hand, the a posteriori error estimate gives a bound for the actual error which reveals, in each particular case, if the error converges or not. In the cases that we have examined in our numerical experiments we have obtained convergence without the δ_j.

6. Adaptive strategy

The goal of the adaptive strategy is to provide a solution to the integro-differential equation within a user defined tolerance, TOL. The adaptive strategy is based on a posteriori error estimates of the total error at the final time $T = t_N$. More precisely, we base the strategy on two different estimates of the error, the exact Galerkin error (5.24) given by

$$E_{\mathrm{G}1} = \sup_{\|\phi\|=1} \left| \sum_{n=1}^N \int_{I_n} (\widehat{R}, \psi)\mathrm{d}t + ([U]_{n-1}, \psi_{n-1}^+) \right|,$$

or the estimated Galerkin error (5.26) given by

$$E_{\mathrm{G}2} = \sup_{\|\phi\|=1} \sum_{n=1}^N k_n^2 r_{\mathrm{G},n} s_{\mathrm{G},n},$$

where, with the choice $\chi = 0$,

$$r_{G,n} = \frac{1}{k_n}\left(\int_{I_n} \|\widehat{R}\| dt + \|[U]_{n-1}\|\right), \quad s_{G,n} = \frac{1}{k_n} \min\left(\int_{I_n} \|\psi_t\| dt, 2\max_{I_n} \|\psi\|\right).$$

Additional error is introduced by sparse quadrature. The estimated quadrature error (5.27) is divided into two parts

$$E_q = \sup_{\|\phi\|=1} \sum_{n=1}^{N} k_n^2 r_{q,n} s_{q,n}, \tag{6.1}$$

where

$$r_{q,n} = \frac{1}{k_n} \sum_{j=1}^{M_L} \epsilon_{nj} k_j \|AU_j\|, \quad s_{q,n} = \max_{I_n} \|\psi\|,$$

and

$$E_\delta = \sup_{\|\phi\|=1} \sum_{n=1}^{N} k_n^2 r_{\delta,n} s_{\delta,n},$$

where

$$r_{\delta,n} = \frac{1}{k_n} \delta_n \|AU_n\|, \quad s_{\delta,n} = \max_{I_n} \|\psi\|.$$

Using this notation, the estimate of the total error becomes

$$E_1 = E_{G1} + E_q + E_\delta,$$

or, using the estimated Galerkin error,

$$E_2 = E_{G2} + E_q + E_\delta = \sup_{\|\phi\|=1} \sum_{n=1}^{N} k_n^2 (r_{G,n} s_{G,n} + r_{q,n} s_{q,n} + r_{\delta,n} s_{\delta,n}) = \sup_{\|\phi\|=1} \sum_{n=1}^{N} E_{2,n}.$$

Finally, we recall the following relationship between the error estimates

$$\|e_N^-\| \leqslant E_1 \leqslant E_2.$$

Note that the first estimate E_1 is sharper, and will be used in the stop criterion. The latter estimate, on the other hand, is used to adapt the mesh.

The stability factors $s_{G,n}$, $s_{q,n}$, $s_{\delta,n}$ must be evaluated. We suggest three possibilities: (i) In simple test problems we may have an analytic solution formula for the adjoint solution $\psi(t) = E(t_N - t)\phi$, see our numerical experiments below. (ii) More generally, in the case of the weakly singular kernel (1.3) we can use a priori estimates of the form (4.4), (4.5), which leads to

$$s_{G,n} \leqslant C_\alpha \frac{1}{k_n} \log \frac{t_N - t_{n-1}}{t_N - t_n}, \quad n \leqslant N - 1,$$
$$s_{G,N} \leqslant \frac{2}{k_N}, \quad s_{q,n} = s_{\delta,n} \leqslant 1.$$

(iii) The adjoint problem may be solved numerically with some guess for the data ϕ with $\|\phi\| = 1$.

