FINITE ELEMENT APPROXIMATION OF THE CAHN-HILLIARD-COOK EQUATION

MIHÁLY KOVÁCS*, STIG LARSSON†, AND ALI MESFORUSH

Abstract. We study the nonlinear stochastic Cahn-Hilliard equation perturbed by additive colored noise. We show almost sure existence and regularity of solutions. We introduce spatial approximation by a standard finite element method and prove error estimates of optimal order on sets of probability arbitrarily close to 1. We also prove strong convergence without known rate.

Key words. Cahn-Hilliard-Cook equation, additive noise, Wiener process, existence, regularity, finite element, error estimate, strong convergence

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1. Introduction.

We study the Cahn-Hilliard equation perturbed by noise, also known as the Cahn-Hilliard-Cook equation (cf. [2, 4]),

$$\begin{aligned}
&\mathrm{d}u - \Delta w \, \mathrm{d}t = \mathrm{d}W && \text{in } \mathcal{D} \times (0,T],\\
&w = -\Delta u + f(u) && \text{in } \mathcal{D} \times (0,T],\\
&\frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0 && \text{on } \partial\mathcal{D} \times (0,T],\\
&u(0) = u_0 && \text{in } \mathcal{D}.
\end{aligned}$$

Here $\mathcal{D}$ is a bounded domain in $\mathbf{R}^d$, $d = 1,2,3$, and $f(s) = s^3 - s$. Using the framework of [6] we write this as an abstract evolution equation of the form

$$\mathrm{d}X + \big(A^2 X + A f(X)\big)\,\mathrm{d}t = \mathrm{d}W, \quad t \in (0,T]; \quad X(0) = X_0, \tag{1.1}$$

where A denotes the negative Neumann Laplacian considered as an unbounded operator in the Hilbert space $H = L_2(\mathcal{D})$ and W is a Q-Wiener process in H with respect to a filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t\geq 0})$. We also write $H^s = H^s(\mathcal{D})$ for the standard Sobolev spaces. See Section 2 for details.

Our goal is to study the convergence properties of the spatially semidiscrete finite element approximation X_h of X, which is defined by an equation of the form

$$\mathrm{d}X_h + \big(A_h^2 X_h + A_h P_h f(X_h)\big)\,\mathrm{d}t = P_h\,\mathrm{d}W, \quad t \in (0,T]; \quad X_h(0) = P_h X_0.$$

In order to do so, we need to prove existence and regularity for solutions of (1.1). Such results were first proved in [5]. Under the assumption that the covariance operator $Q = I$ (space-time white noise, cylindrical noise) it was shown that there is a process which belongs to $C([0,T], H^{-1})$ almost surely and which is the unique solution of (1.1). Under the stronger assumption that A and Q commute and that $\mathrm{Tr}(A^{\delta-1}Q) < \infty$ for some $\delta > 0$ (colored noise) it was shown that the solution belongs to $C([0,T], H)$ almost surely. Such regularity is insufficient for proving convergence of a numerical solution. Our first aim is therefore to prove existence of a solution in $C([0,T], H^{\beta})$ almost surely for some $\beta > 0$.

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Following the semigroup approach of [6] we write the equation (1.1) as the integral equation (mild solution)

$$\begin{aligned} X(t) &= \mathrm{e}^{-tA^2}X_0 - \int_0^t A\mathrm{e}^{-(t-s)A^2} f(X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s) \\ &= Y(t) + W_A(t), \end{aligned}$$

where e^{-tA^2} is the analytic semigroup generated by $-A^2$ (see Corollary 4.2). This naturally splits the solution as $X = Y + W_A$, where $W_A(t) = \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s)$ is a stochastic convolution. This convolution, and its finite element approximation, was studied in [12]. In particular, it was shown there that if $\|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}}^2 < \infty$ for some $\beta \ge 0$, then we have regularity of order β in a mean square sense; that is,

$$\mathbf{E}\big[\|W_A(t)\|_{H^\beta}^2\big] \le \|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}}^2, \quad t \ge 0. \tag{1.2}$$

The other part, Y, solves a differential equation with random coefficient,

$$\dot{Y} + A^2Y + Af(Y + W_A) = 0, \quad t > 0; \quad Y(0) = X_0. \tag{1.3}$$

This can be solved once W_A is known. This approach was also used in [5], but while they used Galerkin's method and energy estimates to solve (1.3), we use a semigroup approach similar to that of [7]. However, published results for the deterministic Cahn-Hilliard equation do not apply directly due to the limited regularity in (1.3).

The nonlinear term is only locally Lipschitz and we need to control the Lipschitz constant. In the deterministic case studied in [7] this is achieved by the Lyapunov functional

$$J(u) = \frac{1}{2}\|\nabla u\|^2 + \int_{\mathcal{D}} F(u)\,\mathrm{d}x, \quad u \in H^1; \quad F(s) = \tfrac14 s^4 - \tfrac12 s^2,$$

which is nonincreasing along paths, so that $\|X(t)\|_{H^1} \le C$ for $t \ge 0$. Due to the stochastic perturbation, this is not true for the stochastic equation (1.1). However, it is possible find a bound for the growth of the expected value of $J(X(t))$,

$$\mathbf{E}[J(X(t))] \le C(t), \quad t \ge 0. \tag{1.4}$$

This was shown in [5] under the assumption that A and Q commute and

$$\mathrm{Tr}(AQ) < \infty, \tag{1.5}$$

which is consistent with $\beta = 3$ in (1.2), since $\|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(AQ)$ in this case. (More generally: if AQ is nuclear, then $\|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(AQ)$, see [11, Theorem 2.1].) We repeat this in Theorem 3.1 with several improvements. First of all we reduce the growth of the bound from exponential to quadratic with respect to t. We also relax the assumptions: we do not assume that A and Q commute; that is, have a common eigenbasis, and we do not assume that the eigenbasis of Q consists of bounded functions. Moreover, we prove the same bound for the finite element solution X_h. Even if A and Q commute, this will not be true for the corresponding finite element approximations A_h and Q_h, so the relaxation of this assumption is necessary for the proof of the bound for X_h.

In Corollary 3.2 we improve (1.4) to a uniform norm bound

$$\mathbf{E}\Big[\sup_{s\in[0,T]}\big(\|X(s)\|_{H^1}^2+\|X_h(s)\|_{H^1}^2\big)\Big]\le K_T.$$

By means of Chebyshev's inequality we may then show that, for each $T>0$ and $\epsilon\in(0,1)$, there are K_T and $\Omega_\epsilon\subset\Omega$ with $\mathbf{P}(\Omega_\epsilon)\ge 1-\epsilon$ and such that

$$\|X(t)\|_{H^1}^2+\|X_h(t)\|_{H^1}^2\le\epsilon^{-1}K_T\quad\text{on }\Omega_\epsilon,\ t\in[0,T].$$

This bound controls the Lipschitz constant of the nonlinear term and we show in Theorem 4.3 that $X\in C([0,T],H^3)$ for $\omega\in\Omega_\epsilon$ under the slightly stronger assumption that $\|A^{\frac{\gamma}{2}}Q^{\frac12}\|_{\mathrm{HS}}<\infty$ for some $\gamma>1$, which is basically consistent with (1.5) and also with (1.2) in case $\beta=3$. We also obtain an error estimate (see Theorem 5.3)

$$\|X_h(t)-X(t)\|\le C(\epsilon^{-1}K_T,T)h^2|\log(h)|\quad\text{on }\Omega_\epsilon,\ t\in[0,T].$$

The constant grows rapidly with $\epsilon^{-1}K_T$, but nevertheless we may use this to show strong convergence (see Theorem 5.4),

$$\mathbf{E}\Big[\sup_{t\in[0,T]}\|X_h(t)-X(t)\|^2\Big]\to 0\quad\text{as }h\to 0.$$

To prove strong convergence with an estimate of the rate remains a challenge for future work. In this connection we note that even for numerical methods for stochastic ordinary differential equations with local Lipschitz nonlinearity there are few results on convergence rates (cf. [9]).

Numerical methods for the deterministic Cahn-Hilliard equation are well covered in the literature. There are few studies of numerical methods for the Cahn-Hilliard-Cook equation. We are only aware of [3] in which convergence in probability was proved for a difference scheme for the nonlinear equation in multiple dimensions. For the linear equation there is [10], where strong convergence estimates were proved for the finite element method for the linear equation in 1-D, and the already mentioned work [12] on the finite element method for the stochastic convolution in multiple dimensions.