We choose a strategy with the purpose to equidistribute the error contributions $E_{2,n}$ from each time-step, i.e., we aim at $E_{2,n} \approx \mathrm{TOL}/N$. The time-steps having an error contribution larger than $E_{2,n}$ are split into C_n smaller elements of equal length $\hat{k}_n$ according to

$$\hat{k}_n = \sqrt{\frac{\mathrm{TOL}/N}{r_{\mathrm{G},n}s_{\mathrm{G},n} + r_{\mathrm{q},n}s_{\mathrm{q},n} + r_{\delta,n}s_{\delta,n}}} = \frac{k_n}{C_n}, \tag{6.2}$$

where k_n is the time-step of element n from the previous mesh. Certainly, the number of elements N is not known a priori, therefore we use the number of elements from the previous calculation. Note that the error estimate E_2 is used here because it is possible to sum the error contribution from the different elements. This choice will also somewhat compensate for using the previous number of time-steps. In this way we obtain a new mesh with steps k_n, in which we choose sparse quadrature steps K_l. A natural choice of the sparse time-steps is (see (5.11) and (5.12))

$$K_l = \sqrt{\bar{k}}, \quad \text{with } \bar{k} = T/N. \tag{6.3}$$

Note that N is here the updated number of elements. For this choice of sparse step length, the first order accuracy of the Galerkin method is preserved (which will be demonstrated by numerical experiments). The strategy follows the procedure:

(1) Start with a uniform mesh and choose K_l according to (6.3) while imposing the margin $t_n - t_{M_L} \in (1,2)$.
(2) Solve the primal problem for $U \in \mathscr{W}_D$.
(3) Evaluate the error estimates E_1 and E_2.
(4) If $E_1 \leqslant \mathrm{TOL}$ then stop, and if not modify the mesh where the error contribution is large, i.e., $E_{2,n} \geqslant \mathrm{TOL}/N$, by splitting these elements according to (6.2). Create K_l as in 1 and return to 2.

7. Numerical experiments

In the following examples we consider (1.4) with $A = 1$ for different values of the fractional integral exponent α. The equation then reads

$$\begin{aligned} &u_t(t) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{u(s)}{(t-s)^{1-\alpha}}\,\mathrm{d}s = f(t), \quad t \in (0,T), \\ &u(0) = u_0. \end{aligned} \tag{7.1}$$

The weights ω_{nj} in (2.4) and $\tilde{\omega}_{nj}$ in (5.7) are integrated analytically. The mean value $\bar{f}_n$ in (2.4) of the source term f and the integrals over I_n in the a posteriori estimates are computed using the trapezoidal rule, which means that additional errors are introduced. These errors are not taken into account in the present study. However, numerical experiments indicate that these errors are negligible.

7.1. Analytical solutions

We will here present an analytical solution to (7.1) with $f(t) = f_0$ in the form of a series. This solution will be used when validating the present numerical method. First we consider the Laplace domain solution

$$\tilde{u}(s) = \frac{f_0}{s^2(1+s^{-\alpha-1})} + \frac{u_0}{s(1+s^{-\alpha-1})}. \tag{7.2}$$

Take $c > 0$ so that $|c^{-\alpha-1}| < 1$. Along the vertical line from $c - \mathrm{i}\infty$ to $c + \mathrm{i}\infty$ in the s-plane, we have

$$\tilde{u}(s) = \frac{f_0}{s^2(1+s^{-\alpha-1})} + \frac{u_0}{s(1+s^{-\alpha-1})} = \sum_{n=0}^{\infty}(-1)^n\left[f_0 s^{-((1+\alpha)n+2)} + u_0 s^{-((1+\alpha)n+1)}\right],$$

which converges uniformly on the vertical line in consideration. For $t > 0$, the inverse $u(t)$ can be found as

$$\begin{aligned} u(t) &= \mathscr{L}^{-1}[\tilde{u}(s)](t) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \tilde{u}(s)\mathrm{e}^{st}\,\mathrm{d}s \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} [f_0 s^{-((1+\alpha)n+2)} + u_0 s^{-((1+\alpha)n+1)}]\mathrm{e}^{st}\,\mathrm{d}s \\ &= \sum_{n=0}^{\infty} \mathscr{L}^{-1}[f_0 s^{-((1+\alpha)n+2)} + u_0 s^{-((1+\alpha)n+1)}]. \end{aligned}$$

Finally, by term-wise inversion (see, e.g., [14, p. 237]), we obtain

$$u(t) = \sum_{n=0}^{\infty} (-1)^n \left[f_0 \frac{t^{(1+\alpha)n+1}}{\Gamma((1+\alpha)n+2)} + u_0 \frac{t^{(1+\alpha)n}}{\Gamma((1+\alpha)n+1)} \right]. \tag{7.3}$$

We are now in the position to investigate the convergence of the series solution. The asymptotic behavior of the partial sum $u_n(t)$ as $n \to \infty$ is obtained by use of the following asymptotic formula for the Gamma function (see, e.g., [1])

$$\Gamma(az+b) \sim \sqrt{2\pi}\mathrm{e}^{-az}(az)^{az+b-1/2}, \quad |z| \to \infty, \ |\arg z| < \pi, \ b > 0. \tag{7.4}$$

Applying the ratio test, we now get

$$\left| \frac{u^{n+1}(t)}{u^n(t)} \right| \sim \left[\frac{t}{(\alpha+1)(n+1)} \right]^{\alpha+1} \to 0, \quad \text{as } n \to \infty, \tag{7.5}$$

which means that the series is convergent. But the convergence rate is rather slow (in particular for large times). For given t we need to add

$$n > \frac{t}{\alpha+1} - 1$$

terms before we can expect terms to begin to fall off in size. If a large number of terms is to be included in the sum when evaluating $u(t)$ the numerical stability can be lost. Obviously, there is a need for an asymptotic expression for $u(t)$ for large times. Concerning the asymptotic behavior of $u(t)$, we formally use the asymptotic theorem for the Laplace transform as $s \to 0$ (see, e.g., [17]). Thus, expanding $\tilde{u}(s)$ in (7.2) for small $s > 0$ gives

$$\tilde{u}(s) \sim \sum_{n=0}^{\infty} (-1)^n \left(f_0 s^{-((1+\alpha)n+2)} + u_0 s^{-((1+\alpha)n+1)} \right), \quad \text{as } s \to 0.$$

Formal term-wise Laplace inversion then yields an asymptotic series for $u(t)$:

$$u(t) \sim \sum_{n=0}^{\infty} (-1)^n \left(f_0 \frac{t^{-(1+\alpha)n-\alpha}}{\Gamma[-(1+\alpha)n+1-\alpha]} + u_0 \frac{t^{-(1+\alpha)n-\alpha-1}}{\Gamma[-(1+\alpha)n-\alpha]} \right), \quad \text{as } t \to \infty.$$

Note that the above derivation is purely formal and requires detailed verification which we will not attempt here. However, numerical calculations verify that the sum of the first few terms approximate $u(t)$ well for large times.

For comparison we consider the two extreme cases of $\alpha = 0$ and $\alpha = 1$. In the case of $\alpha = 0$, (7.1) with $f(t) = f_0$ becomes the parabolic equation

$$u_t + u = f_0, \quad u(0) = u_0,$$

with solution

$$u(t) = f_0(1 - \mathrm{e}^{-t}) + u_0\mathrm{e}^{-t}.$$

For $\alpha = 1$, (7.1) with $f(t) = f_0$ becomes the hyperbolic equation

$$u_t(t) + \int_0^t u(s)\mathrm{d}s = f_0, \quad u(0) = u_0,$$

with solution

$$u(t) = f_0 \sin(t) + u_0 \cos(t).$$

7.1.1. The adjoint solution

In the calculation of the error estimates we need the solution of the adjoint problem (cf. (4.1) with $\phi = 1$ as the error is to be calculated at the final time T)

$$\begin{aligned} &-\psi_t(t) + \int_t^T \beta(s-t)\psi(s)\mathrm{d}s = 0, \quad \beta(t) = \frac{1}{\Gamma(\alpha)}\frac{1}{t^{1-\alpha}}, \quad t \in (0,T), \\ &\psi(T) = \phi = 1. \end{aligned} \tag{7.6}$$

In general we need to solve this problem numerically. However, in case of a fractional integral it is possible to find an analytical solution. The solution takes the same form as the solution to the primal problem in (7.3) with $f_0 = 0$, $u_0 = 1$ and $t \mapsto T - t$

$$\psi(t) = E(T-t)\phi = \sum_{n=0}^{\infty} (-1)^n \frac{(T-t)^{(1+\alpha)n}}{\Gamma((1+\alpha)n+1)}, \tag{7.7}$$

with asymptotic series for large $T - t$

$$\psi(t) \sim \sum_{n=0}^{\infty} (-1)^n \frac{(T-t)^{-(1+\alpha)n-\alpha-1}}{\Gamma[-(1+\alpha)n-\alpha]}. \tag{7.8}$$