2. Preliminaries.

2.1. Norms. Let $\mathcal{D}\subset\mathbf{R}^d$, $d=1,2,3$, be a bounded convex domain with polygonal boundary $\partial\mathcal{D}$. Let $H=L_2(\mathcal{D})$ with standard inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and

$$\dot H=\Big\{v\in H:\int_{\mathcal{D}}v\,\mathrm{d}x=0\Big\}.$$

Let $P\colon H\to\dot H$ define the orthogonal projector. Then

$$(I-P)v=|\mathcal{D}|^{-1}\int_{\mathcal{D}}v\,\mathrm{d}x,$$

is the average of v. We also denote by $H^k=H^k(\mathcal{D})$ the standard Sobolev space. We define $A=-\Delta$ with domain of definition

$$D(A)=\Big\{v\in H^2:\frac{\partial v}{\partial n}=0\text{ on }\partial\mathcal{D}\Big\}.$$

Then A is a positive definite, selfadjoint, unbounded, linear operator on $\dot{H}$ with compact inverse. When extended to H as $Av = APv$ it has an orthonormal eigenbasis $\{\varphi_j\}_{j=0}^\infty$ with corresponding eigenvalues $\{\lambda_j\}_{j=0}^\infty$ such that

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots, \quad \lambda_j \to \infty.$$

The first eigenfunction is constant, $\varphi_0 = |\mathcal{D}|^{-\frac{1}{2}}$.

We define seminorms and norms

$$|v|_\alpha = \Big(\sum_{j=1}^\infty \lambda_j^\alpha |\langle v, \varphi_j\rangle|^2\Big)^{\frac{1}{2}}, \quad \alpha \in \mathbf{R}, \tag{2.1}$$

$$\|v\|_\alpha = \big(|v|_\alpha^2 + |\langle v, \varphi_0\rangle|^2\big)^{\frac{1}{2}}, \quad \alpha \in \mathbf{R}, \tag{2.2}$$

and corresponding spaces

$$\dot{H}^\alpha = D(A^{\frac{\alpha}{2}}) = \Big\{v \in \dot{H} : |v|_\alpha < \infty\Big\}, \quad H^\alpha = \Big\{v \in H : \|v\|_\alpha < \infty\Big\}.$$

For integer order $\alpha = k \ge 0$, H^k coincides with the standard Sobolev spaces with $\|\cdot\|_k$ equivalent to the standard norm $\|\cdot\|_{H^k}$. For example,

$$\|v\|_1^2 = |v|_1^2 + |\langle v, \varphi_0\rangle|^2 = \|\nabla v\|^2 + |\langle v, \varphi_0\rangle|^2 \tag{2.3}$$

is equivalent to the standard norm $\|v\|_{H^1}^2$ by the Poincaré inequality.

2.2. The semigroup. The operator $-A^2$ is the infinitesimal generator of an analytic semigroup e^{-tA^2} on H,

$$\begin{aligned}\mathrm{e}^{-tA^2} v &= \sum_{j=0}^\infty \mathrm{e}^{-t\lambda_j^2}\langle v, \varphi_j\rangle\varphi_j = \sum_{j=1}^\infty \mathrm{e}^{-t\lambda_j^2}\langle v, \varphi_j\rangle\varphi_j + \langle v, \varphi_0\rangle\varphi_0 \\ &= \mathrm{e}^{-tA^2}Pv + (I-P)v.\end{aligned} \tag{2.4}$$

The analyticity implies that

$$\|A^\alpha \mathrm{e}^{-tA^2} v\| \le C t^{-\frac{\alpha}{2}} \mathrm{e}^{-ct}\|v\|, \quad v \in H, \ \alpha > 0. \tag{2.5}$$

2.3. The finite element method. Let $\{\mathcal{T}_h\}_{h>0}$ denote a family of regular triangulations of $\mathcal{D}$ with maximal mesh size h. Let S_h be the space of continuous functions on $\mathcal{D}$, which are piecewise polynomials of degree ≤ 1 with respect to $\mathcal{T}_h$. Hence, $S_h \subset H^1$. We also define $\dot{S}_h = PS_h$; that is,

$$\dot{S}_h = \Big\{v_h \in S_h : \int_{\mathcal{D}} v_h \,\mathrm{d}x = 0\Big\}.$$

The space $\dot{S}_h$ is introduced only for the purpose of theory but not for computation. Now we define the "discrete Laplacian" $A_h \colon S_h \to \dot{S}_h$ by

$$\langle A_h v_h, w_h\rangle = \langle \nabla v_h, \nabla w_h\rangle, \quad \forall v_h \in S_h,\ w_h \in \dot{S}_h.$$

We note that

$$|v_h|_1 = \|A^{\frac{1}{2}} v_h\| = \|\nabla v_h\| = \|A_h^{\frac{1}{2}} v_h\|, \quad v_h \in S_h. \tag{2.6}$$

The operator A_h is selfadjoint, positive definite on $\dot S_h$, positive semidefinite on S_h, and A_h has an orthonormal eigenbasis $\{\varphi_{h,j}\}_{j=0}^{N_h}$ with corresponding eigenvalues $\{\lambda_{h,j}\}_{j=0}^{N_h}$. We have

$$0 = \lambda_{h,0} < \lambda_{h,1} \le \cdots \le \lambda_{h,j} \le \cdots \le \lambda_{h,N_h},$$

and $\varphi_{h,0} = \varphi_0 = |\mathcal{D}|^{-\frac{1}{2}}$. Moreover, we define $\mathrm{e}^{-tA_h^2} \colon S_h \to S_h$ by

$$\mathrm{e}^{-tA_h^2} v_h = \sum_{j=0}^{N_h} \mathrm{e}^{-t\lambda_{h,j}} \langle v_h, \varphi_{h,j}\rangle \varphi_{h,j} = \sum_{j=1}^{N_h} \mathrm{e}^{-t\lambda_{h,j}} \langle v_h, \varphi_{h,j}\rangle \varphi_{h,j} + \langle v_h, \varphi_0\rangle \varphi_0,$$

and the orthogonal projector $P_h \colon H \to S_h$ by

$$\langle P_h v, w_h\rangle = \langle v, w_h\rangle \quad \forall v \in H,\, w_h \in S_h. \tag{2.7}$$

Clearly, $P_h \colon \dot H \to \dot S_h$ and

$$\mathrm{e}^{-tA_h^2} P_h v = \mathrm{e}^{-tA_h^2} P_h P v + (I - P)v. \tag{2.8}$$

We have a discrete analog of (2.5),

$$\|A_h^{\alpha} \mathrm{e}^{-tA_h^2} v_h\| \le C t^{-\frac{\alpha}{2}} \mathrm{e}^{-ct} \|v_h\|, \quad v_h \in S_h,\ \alpha > 0. \tag{2.9}$$

Finally, we define the Ritz projector $R_h \colon \dot H^1 \to \dot S_h$ by

$$\langle \nabla R_h v, \nabla w_h\rangle = \langle \nabla v, \nabla w_h\rangle, \quad \forall v \in \dot H^1,\, w_h \in \dot S_h.$$

We extend it to $R_h \colon H^1 \to S_h$ by

$$R_h v = R_h P v + (I - P)v, \quad v \in H^1. \tag{2.10}$$

We then have the following bound for $R_h v - v = (R_h - I)Pv$ (cf. [13, Ch. 1])

$$\|R_h v - v\| \le C h^{\beta} |v|_{\beta}, \quad v \in H^{\beta},\ \beta \in [1, 2]. \tag{2.11}$$

In order to simplify the presentation, we assume that P_h is bounded with respect to the H^1 and L_4 norms, and that we have an inverse bound for A_h,

$$\begin{aligned} &\|P_h v\|_1 \le C\|v\|_1, && v \in H^1,\\ &\|P_h v\|_{L_4} \le C\|v\|_{L_4}, && v \in L_4(\mathcal{D}),\\ &\|A_h v_h\| \le C h^{-2}\|v_h\|, && v_h \in S_h. \end{aligned} \tag{2.12}$$

This holds, for example, if the mesh family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform; that is, the area of $\tau \in \mathcal{T}_h$ is bounded below by ch^d, with $c > 0$ independent of h.

2.4. The Wiener process. We recall the definitions of the trace and the Hilbert-Schmidt norm of a linear operator T on H:

$$\mathrm{Tr}(T) = \sum_{k=1}^{\infty} \langle T f_k, f_k\rangle, \quad \|T\|_{\mathrm{HS}} = \Big(\sum_{k=1}^{\infty} \|T f_k\|^2\Big)^{\frac{1}{2}}, \tag{2.13}$$

where $\{f_k\}_{k=1}^{\infty}$ is an arbitrary orthonormal basis of H.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let Q be a selfadjoint, positive semidefinite, bounded, linear operator on H with $\mathrm{Tr}(Q) < \infty$. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal eigenbasis for Q with eigenvalues $\{\gamma_k\}_{k=1}^{\infty}$. Then we define the Q-Wiener process

$$W(t) = \sum_{k=1}^{\infty} \gamma_k^{\frac{1}{2}} \beta_k(t) e_k,$$

where the β_k are real-valued, independent Brownian motions. The series converges in $L_2(\Omega, H)$; that is, with respect to the norm $\|v\|_{L_2(\Omega,H)} = (\mathbf{E}[\|v\|^2])^{\frac{1}{2}}$. The process W generates a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ so that it becomes a square integrable martingale and so that we can integrate with respect to W. In the sequel we work in the resulting filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t\geq 0})$. We refer to [6] for the details. The Q-Wiener process can be defined also when the covariance operator has infinite trace but this is not needed in the present work.