7.2. Uniform mesh

To validate the discontinuous Galerkin method for integro-differential equations with weakly singular kernel, we solve (7.1) using a uniform mesh with $k_n = k = 0.025$ on $t \in (0, 10)$ without sparse quadrature. Different values of $\alpha \in (0, 1)$, which is the relevant interval for viscoelastic applications, are used together with $f(t) = 1$ and $u_0 = 0.5$. Fig. 2 shows the numerical solutions and the analytical solutions (7.3) to (7.1) for three different α. As mentioned before, the case $\alpha = 0$ is a parabolic equation and the case $\alpha = 1$ is a hyperbolic equation, whereas the intermediate values represent a mixed behavior, as $\alpha = 0.67$ indicates. The corresponding adjoint solution according to (7.7) is displayed in Fig. 3. We note that most of the error arises from the last time-steps, and that this behavior becomes more distinct then the equation is parabolic. Further, Table 1 shows that the computed error E_1 is in close agreement with the exact error $e(t_N) = U_N - u(t_N)$ (obtained using the analytical solution (7.3)), while notably E_2 is an over-estimate. As expected, the error becomes comparatively larger for increasing α. The reason for this is that (7.1) becomes more "parabolic" with decreasing α, which means that errors are strongly damped. Moreover, we observe that in the case of $\alpha = 1$ numerical damping is imposed, which can be understood by the fact that the discontinuous Galerkin method is similar to the backward Euler method.

We are now in the position to introduce sparse quadrature and investigate its effects on the total error. The factor $\max_{s \in I_{nl}} |\beta''(s)|$, see (5.14), that enters in the quadrature error (6.1) shows that this error depends

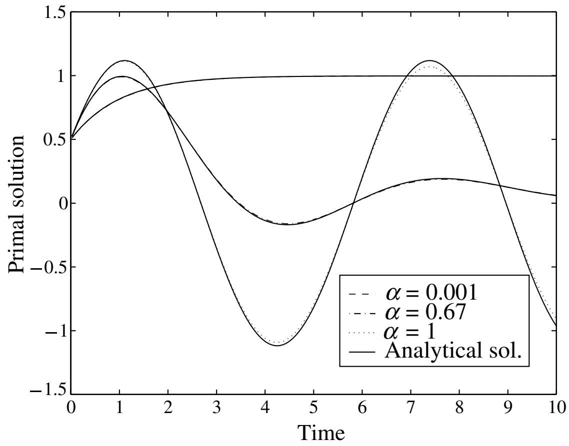

Fig. 2. Comparison of the analytical solution and the numerical solution for different values of α using constant time-steps $k_n = k = 0.025$.

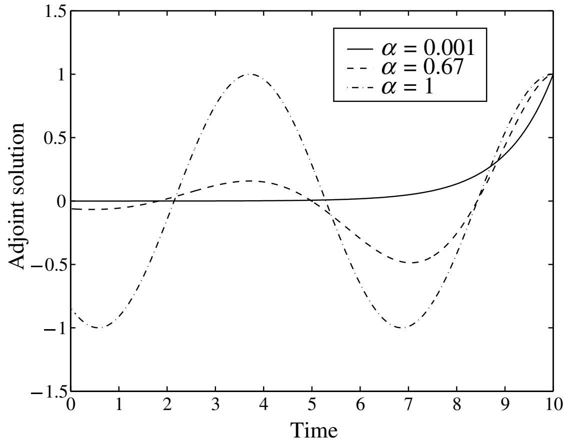

Fig. 3. The adjoint solution.

Table 1
Exact errors and computed errors for different values of α

α	$\lvert e(t_N)\rvert$	E_1	E_2
0.001	9.98×10^{-7}	1.19×10^{-6}	1.06×10^{-5}
0.67	3.14×10^{-3}	3.21×10^{-3}	1.34×10^{-2}
1	5.71×10^{-2}	5.73×10^{-2}	2.37×10^{-1}

strongly on α (note that the error vanishes for $\alpha = 0$ and $\alpha = 1$). We compute the solution to (7.1) on $t \in (0, 10)$ with $\alpha = 0.67$, $f(t) = 1$ and $u_0 = 0$ for three uniform meshes with time-steps decreasing by a factor ten. Two different choices of the parameter δ_n are used. First we choose δ_n as in Lemma 5.2, which guarantees stability. This results in the error contribution E_δ. Then we choose $\delta_n = 0$, and consequently $E_\delta = 0$, which gives sharper error estimates. The computed errors and the contributions from their various parts are shown in Table 2. We see that the solution converges for decreasing time-steps in both cases. Also note that the total error decreases by the same factor as the time-step, reflecting that the first order accuracy is retained. The first order accuracy of the Galerkin error is expected. The quadrature error, however, requires the quadrature steps K_l to be chosen as in (6.3) while imposing $t_n - t_{M_L} \geqslant 1$ for the accuracy to be preserved. Table 3 shows the error without using sparse quadrature. The maximum number of quadrature steps L during the calculation is also shown in Table 2. Having in mind that we always have a margin $t_n - t_{M_L} \in (1, 2)$, we observe that the major part of the time interval is covered by large quadrature steps. This means that, for sufficiently large times the number of data that need to be stored and included in each calculation is significantly reduced from $O(n)$ to $O(L)$. It is also worth mentioning that if a spatial domain is included we would benefit even more by using sparse quadrature.

Table 2
The total errors and their different parts for different step lengths when using sparse quadrature

N	L	E_1	E_{G1}	E_q	E_δ
100	28	9.62×10^{-3}	6.59×10^{-3}	1.47×10^{-3}	1.57×10^{-3}
1000	89	9.66×10^{-4}	7.05×10^{-4}	1.42×10^{-4}	1.19×10^{-4}
10 000	284	9.42×10^{-5}	6.98×10^{-5}	1.40×10^{-5}	1.05×10^{-5}
100	28	8.12×10^{-3}	6.65×10^{-3}	1.47×10^{-3}	0
1000	89	8.47×10^{-4}	7.05×10^{-4}	1.42×10^{-4}	0
10 000	284	8.37×10^{-5}	6.98×10^{-5}	1.40×10^{-5}	0

Table 3
The computed errors for different step lengths when using full quadrature

N	E_1
100	6.63×10^{-3}
1000	7.05×10^{-4}
10 000	6.98×10^{-5}

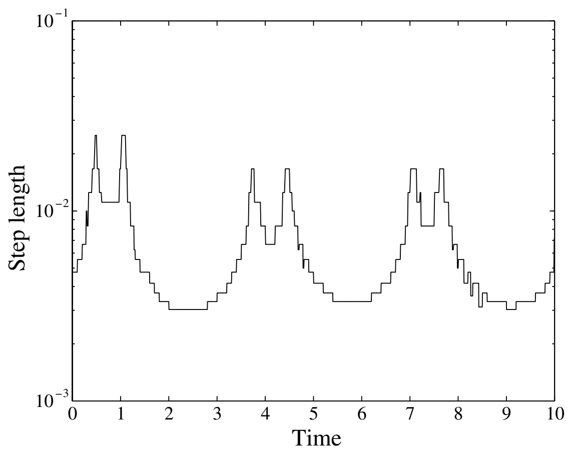

Fig. 4. The time-steps after using the adaptive strategy.

7.3. Adaptivity

Again we consider (7.1) on $t \in (0, 10)$ with $\alpha = 0.67$, $f(t) = 1$ and $u_0 = 0.5$. The capability of the method to handle variable step lengths and predict solutions within a given tolerance is investigated here. For this purpose the adaptive strategy outlined in Section 6 with δ_n as in Lemma 5.2 is employed. When the tolerance is set to 1×10^{-3} and the number of time-steps initially are 100, three iterations are required (i.e., two refinements are used) giving an upper limit to the error of $E_1 = 5.96 \times 10^{-4}$ and a number of time-steps of 2093. Note that this is the error at the final time, it does not tell us anything about the error in the interior of the interval. Further, Fig. 4 shows the time-mesh used in the last iteration suggested by the adaptive strategy aiming to equidistribute the error contributions. For comparison, to obtain the same error with a uniform mesh, 2573 time-steps need to be used. This means that we do not significantly benefit from using adaptivity in this particular case.

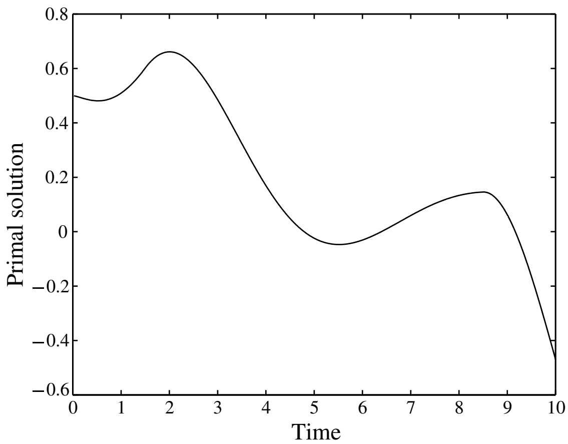

Fig. 5. The primal solution after using the adaptive strategy.