2.5. The stochastic convolution. We now define (cf. [6])

$$W_A(t) = \int_0^t \mathrm{e}^{-(t-s)A^2} \, \mathrm{d}W(s) = \int_0^t \mathrm{e}^{-(t-s)A^2} P \, \mathrm{d}W(s) + (I-P)W(t), \tag{2.14}$$

where (2.4) was also used, and similarly, by (2.8),

$$\begin{aligned} W_{A_h}(t) &= \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h \, \mathrm{d}W(s) \\ &= \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h P \, \mathrm{d}W(s) + (I-P)W(t). \end{aligned} \tag{2.15}$$

Hence, the constant eigenmodes cancel:

$$W_{A_h}(t) - W_A(t) = \int_0^t \big(\mathrm{e}^{-(t-s)A_h^2} P_h - \mathrm{e}^{-(t-s)A^2}\big) P \, \mathrm{d}W(s). \tag{2.16}$$

These convolutions were studied in [12]. We quote the following results from there. We use the norms

$$\|v\|_{L_2(\Omega, \dot{H}^\beta)} = \big(\mathbf{E}\big[\,|v|_\beta^2\,\big]\big)^{\frac{1}{2}}.$$

Theorem 2.1. *If $\|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 2$, then*

$$\|W_A(t)\|_{L_2(\Omega, \dot{H}^\beta)} \leq C \|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}, \quad t \geq 0.$$

Theorem 2.2. *If $\|Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$, then*

$$\|W_{A_h}(t) - W_A(t)\|_{L_2(\Omega,H)} \leq C h^2 |\log h| \|Q^{\frac{1}{2}}\|_{\mathrm{HS}}, \quad t \geq 0.$$

Note that $\beta = 2$ in the latter theorem. In [12] these are stated with a wider range of the order β, but this is not needed in the present work.

2.6. Gronwall's lemma. We need a generalized Gronwall lemma, see [8, Lemma 7.1.1]. A shorter proof can found in [7].

LEMMA 2.3 (Generalized Gronwall lemma). *Let $\varphi \in L_1([0,T],\mathbf{R})$ be a nonnegative function. If*

$$\varphi(t) \le At^{-1+\alpha} + B\int_0^t (t-s)^{-1+\beta}\varphi(s)\,\mathrm{d}s, \quad t \in (0,T],$$

with constants $A, B \ge 0$ and $\alpha, \beta > 0$, then there is a constant $C = C(B,T,\alpha,\beta)$ such that

$$\varphi(t) \le CAt^{-1+\alpha}, \quad t \in (0,T].$$

We also use the standard Gronwall lemma:

LEMMA 2.4 (Gronwall's lemma). *Let $\varphi \in L_1([0,T],\mathbf{R})$. If*

$$\varphi(t) \le A + Ct + B\int_0^t \varphi(s)\,\mathrm{d}s, \quad t \in [0,T],$$

for some constants $A, C \ge 0$ and $B > 0$, then

$$\varphi(t) \le \Big(A + \frac{C}{B}\Big)\mathrm{e}^{Bt}, \quad t \in [0,T].$$

2.7. Bounds for the nonlinear term. Recall that the standard Sobolev norm $\|\cdot\|_{H^k}$ is equivalent to the norm $\|\cdot\|_k$ in (2.2) for integer $k \ge 0$. Most of the following estimates are well known and the detailed proofs are just included for pedagogical reasons, for example, to clarify the role of the projector P and the difference between the standard Sobolev norm $\|\cdot\|_{H^k}$ and the "abstract" norm $\|\cdot\|_k$.

LEMMA 2.5. *For $u, v \in H^3$ and $f(s) = s^3 - s$ we have*

$$\|\Delta f(u)\| \le C\big(1 + \|u\|_1^2\big)\|u\|_3, \tag{2.17}$$

$$\|A_h^{-\frac{1}{2}}P\big(f(u) - f(v)\big)\| \le C\big(1 + \|u\|_1^2 + \|v\|_1^2\big)\|u - v\|. \tag{2.18}$$

Proof. We have $f'(s) = 3s^2 - s$, $f''(s) = 6s$. Using Hölder's inequality, Sobolev's inequality $\|u\|_{L_6} \le C\|u\|_{H^1}$ (for $d \le 3$), and $\|u\|_{H^k} \le C\|u\|_k$, we get

$$\begin{aligned}
\|\Delta f(u)\| &= \|f'(u)\Delta u + f''(u)|\nabla u|^2\| \\
&\le \|f'(u)\|_{L_3}\|\Delta u\|_{L_6} + \|f''(u)\|_{L_6}\|\nabla u\|_{L_6}^2 \\
&\le C\big(1 + \|u\|_{L_6}^2\big)\|\Delta u\|_{L_6} + C\|u\|_{L_6}\|\nabla u\|_{L_6}^2 \\
&\le C\big(1 + \|u\|_{H^1}^2\big)\|u\|_{H^3} + C\|u\|_{H^1}\|u\|_{H^2}^2 \\
&\le C\big(1 + \|u\|_1^2\big)\|u\|_3 + C\|u\|_1\|u\|_2^2 \\
&\le C\big(1 + \|u\|_1^2\big)\|u\|_3,
\end{aligned}$$

where we used $\|u\|_2 \le C\|u\|_1^{\frac{1}{2}}\|u\|_3^{\frac{1}{2}}$ in the last step. This proves (2.17).

For (2.18) we apply (2.6) and the Hölder and Sobolev inequalities ($d \le 3$) to get

$$\begin{aligned}\|A_h^{-\frac12}P\varphi\| &= \sup_{v_h\in S_h}\frac{\langle A_h^{-\frac12}P\varphi, v_h\rangle}{\|v_h\|} = \sup_{v_h\in S_h}\frac{\langle \varphi, A_h^{-\frac12}Pv_h\rangle}{\|v_h\|}\\ &= \sup_{w_h\in \dot S_h}\frac{\langle\varphi, w_h\rangle}{|w_h|_1} \le \sup_{w_h\in\dot S_h}\frac{\|\varphi\|_{L_{6/5}}\|w_h\|_{L_6}}{|w_h|_1} \le C\|\varphi\|_{L_{6/5}}.\end{aligned}$$

We use this with $\varphi = f(u)-f(v) = \int_0^1 f'(u_s)\,\mathrm{d}s\,(u-v)$, where $u_s = su+(1-s)v$, and Hölder's and Sobolev's inequalities to get

$$\begin{aligned}\|A_h^{-\frac12}P\big(f(u)-f(v)\big)\| &= \|A_h^{-\frac12}P\varphi\| \le C\|\varphi\|_{L_{6/5}}\\ &\le C\int_0^1 \|f'(u_s)\|_{L_3}\,\mathrm{d}s\,\|u-v\| \le C\int_0^1\big(1+\|u_s\|_{L_6}^2\big)\,\mathrm{d}s\,\|u-v\|\\ &\le C\int_0^1\big(1+\|u_s\|_1^2\big)\,\mathrm{d}s\,\|u-v\| \le C\big(1+\|u\|_1^2+\|v\|_1^2\big)\|u-v\|.\end{aligned}$$

This is (2.18). □

3. The Cahn-Hilliard-Cook equation.

3.1. The continuous problem. The Cahn-Hilliard-Cook equation is

$$\begin{aligned}&\mathrm{d}u - \Delta w\,\mathrm{d}t = \mathrm{d}W && \text{in } \mathcal{D}\times(0,T],\\ &w = -\Delta u + f(u) && \text{in } \mathcal{D}\times(0,T],\\ &\frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0 && \text{on } \partial\mathcal{D}\times(0,T],\\ &u(0)=u_0 && \text{in } \mathcal{D}.\end{aligned}\tag{3.1}$$

The finite element approximation is based on its weak form, which is (formally)

$$\begin{aligned}&\langle u(t),v\rangle - \langle u_0,v\rangle + \int_0^t\langle\nabla w(s),\nabla v\rangle\,\mathrm{d}s = \langle W(t),v\rangle, && t\in(0,T],\\ &\langle w,v\rangle = \langle\nabla u,\nabla v\rangle + \langle f(u),v\rangle, && t\in(0,T],\end{aligned}\tag{3.2}$$

for all $v\in\dot H^1$. With the operator A, defined in § 2.1, we write (3.1) in the formal abstract form on $H = L_2(\mathcal{D})$:

$$\mathrm{d}X + \big(A^2X + Af(X)\big)\,\mathrm{d}t = \mathrm{d}W,\quad t\in(0,T];\quad X(0)=X_0.\tag{3.3}$$

A *weak solution* of (3.3) is an adapted H-valued process X, which is continuous almost surely and satisfies the equation

$$\langle X(t),v\rangle - \langle X_0,v\rangle + \int_0^t\big(\langle X(s),A^2v\rangle + \langle f(X(s)),Av\rangle\big)\,\mathrm{d}s = \langle W(t),v\rangle\tag{3.4}$$

almost surely for all $v\in\dot H^4 = D(A^2)$, $t\in[0,T]$, where we also require the integrand in the deterministic integral to be in $L_1([0,T],\mathbf{R})$ almost surely. A *mild solution* of (3.3) is an adapted H-valued process X, continuous almost surely, which satisfies

$$X(t) = \mathrm{e}^{-tA^2}X_0 - \int_0^t A\mathrm{e}^{-(t-s)A^2}f(X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s),\tag{3.5}$$

almost surely for $t\in[0,T]$, where we also require that the first integrand is in $L_1([0,T],H)$ and the stochastic integral exists almost surely.

3.2. The finite element problem. Recalling (3.2), we define the finite element solution $u_h(t), w_h(t) \in S_h$ of (3.1) by

$$\begin{aligned} &\langle u_h(t), v_h\rangle - \langle u_0, v_h\rangle + \int_0^t \langle \nabla w_h(s), \nabla v_h\rangle \,\mathrm{d}s = \langle W(t), v_h\rangle, && t \in (0,T],\\ &\langle w_h, v_h\rangle = \langle \nabla u_h, \nabla v_h\rangle + \langle f(u_h), v_h\rangle, && t \in (0,T], \end{aligned}$$

for all $v_h \in S_h$. With the operators A_h, P_h from § 2.3 we write this as an abstract equation in S_h:

$$\mathrm{d}X_h + \big(A_h^2 X_h + A_h P_h f(X_h)\big)\,\mathrm{d}t = P_h\,\mathrm{d}W, \quad t \in (0,T]; \quad X_h(0) = P_h X_0. \tag{3.6}$$

Since S_h is finite-dimensional and f is a polynomial, it is easy to see using standard arguments that (3.6) has a unique solution X_h, adapted, continuous almost surely, satisfying both

$$X_h(t) - P_h X_0 + \int_0^t \big(A_h^2 X_h(s) + A_h P_h f(X_h(s))\big)\,\mathrm{d}s = P_h W(t),$$

and

$$X_h(t) = \mathrm{e}^{-tA_h^2} P_h X_0 - \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h P_h f(X_h(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h\,\mathrm{d}W(s),$$

almost surely for $t \in [0,T]$.

3.3. A Lyapunov functional. The deterministic Cahn-Hilliard equation defines a gradient flow in $\dot H^{-1}$ for the energy functional

$$J(u) = \frac{1}{2}\|\nabla u\|^2 + \int_{\mathcal{D}} F(u)\,\mathrm{d}x, \quad u \in H^1, \tag{3.7}$$

where $F(s) = \frac14 s^4 - \frac12 s^2$ is a primitive of $f(s) = s^3 - s$. This is a Lyapunov functional for the deterministic Cahn-Hilliard equation, which implies that $J(X(t))$ does not increase along solution paths. For the stochastic equation this is not true, but we have a bound for the expected value of $J(X(t))$.

Theorem 3.1. *Assume that $\|A^{\frac12} Q^{\frac12}\|_{\mathrm{HS}} < \infty$ and that X_0 is $\mathcal{F}_0$-measurable with values in H^1 satisfying $\mathbf{E}[J(X_0)] < \infty$. If X is a weak solution of* (3.3) *and X_h is the solution of* (3.6)*, then, for all $t > 0$, we have*

$$\mathbf{E}[J(X(t))] + \mathbf{E}\Big[\int_0^t |J'(X(s))|_1^2\,\mathrm{d}s\Big] \le C\Big(\mathbf{E}[J(X_0)] + K_Q t + K_Q^2 t^2\Big) \tag{3.8}$$

and

$$\mathbf{E}[J(X_h(t))] + \mathbf{E}\Big[\int_0^t |J'(X_h(s))|_1^2\,\mathrm{d}s\Big] \le C\Big(\mathbf{E}[J(P_h X_0)] + K_Q t + K_Q^2 t^2\Big), \tag{3.9}$$

where $K_Q = \|A^{\frac12} Q^{\frac12}\|_{\mathrm{HS}}^2 + \|Q^{\frac12}\|_{\mathrm{HS}}^2$.

Proof. We prove (3.9); the proof of (3.8) is obtained in a similar way by approximating (3.3) by Galerkin's method based on the eigenbasis of A instead of the finite element Galerkin method used in (3.6) (see also [5]).

We consider (3.6) as an Itô differential equation in S_h driven by $P_h W$, which is a Q_h-Wiener process in S_h with $Q_h = P_h Q P_h$. By assumption (2.12) it follows that

$\mathbf{E}[J(P_hX_0)] < \infty$, if $\mathbf{E}[J(X_0)] < \infty$. By applying Itô's formula ([6, Theorem 4.17]) to $J(X_h(t))$, we obtain

$$\begin{aligned} J(X_h(t)) &= J(X_h(0)) + \int_0^t \langle J'(X_h(s)), \mathrm{d}X_h(s)\rangle + \frac{1}{2}\int_0^t \mathrm{Tr}(J''(X_h(s))Q_h)\,\mathrm{d}s \\ &= J(P_hX_0) + \int_0^t \langle J'(X_h(s)), -A_h^2X_h(s) - A_hP_hf(X_h(s))\rangle\,\mathrm{d}s \\ &\quad + \int_0^t \langle J'(X_h(s)), P_h\,\mathrm{d}W(s)\rangle + \frac{1}{2}\int_0^t \mathrm{Tr}(J''(X_h(s))Q_h)\,\mathrm{d}s. \end{aligned}$$

With a slight abuse of notation we consider here J as a function $S_h \to \mathbf{R}$ and we compute $J'(u_h) \in S_h$ and $J''(u_h)\colon S_h \to S_h$ as follows:

$$\langle J'(u_h), v_h\rangle = \langle \nabla u_h, \nabla v_h\rangle + \langle f(u_h), v_h\rangle = \langle A_hu_h + P_hf(u_h), v_h\rangle$$

and

$$\langle J''(u_h)v_h, w_h\rangle = \langle \nabla v_h, \nabla w_h\rangle + \langle f'(u_h)v_h, w_h\rangle = \langle A_hv_h + P_h[f'(u_h)v_h], w_h\rangle$$

for $u_h, v_h, w_h \in S_h$, so that

$$J'(u_h) = A_hu_h + P_hf(u_h), \quad J''(u_h) = A_h + P_h[f'(u_h)\,\cdot\,]. \tag{3.10}$$

Hence, by (2.6),

$$\begin{aligned} J(X_h(t)) + \int_0^t |J'(X_h(s))|_1^2\,\mathrm{d}s = J(P_hX_0) &+ \int_0^t \langle J'(X_h(s)), P_h\,\mathrm{d}W(s)\rangle \\ &+ \frac{1}{2}\int_0^t \mathrm{Tr}(J''(X_h(s))Q_h)\,\mathrm{d}s. \end{aligned} \tag{3.11}$$

The stochastic integral is a martingale, so that $\mathbf{E}[\int_0^t\langle J'(X_h), P_h\,\mathrm{d}W\rangle] = 0$, and hence

$$\begin{aligned} &\mathbf{E}[J(X_h(t))] + \mathbf{E}\Big[\int_0^t |J'(X_h(s))|_1^2\,\mathrm{d}s\Big] \\ &\quad = \mathbf{E}[J(P_hX_0)] + \frac{1}{2}\mathbf{E}\Big[\int_0^t \mathrm{Tr}(J''(X_h(s))Q_h)\,\mathrm{d}s\Big]. \end{aligned} \tag{3.12}$$

We now compute

$$\mathrm{Tr}(J''(X_h(s))Q_h) = \mathrm{Tr}(A_hQ_h) + \mathrm{Tr}(P_h[f'(X_h(s))\,\cdot\,]Q_h)$$

by the definition in (2.13). Recall that for $S, T \in \mathcal{B}(H)$ we have $\mathrm{Tr}(TS) = \mathrm{Tr}(ST)$ provided that either S or T has finite trace. If, in addition, the operators S and T are positive semidefinite as well, then $\mathrm{Tr}(ST) = \|S^{\frac{1}{2}}T^{\frac{1}{2}}\|_{\mathrm{HS}}^2$. Thus,

$$\begin{aligned} \mathrm{Tr}(A_hQ_h) = \mathrm{Tr}(A_hP_hQP_h) = \mathrm{Tr}(P_hA_hP_hQ) &= \|A_h^{\frac{1}{2}}P_hQ^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \\ &\le \|A_h^{\frac{1}{2}}P_hA^{-\frac{1}{2}}\|_{B(\dot{H})}^2\|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2. \end{aligned}$$