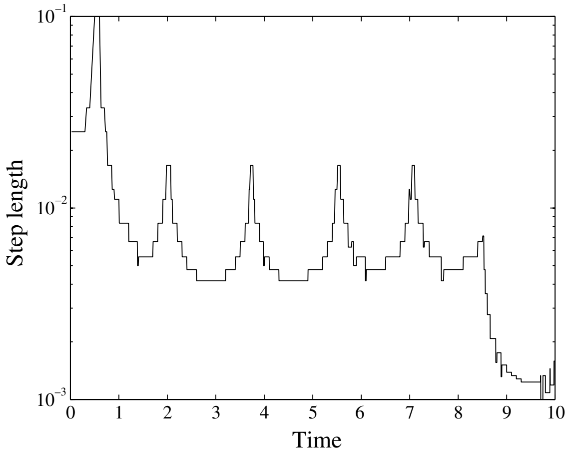

Fig. 6. The time-steps after using the adaptive strategy.

Next we consider the same problem with a variable source term,

$$f(t) = \begin{cases} (1/1.5)t, & t \in (0, 1.5), \\ 1, & t \in (1.5, 8.5), \\ 1 - (1/1.5)(t - 8.5), & t \in (8.5, 10). \end{cases}$$

Also in this case three iterations are needed to meet the tolerance 1×10^{-3}, resulting in 2406 time-steps and the error $E_1 = 4.50 \times 10^{-4}$. Figs. 5 and 6 show the primal solution and the time-mesh from the last iteration, respectively. By comparing Figs. 4 and 6, we see that the time-steps vary more in the latter case, which gives an indication that it is more preferable to use adaptivity. Computing with a uniform mesh, we find that it takes 4511 time-steps to reach the same error. Due to the fact that the integro-differential equation contains a convolution term this is a considerable gain in computational effort.

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References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Tables, National Bureau of Standards—Applied Mathematics Series No. 55, Washington, DC, 1965.
[2] K. Adolfsson, M. Enelund, S. Larsson, Adaptive discretization of fractional order viscoelasticity using sparse time history, Chalmers Finite Element Center, 2003, preprint.
[3] R.L. Bagley, P.J. Torvik, Fractional calculus—a different approach to the analysis of viscoelastically damped structures, AIAA J. 21 (1983) 741–748.
[4] M. Caputo, F. Mainardi, A new dissipation model based on memory mechanism, Pure Appl. Geophys. 91 (1971) 134–147.
[5] K. Diethelm, An algorithm for the numerical solution of differential equations of fractional order, Electron. Trans. Numer. Anal. 5 (1997) 1–6.
[6] K. Diethelm, N.J. Ford, Analysis of fractional differential equations, J. Math. Anal. Appl. 265 (2002) 229–248.
[7] K. Diethelm, N.J. Ford, Numerical solution of the Bagley–Torvik equation, BIT 42 (2002) 490–507.
[8] M. Enelund, L. Mähler, K. Runesson, B.L. Josefson, Formulation and integration of the standard linear viscoelastic solid with fractional order rate laws, Int. J. Solids Struct. 36 (1999) 2417–2442.
[9] N.J. Ford, A.C. Simpson, The numerical solution of fractional differential equations: Speed versus accuracy, Numer. Algorithms 26 (2001) 333–346.
[10] W.G. Glöckle, T.F. Nonnenmacher, Fractional integral operators and Fox functions in the theory of viscoelasticity, Macromolecules 24 (1991) 6426–6434.
[11] Ch. Lubich, Discretized fractional calculus, SIAM J. Math. Anal. 17 (1986) 704–719.
[12] W. McLean, V. Thomée, Numerical solution of an evolution equation with a positive-type memory term, J. Austral. Math. Soc. Ser. B 35 (1993) 23–70.
[13] W. McLean, V. Thomée, L.B. Wahlbin, Discretization with variable time steps of an evolution equation with a positive-type memory term, J. Comput. Appl. Math. 69 (1996) 49–69.
[14] F. Oberhettinger, L. Badii, Tables of Laplace Transforms, Springer-Verlag, New York, 1973.
[15] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.
[16] I.H. Sloan, V. Thomée, Time discretization of an integro-differential equation of parabolic type, SIAM J. Numer. Anal. 23 (1986) 1052–1061.
[17] B. van der Pol, H. Bremmer, Operational Calculus Based on the Two-Sided Laplace Integral, Cambridge University Press, Cambridge, 1975.
[18] N.Y. Zhang, On fully discrete Galerkin approximations for partial integro-differential equations of parabolic type, Math. Comput. 60 (1993) 133–166.