Here we use (2.6) and (2.12) to get

$$\|A_h^{\frac{1}{2}}P_hA^{-\frac{1}{2}}v\| = |P_hA^{-\frac{1}{2}}v|_1 \le C|A^{-\frac{1}{2}}v|_1 = C\|v\|, \quad v \in \dot{H},$$

so that $\|A_h^{\frac12} P_h A^{-\frac12}\|_{B(\dot H)} \le C$. Hence, with $K_Q = \|A^{\frac12} Q^{\frac12}\|_{\mathrm{HS}}^2 + \|Q^{\frac12}\|_{\mathrm{HS}}^2$,

$$\mathrm{Tr}(A_h Q_h) \le C\|A^{\frac12} Q^{\frac12}\|_{\mathrm{HS}}^2 \le CK_Q. \tag{3.13}$$

Let $\{e_{h,k}\}_{k=0}^{N_h}$ be an orthonormal eigenbasis of Q_h and $\{\gamma_{h,k}\}_{k=0}^{N_h}$ the corresponding eigenvalues. We get

$$\begin{aligned}\mathrm{Tr}\left(P_h[f'(X_h)\cdot]Q_h\right) &= \sum_{k=0}^{N_h} \langle P_h[f'(X_h)Q_h e_{h,k}], e_{h,k}\rangle \\ &= \sum_{k=0}^{N_h} \gamma_{h,k}\langle f'(X_h)e_{h,k}, e_{h,k}\rangle = \sum_{k=0}^{N_h}\langle f'(X_h)Q_h^{\frac12}e_{h,k}, Q_h^{\frac12}e_{h,k}\rangle.\end{aligned} \tag{3.14}$$

By using the bound $|f'(s)| \le C(1+s^2)$ and Hölder's and Sobolev's inequalities we get

$$|\langle f'(u)v, v\rangle| \le C(1+\|u\|_{L_4}^2)\|v\|_{L_4}^2 \le C(1+\|u\|_{L_4}^2)\|v\|_{H^1}^2 \le C(1+\|u\|_{L_4}^2)\|v\|_1^2.$$

By (2.3) and (2.6) we have, for $v_h \in S_h$,

$$\|v_h\|_1^2 = |v_h|_1^2 + \langle v_h, \varphi_0\rangle^2 = \|A_h^{\frac12} v_h\|^2 + \langle v_h, \varphi_0\rangle^2,$$

so that, by (3.13),

$$\begin{aligned}\sum_{k=0}^{N_h}\|Q_h^{\frac12}e_{h,k}\|_1^2 &= \sum_{k=0}^{N_h}\|A_h^{\frac12}Q_h^{\frac12}e_{h,k}\|^2 + \sum_{k=0}^{N_h}\langle Q_h^{\frac12}e_{h,k}, \varphi_0\rangle^2 \\ &= \sum_{k=0}^{N_h}\gamma_{h,k}\langle A_h e_{h,k}, e_{h,k}\rangle + \sum_{k=0}^{N_h}\gamma_{h,k}\langle e_{h,k}, \varphi_0\rangle^2 \\ &\le \mathrm{Tr}(A_h Q_h) + \mathrm{Tr}(Q_h) \le \mathrm{Tr}(A_h Q_h) + \mathrm{Tr}(Q) \\ &\le C\|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}}^2 + \|Q^{\frac12}\|_{\mathrm{HS}}^2 \le CK_Q.\end{aligned}$$

Returning to (3.14), we now have

$$|\,\mathrm{Tr}(P_h[f'(X_h)\cdot]Q_h)| \le C\big(1+\|X_h\|_{L_4}^2\big)\sum_{k=0}^{N_h}\|Q_h^{\frac12}e_{h,k}\|_1^2 \le C\big(1+\|X_h\|_{L_4}^2\big)K_Q,$$

Using also (3.13) we conclude

$$|\,\mathrm{Tr}(J''(X_h)Q_h)| \le CK_Q\big(1+\|X_h\|_{L_4}^2\big). \tag{3.15}$$

It remains to relate $\|X_h\|_{L_4}$ to $J(X_h)$. By definition of the Lyapunov functional (3.7) and noting that $F(s) = \frac14 s^4 - \frac12 s^2 \ge c_1 s^4 - c_2$, we get (with new constants)

$$J(u) \ge \frac12\|\nabla u\|^2 + C_1\|u\|_{L_4}^4 - C_2,$$

which implies

$$\|\nabla u\|^2 + \|u\|_{L_4}^4 \le C_3\big(1+J(u)\big). \tag{3.16}$$

Hence, in view of (3.15),

$$|\operatorname{Tr}(J''(X_h)Q_h)| \le CK_Q\big(1 + J(X_h)^{\frac{1}{2}}\big). \tag{3.17}$$

Inserting this into (3.12) gives

$$\begin{aligned} &\mathbf{E}[J(X_h(t))] + \mathbf{E}\Big[\int_0^t |J'(X_h(s))|_1^2 \,\mathrm{d}s\Big] \\ &\qquad \le \mathbf{E}[J(P_h X_0)] + CK_Q\Big(t + \int_0^t \mathbf{E}\big[J(X_h(s))^{\frac{1}{2}}\big]\,\mathrm{d}s\Big). \end{aligned} \tag{3.18}$$

Here, by Hölder's and Young's inequalities, we have, for $\epsilon > 0$,

$$\begin{aligned} CK_Q \int_0^t \mathbf{E}[J(X_h(s))^{\frac{1}{2}}]\,\mathrm{d}s &\le CK_Q t^{\frac{1}{2}}\Big(\int_0^t \mathbf{E}[J(X_h(s))]\,\mathrm{d}s\Big)^{\frac{1}{2}} \\ &\le \epsilon \int_0^t \mathbf{E}[J(X_h(s))]\,\mathrm{d}s + C\epsilon^{-1} t K_Q^2. \end{aligned}$$

Putting this in (3.18) gives

$$\begin{aligned} &\mathbf{E}[J(X_h(t))] + \mathbf{E}\Big[\int_0^t |J'(X_h(s))|_1^2 \,\mathrm{d}s\Big] \\ &\qquad \le \mathbf{E}[J(P_h X_0)] + C\big(K_Q + \epsilon^{-1}K_Q^2\big)t + \epsilon \int_0^t \mathbf{E}[J(X_h(s))]\,\mathrm{d}s. \end{aligned}$$

We apply the Gronwall Lemma 2.4 to get,

$$\begin{aligned} &\mathbf{E}[J(X_h(t))] + \mathbf{E}\Big[\int_0^t |J'(X_h(s))|_1^2 \,\mathrm{d}s\Big] \\ &\qquad \le \mathrm{e}^{\epsilon t}\Big(\mathbf{E}[J(P_h X_0)] + C(\epsilon^{-1}K_Q + \epsilon^{-2}K_Q^2)\Big) \\ &\qquad \le \mathrm{e}\Big(\mathbf{E}[J(P_h X_0)] + C(tK_Q + t^2 K_Q^2)\Big), \end{aligned}$$

where for each fixed t we have chosen $\epsilon = t^{-1}$ to get an optimal bound. □

This theorem is adapted from [5]. We have improved it in several ways. First the growth of the bound is reduced from exponential to quadratic with respect to t. Most importantly, we have removed the assumption that A and Q have a common eigenbasis and that the eigenbasis satisfies $\|e_k\|_{L_\infty} \le C$ for all k. This is important because even if A and Q commute, this will not be true for A_h and Q_h. This is crucial for the proof of the bound for X_h.

Note that $\|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ implies $K_Q = \|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 + \|Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 < \infty$. This is because of the boundedness of $A^{-\frac{1}{2}}$ and

$$\begin{aligned}\|Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 &= \sum_{j=1}^{\infty}\langle Q\varphi_j,\varphi_j\rangle + \langle Q\varphi_0,\varphi_0\rangle = \sum_{j=1}^{\infty}\|PQ^{\frac{1}{2}}\varphi_j\|^2 + \langle Q\varphi_0,\varphi_0\rangle \\ &= \sum_{j=1}^{\infty}\|A^{-\frac{1}{2}}A^{\frac{1}{2}}PQ^{\frac{1}{2}}\varphi_j\|^2 + \langle Q\varphi_0,\varphi_0\rangle \\ &\le C\sum_{j=1}^{\infty}\|A^{\frac{1}{2}}PQ^{\frac{1}{2}}\varphi_j\|^2 + \langle Q\varphi_0,\varphi_0\rangle \\ &\le C\|A^{\frac{1}{2}}PQ^{\frac{1}{2}}\|_{\mathrm{HS}}^2 + \langle Q\varphi_0,\varphi_0\rangle = C\|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 + \langle Q\varphi_0,\varphi_0\rangle < \infty.\end{aligned} \tag{3.19}$$

This condition is therefore the same as the condition for regularity of order $\beta = 3$ for $W_A(t)$ in Theorem 2.1.

We now use the previous theorem together with Chebyshev's inequality to obtain pathwise norm bounds uniformly on subsets of Ω with probability arbitrarily close to 1. In order to achieve this we first replace the bound of $\sup_{s\in[0,t]}\mathbf{E}[J(X(s))]$ from Theorem 3.1 by a bound for $\mathbf{E}[\sup_{s\in[0,t]}(\|\nabla X(s)\|^2 + \|X(s)\|_{L_4}^4)]$.

Corollary 3.2. *Assume that $\|A^{\frac{\gamma}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ for some $\gamma > 1$ and that X_0 is $\mathcal{F}_0$-measurable with values in H^1 satisfying*

$$\|X_0\|_{L_2(\Omega,H^1)}^2 + \|X_0\|_{L_4(\Omega,L_4)}^4 \le \rho \tag{3.20}$$

for some $\rho \ge 0$. If X is a weak solution of (3.3) *and X_h is the solution of* (3.6), *then, for $T \ge 0$,*

$$\mathbf{E}\Big[\sup_{t\in[0,T]}\big(\|\nabla X(t)\|^2 + \|X(t)\|_{L_4}^4\big)\Big] \le K_T, \tag{3.21}$$

$$\mathbf{E}\Big[\sup_{t\in[0,T]}\big(\|\nabla X_h(t)\|^2 + \|X_h(t)\|_{L_4}^4\big)\Big] \le K_T, \tag{3.22}$$

where K_T depends on ρ, K_Q, T. Moreover, for every $\epsilon \in (0,1)$, there is $\Omega_\epsilon \subset \Omega$ with $\mathbf{P}(\Omega_\epsilon) \ge 1-\epsilon$ and

$$\|\nabla X(t)\|^2 + \|X(t)\|_{L_4}^4 \le \epsilon^{-1}K_T \quad \text{on } \Omega_\epsilon,\ t\in[0,T], \tag{3.23}$$

$$\|\nabla X_h(t)\|^2 + \|X_h(t)\|_{L_4}^4 \le \epsilon^{-1}K_T \quad \text{on } \Omega_\epsilon,\ t\in[0,T], \tag{3.24}$$

$$\|X(t)\|_1^2 + \|X_h(t)\|_1^2 \le \epsilon^{-1}K_T \quad \text{on } \Omega_\epsilon,\ t\in[0,T], \tag{3.25}$$

$$\|W_A(t)\|_3^2 \le \epsilon^{-1}K_T \quad \text{on } \Omega_\epsilon,\ t\in[0,T]. \tag{3.26}$$

We remark that the stronger assumption $\gamma > 1$ is only used in the proof of (3.26).

Proof. It is enough to prove the existence of an Ω_ϵ for every $\epsilon > 0$ individually in (3.23)–(3.26). We prove the bounds for X_h and W_A; the others are proved similarly.

From (3.20) and (2.12) there follows $\mathbf{E}[J(P_hX_0)] \le C(1+\rho)$. Using also (3.16) in (3.11), we obtain

$$\begin{aligned}&\mathbf{E}\Big[\sup_{t\in[0,T]}\big(\|\nabla X_h(t)\|^2 + \|X_h(t)\|_{L_4}^4\big)\Big] \\ &\quad\le C\big(1+\rho\big) + C\,\mathbf{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t\langle J'(X_h(s)), P_h\,\mathrm{d}W(s)\rangle\Big|\Big] \\ &\qquad + C\,\mathbf{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t\mathrm{Tr}(J''(X_h(s)Q_h))\,\mathrm{d}s\Big|\Big].\end{aligned}$$

The stochastic integral is $\int_0^t \langle J'(X_h(s)), P_h \,\mathrm{d}W(s)\rangle = \int_0^t \widetilde{J'(X_h(s))} P_h \,\mathrm{d}W(s)$, where $\widetilde{J'(X_h(s))}\colon H \to \mathbf{R}$ is defined by $\widetilde{J'(X_h(s))}v = \langle J'(X_h(s)), v\rangle$. This integral is a martingale. Hence, we may use Hölder's inequality, the martingale inequality ([6, Theorem 3.8]), and the Itô isometry ([6, Corollary 4.14]) to get

$$\begin{aligned}
&\Big(\mathbf{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t \widetilde{J'(X_h(s))} P_h \,\mathrm{d}W(s)\Big|\Big]\Big)^2 \le \mathbf{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t \widetilde{J'(X_h(s))} P_h \,\mathrm{d}W(s)\Big|^2\Big] \\
&\quad\le 4\sup_{t\in[0,T]}\mathbf{E}\Big[\Big|\int_0^t \widetilde{J'(X_h(s))} P_h \,\mathrm{d}W(s)\Big|^2\Big] = 4\sup_{t\in[0,T]}\mathbf{E}\Big[\int_0^t \|\widetilde{J'(X_h(s))} Q_h^{\frac12}\|_{\mathrm{HS}}^2 \,\mathrm{d}s\Big] \\
&\quad= 4\mathbf{E}\Big[\int_0^T \|\widetilde{J'(X_h(s))} Q_h^{\frac12}\|_{\mathrm{HS}}^2 \,\mathrm{d}s\Big].
\end{aligned}$$

Here, by (2.2),

$$\begin{aligned}
\|\widetilde{J'(X_h(s))} Q_h^{\frac12}\|_{\mathrm{HS}}^2 &\le \|J'(X_h(s))\|^2 \|Q_h^{\frac12}\|_{\mathrm{HS}}^2 \\
&\le \big(|J'(X_h(s))|_0^2 + \langle J'(X_h(s)), \varphi_0\rangle^2\big)\,\mathrm{Tr}(Q) \\
&\le C\big(1 + |J'(X_h(s))|_1^2 + J(X_h(s))\big)\,\mathrm{Tr}(Q),
\end{aligned}$$

where we used (3.10) and the (rough) bounds $|u|_0 \le |u|_1$ and

$$\begin{aligned}
\langle J'(u_h), \varphi_0\rangle^2 &= |\mathcal{D}|^{-1}\Big(\int_{\mathcal{D}} \big(A_h u_h + P_h(u_h^3 - u_h)\big)\,\mathrm{d}x\Big)^2 \\
&= |\mathcal{D}|^{-1}\Big(\int_{\mathcal{D}} P_h(u_h^3 - u_h)\,\mathrm{d}x\Big)^2 \le C\big(1 + \|u_h\|_{L_4}^4\big) \le C\big(1 + J(u_h)\big).
\end{aligned}$$

By using (3.9), we conclude that

$$\begin{aligned}
&\Big(\mathbf{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t \widetilde{J'(X_h(s))} P_h \,\mathrm{d}W(s)\Big|\Big]\Big)^2 \\
&\quad\le C\,\mathrm{Tr}(Q)\Big(T + \mathbf{E}\Big[\int_0^T |J'(X_h(s))|_1^2 \,\mathrm{d}s\Big] + T\sup_{s\in[0,T]}\mathbf{E}\big[J(X_h(s))\big]\Big) \le K_T^2.
\end{aligned}$$

Next, using (3.17) and Hölder's inequality, we have

$$\begin{aligned}
\mathbf{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t \mathrm{Tr}(J''(X_h(s)Q_h))\,\mathrm{d}s\Big|\Big] &\le \mathbf{E}\Big[\int_0^T |\mathrm{Tr}(J''(X_h(s)Q_h))|\,\mathrm{d}s\Big] \\
&\le CK_Q\mathbf{E}\Big[\int_0^T \big(1 + J(X_h(s))^{\frac12}\big)\,\mathrm{d}s\Big] \\
&\le CK_Q T\big(1 + \sup_{s\in[0,T]}\mathbf{E}[J(X_h(s))]\big) \le K_T,
\end{aligned}$$

which finishes the proof of (3.22).

In order to prove (3.24) we denote

$$F_h = \sup_{t\in[0,T]}\big(\|\nabla X_h(t)\|^2 + \|X_h(t)\|_{L_4}^4\big).$$

We apply Chebyshev's inequality and (3.22) to get, for every $\alpha > 0$,

$$\mathbf{P}\Big(\big\{\omega\in\Omega : F_h > \alpha\big\}\Big) \le \frac{1}{\alpha}\mathbf{E}\big[F_h\big] \le \frac{K_T}{\alpha}.$$

We choose $\alpha = \epsilon^{-1}K_T$ and set $\Omega_\epsilon = \{\omega \in \Omega : F_h \le \epsilon^{-1}K_T\}$. Then

$$\mathbf{P}(\Omega_\epsilon) = 1 - \mathbf{P}\Big(\{\omega \in \Omega : F_h > \alpha\}\Big) \ge 1 - \epsilon$$

and (3.24) holds. For (3.25) we note that

$$\|u\|_1^2 \le \|\nabla u\|^2 + \|u\|^2 \le \|\nabla u\|^2 + C\big(1 + \|u\|_{L_4}^4\big)$$

and hence (3.25) follows from (3.21) and (3.22) after an adjustment of K_T. Finally, (3.26) follows by first employing a factorization method argument as in the proof of [6, Remark 5.11] but in the H^3-norm using the analyticity of the semigroup (this is where $\gamma > 1$ is needed), and then using Chebychev's inequality as above. □

4. Regularity of the solution. We quote the following from [5]. There it is assumed that A and Q commute and that the eigenfunctions of A are uniformly bounded in the sup norm but it can be verified that these are not necessary for the following result. Recall the definitions of weak and mild solutions in (3.4) and (3.5).

THEOREM 4.1. *Let $T > 0$ and assume that $\mathrm{Tr}(A^{\delta-1}Q) < \infty$ for some $\delta > 0$ and that X_0 is $\mathcal{F}_0$-measurable with values in H. Then there is a unique weak solution X of* (3.3).

COROLLARY 4.2. *Assume that $\|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}} < \infty$ and that X_0 is $\mathcal{F}_0$-measurable with values in H^1 satisfying $\|X_0\|^2_{L_2(\Omega,H^1)} + \|X_0\|^4_{L_4(\Omega,L_4)} \le \rho$ for some $\rho \ge 0$. Then the weak solution X of* (3.3) *is also a mild solution.*

Proof. By (3.19), the condition $\|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}} < \infty$ implies $\mathrm{Tr}(A^{\delta-1}Q) < \infty$ with $\delta = 1$ and hence there is a unique weak solution X of (3.3) by Theorem 4.1. Let $\epsilon > 0$ and Ω_ϵ the set defined in Corollary 3.2. We first show that X satisfies (3.5) on Ω_ϵ. By the uniqueness weak solutions of (3.3), we only have to show that the right-hand side of (3.5) satisfies (3.4) on Ω_ϵ. Since W_A is the unique weak solution of $\mathrm{d}Z + A^2 Z\,\mathrm{d}t = \mathrm{d}W,\ Z(0) = 0$ (by [6, Theorem 5.4]), it is enough to show that

$$Y(t) := \mathrm{e}^{-tA^2}X_0 - \int_0^t A\mathrm{e}^{-(t-s)A^2} f(X(s))\,\mathrm{d}s$$

satisfies

$$\langle Y(t), v\rangle - \langle X_0, v\rangle + \int_0^t \big(\langle Y(s), A^2 v\rangle + \langle f(X(s)), Av\rangle\big)\,\mathrm{d}s = 0 \text{ on } \Omega_\epsilon,\ v \in D(A^2).$$

This follows by standard arguments (see, e.g., [1]) as, by Sobolev's inequality and Corollary 3.2, there is C depending on ϵ, ρ, Q, and T such that

$$\|f(X(s))\| \le C(\|X(s)\|_{L^2} + \|X(s)\|^3_{L^6}) \le C(\|X(s)\|_{L^2} + \|X(s)\|^3_{H^1}) \le C$$

for $t \in [0,T],\ \omega \in \Omega_\epsilon$. This also shows that the integrand of the deterministic integral in (3.5) is in $L_1([0,T],H)$ on Ω_ϵ by the analyticity of the semigroup e^{-tA^2}. Finally, since $\epsilon > 0$ is arbitrary and $P(\Omega_\epsilon) > 1 - \epsilon$, the statement follows. □

We now show that, under the stronger assumption $\|A^{\frac{\gamma}{2}}Q^{\frac12}\|_{\mathrm{HS}} < \infty$ for some $\gamma > 1$, the solution $X(t)$ is actually in H^3 almost surely. In order to do this we write, as in the previous proof, $X(t) = Y(t) + W_A(t)$, where we already know from Theorem 2.1 that $W_A(t)$ is in H^3 almost surely. The regularity of Y is studied in the next theorem. Note that we saw in the proof of Corollary 4.2 that

$$Y(t) = X(t) - W_A(t) = \mathrm{e}^{-tA^2}X_0 - \int_0^t A\mathrm{e}^{-(t-s)A^2} f(X(s))\,\mathrm{d}s$$

is a weak solution of

$$\dot{Y} + A^2 Y + Af(X) = 0, \quad t \in (0,T]; \quad Y(0) = X_0, \tag{4.1}$$

almost surely.

THEOREM 4.3. *Assume that* $\|A^{\frac{\gamma}{2}} Q^{\frac12}\|_{\mathrm{HS}} < \infty$ *for some* $\gamma > 1$ *and that* X_0 *is* $\mathcal{F}_0$*-measurable with values in* H^3 *satisfying* $\|X_0\|^2_{L_2(\Omega,H^1)} + \|X_0\|^4_{L_4(\Omega,L_4)} \le \rho$ *for some* $\rho \ge 0$*. Let* $T > 0$ *and* $\epsilon \in (0,1)$ *and let* Ω_ϵ *and* K_T *be as in Corollary 3.2 such that also* $\|X_0\|_3 \le C$ *on* Ω_ϵ*. Let* X *be the solution from Theorem 4.1 and* $Y = X - W_A$*. Then* $X, Y \in C([0,T],H) \cap L_\infty([0,T],H^3)$ *almost surely, and, for each* $\omega \in \Omega_\epsilon$*,*

$$\|Y(t)\|_3 \le C(\|X_0\|_3, \epsilon^{-1}K_T, T) \quad \text{on } \Omega_\epsilon,\ t \in [0,T], \tag{4.2}$$
$$\|X(t)\|_3 \le C(\|X_0\|_3, \epsilon^{-1}K_T, T) \quad \text{on } \Omega_\epsilon,\ t \in [0,T]. \tag{4.3}$$

Proof. First note that since X_0 is H^3-valued almost surely, it is always possible to choose Ω_ϵ in Corollary 3.2 such that also $\|X_0\|_3 \le C$ on Ω_ϵ. The continuity of X is already contained in Theorem 4.1 and the continuity of Y follows from the continuity of X and W_A. To show that $X, Y \in L_\infty([0,T],H^3)$ almost surely it is enough to show (4.2) and (4.3) as $\epsilon > 0$ is arbitrary and $P(\Omega_\epsilon) \ge 1-\epsilon$. Let $t \in [0,T]$ and $\omega \in \Omega_\epsilon$. From Corollary 3.2 we have

$$\|X(t)\|_1^2 \le \epsilon^{-1}K_T, \quad \|W_A(t)\|_3 \le \epsilon^{-1}K_T. \tag{4.4}$$

We take seminorms in

$$Y(t) = \mathrm{e}^{-tA^2}X_0 - \int_0^t A\mathrm{e}^{-(t-s)A^2} f(X(s))\,\mathrm{d}s \tag{4.5}$$

and use (2.5) to get

$$\begin{aligned}
|Y(t)|_3 &\le |\mathrm{e}^{-tA^2}X_0|_3 + \int_0^t |\mathrm{e}^{-(t-s)A^2} Af(X(s))|_3\,\mathrm{d}s \\
&= \|\mathrm{e}^{-tA^2}A^{\frac32}X_0\| + \int_0^t \|A^{\frac32}\mathrm{e}^{-(t-s)A^2}Af(X(s))\|\,\mathrm{d}s \\
&\le |X_0|_3 + C\int_0^t (t-s)^{-\frac34}\|Af(X(s))\|\,\mathrm{d}s.
\end{aligned}$$

We apply (2.17) to $\|Af(X(s))\| = \|\Delta f(X(s))\|$ to get

$$\begin{aligned}
|Y(t)|_3 &\le |X_0|_3 + C\int_0^t (t-s)^{-\frac34}\big(1+\|X(s)\|_1^2\big)\|X(s)\|_3\,\mathrm{d}s \\
&\le |X_0|_3 + C\int_0^t (t-s)^{-\frac34}\big(1+\|X(s)\|_1^2\big)\big(\|Y(s)\|_3 + \|W_A(s)\|_3\big)\,\mathrm{d}s.
\end{aligned}$$

Since $(I-P)Y(t) = (I-P)X_0$ is constant, we get the same bound for the norm $\|Y(t)\|_3$. Using also (4.4) gives

$$\begin{aligned}
\|Y(t)\|_3 &\le \|X_0\|_3 + C\int_0^t (t-s)^{-\frac34}\big(1+\epsilon^{-1}K_T\big)\big(\|Y(s)\|_3 + \epsilon^{-1}K_T\big)\,\mathrm{d}s \\
&\le \|X_0\|_3 + C\epsilon^{-1}K_T\big(1+\epsilon^{-1}K_T\big)T^{\frac14} \\
&\quad + C\big(1+\epsilon^{-1}K_T\big)\int_0^t (t-s)^{-\frac34}\|Y(s)\|_3\,\mathrm{d}s.
\end{aligned}$$

Applying Gronwall's Lemma 2.3 with $\alpha = 1$, $\beta = \frac{1}{4}$ and

$$A = \|X_0\|_3 + C\epsilon^{-1}K_T\big(1+\epsilon^{-1}K_T\big),\ B = C\big(1+\epsilon^{-1}K_T\big), \tag{4.6}$$

gives

$$\|Y(t)\|_3 \le AC(B,T) = C(\|X_0\|_3, \epsilon^{-1}K_T, T), \quad t \in [0,T].$$

The bound for $\|X(t)\|_3$ then follows in view of (4.4). $\square$

The constant $C(\|X_0\|_3, \epsilon^{-1}K_T, T)$ grows rapidly with $\epsilon^{-1}K_T$ and T. Hence, it is important that K_T grows only quadratically with T. Also note that the proof of Theorem 4.3 shows that under the assumptions of the theorem, in fact, $f(X(t)) \in D(A)$ almost surely and $\|Af(X(t))\| < \infty$ almost surely for $t \in [0,T]$. Therefore, X satisfies a more strict (in comparison to (3.5)) mild form of (3.3):

$$X(t) = \mathrm{e}^{-tA^2}X_0 - \int_0^t \mathrm{e}^{-(t-s)A^2}Af(X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s).$$

5. Error estimates.

5.1. The linear deterministic Cahn-Hilliard equation. Consider the linear Cahn-Hilliard equation

$$\dot{u} + Av = 0,\ v - Au - f = 0,\ t > 0;\ u(0) = u_0, \tag{5.1}$$

where f is a function of x, t, and the corresponding finite element problem

$$\dot{u}_h + A_hv_h = 0,\ v_h - A_hu_h - P_hf = 0,\ t > 0;\ u_h(0) = P_hu_0. \tag{5.2}$$

We have the following error estimate. We will later use this for fixed $\omega \in \Omega_\epsilon$ with f replaced by $f(X)$ and u by the solution Y of (4.1). The error estimate differs from the corresponding error estimates in [7, 12] in that it contains no time derivative. This is important since Y has limited temporal regularity.

THEOREM 5.1. *Assume that u, v and u_h, v_h are weak solutions of* (5.1) *and* (5.2)*, respectively. Then, for $t \ge 0$ and $h \in (0, \frac{1}{2}]$, we have*

$$\|u_h(t) - u(t)\| \le Ch^2\Big(|\log(h)| \max_{0\le s\le t}|u(s)|_2 + \Big(\int_0^t |v(s)|_2^2\,\mathrm{d}s\Big)^{\frac{1}{2}}\Big). \tag{5.3}$$

Proof. The weak forms of (5.1) and (5.2) are

$$\begin{aligned}
&\langle \dot{u}, \varphi_1\rangle + \langle \nabla v, \nabla\varphi_1\rangle = 0 && \forall \varphi_1 \in H^1,\\
&\langle v, \varphi_2\rangle - \langle \nabla u, \nabla\varphi_2\rangle - \langle f, \varphi_2\rangle = 0 && \forall \varphi_2 \in H^1,\\
&u(0) = u_0,
\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}
&\langle \dot{u}_h, \varphi_{h,1}\rangle + \langle \nabla v_h, \nabla\varphi_{h,1}\rangle = 0 && \forall \varphi_{h,1} \in S_h,\\
&\langle v_h, \varphi_{h,2}\rangle - \langle \nabla u_h, \nabla\varphi_{h,2}\rangle - \langle f, \varphi_{h,2}\rangle = 0 && \forall \varphi_{h,2} \in S_h,\\
&u_h(0) = P_hu_0.
\end{aligned} \tag{5.5}$$

Let P_h and R_h be as in (2.7) and (2.10) and set

$$e_u = u_h - u = (u_h - P_h u) + (P_h u - u) = \theta_u + \rho_u, \tag{5.6}$$
$$e_v = v_h - v = (v_h - R_h v) + (R_h v - v) = \theta_v + \rho_v. \tag{5.7}$$

We want to compute

$$\|e_u\| \leq \|\theta_u\| + \|\rho_u\|. \tag{5.8}$$

In (5.4) choose $\varphi_1 = \varphi_{h,1}$ and $\varphi_2 = \varphi_{h,2}$ and subtract the first two equations of (5.4) from the corresponding equations in (5.5) to get

$$\begin{aligned} \langle \dot{e}_u, \varphi_{h,1} \rangle + \langle \nabla e_v, \nabla \varphi_{h,1} \rangle &= 0 \quad \forall \varphi_{h,1} \in S_h, \\ \langle e_v, \varphi_{h,2} \rangle - \langle \nabla e_u, \nabla \varphi_{h,2} \rangle &= 0 \quad \forall \varphi_{h,2} \in S_h. \end{aligned}$$

Hence, by (5.6) and (5.7),

$$\begin{aligned} \langle \dot{\theta}_u, \varphi_{h,1} \rangle + \langle \nabla \theta_v, \nabla \varphi_{h,1} \rangle &= -\langle \dot{\rho}_u, \varphi_{h,1} \rangle - \langle \nabla \rho_v, \nabla \varphi_{h,1} \rangle \quad \forall \varphi_{h,1} \in S_h, \\ \langle \theta_v, \varphi_{h,2} \rangle - \langle \nabla \theta_u, \nabla \varphi_{h,2} \rangle &= -\langle \rho_v, \varphi_{h,2} \rangle + \langle \nabla \rho_u, \nabla \varphi_{h,2} \rangle \quad \forall \varphi_{h,2} \in S_h. \end{aligned}$$

By the definitions of P_h and R_h we have

$$\begin{aligned} \langle \dot{\rho}_u, \varphi_{h,1} \rangle &= \langle P_h \dot{u} - \dot{u}, \varphi_{h,1} \rangle = 0 \qquad \forall \varphi_{h,1} \in S_h, \\ \langle \nabla \rho_v, \nabla \varphi_{h,1} \rangle &= \langle \nabla R_h v - v, \nabla \varphi_{h,1} \rangle = 0 \quad \forall \varphi_{h,2} \in S_h, \end{aligned}$$

so that

$$\begin{aligned} &\langle \dot{\theta}_u, \varphi_{h,1} \rangle + \langle \nabla \theta_v, \nabla \varphi_{h,1} \rangle = 0 && \forall \varphi_{h,1} \in S_h, \\ &\langle \theta_v, \varphi_{h,2} \rangle - \langle \nabla \theta_u, \nabla \varphi_{h,2} \rangle = -\langle P_h \rho_v, \varphi_{h,2} \rangle + \langle \nabla R_h \rho_u, \nabla \varphi_{h,2} \rangle && \forall \varphi_{h,2} \in S_h. \end{aligned}$$

In the second equation we set $\varphi_{h,2} = A_h \varphi_{h,1}$ to get

$$\langle \nabla \theta_v, \nabla \varphi_{h,1} \rangle = \langle A_h^2 \theta_u, \varphi_{h,1} \rangle - \langle A_h P_h \rho_v, \varphi_{h,1} \rangle + \langle A_h^2 R_h \rho_u, \varphi_{h,1} \rangle.$$

Inserting this into the first equation gives

$$\langle \dot{\theta}_u, \varphi_{h,1} \rangle + \langle A_h^2 \theta_u, \varphi_{h,1} \rangle = \langle A_h P_h \rho_v, \varphi_{h,1} \rangle - \langle A_h^2 R_h \rho_u, \varphi_{h,1} \rangle,$$

so the strong form is

$$\dot{\theta}_u + A_h^2 \theta_u = A_h P_h \rho_v - A_h^2 R_h \rho_u, \quad t > 0; \quad \theta_u(0) = 0,$$

with solution

$$\theta_u(t) = \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h P_h \rho_v(s)\,\mathrm{d}s - \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h^2 R_h \rho_u(s)\,\mathrm{d}s.$$

Taking norms here gives

$$\begin{aligned} \|\theta_u(t)\| \leq &\Big\| \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h P_h \rho_v(s)\,\mathrm{d}s \Big\| \\ &+ \Big\| \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h^2 R_h \rho_u(s)\,\mathrm{d}s \Big\| = I + II. \end{aligned} \tag{5.9}$$

For I we define

$$w_h(t) = \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h \rho_v(s)\,\mathrm{d}s,$$

which satisfies the equation

$$\dot{w}_h + A_h^2 w_h = P_h \rho_v, \quad t > 0; \quad w_h(0) = 0.$$

We multiply by $\dot{w}_h$ to get

$$\|\dot{w}_h\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A_h w_h\|^2 = \langle P_h \rho_v, \dot{w}_h\rangle \le \|\rho_v\|\|\dot{w}_h\| \le \frac{1}{2}\|\rho_v\|^2 + \frac{1}{2}\|\dot{w}_h\|^2,$$

so that

$$\|\dot{w}_h\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|A_h w_h\|^2 \le \|\rho_v\|^2.$$

Integration and ignoring $\int_0^t \|\dot{w}_h(s)\|^2\,\mathrm{d}s$ leads to

$$\Big\|A_h \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h \rho_v(s)\,\mathrm{d}s\Big\| = \|A_h w_h(t)\| \le \Big(\int_0^t \|\rho_v(s)\|^2\,\mathrm{d}s\Big)^{\frac{1}{2}},$$

where, from (2.11),

$$\|\rho_v\| = \|(R_h - I)v\| \le Ch^2 |v|_2.$$

Hence,

$$\Big\|A_h \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h \rho_v(s)\,\mathrm{d}s\Big\| \le Ch^2 \Big(\int_0^t |v(s)|_2^2\,\mathrm{d}s\Big)^{\frac{1}{2}}. \tag{5.10}$$

For the term II we use

$$R_h \rho_u = R_h(P_h u - u) = P_h u - R_h u = P_h(u - R_h u).$$

Then

$$\begin{aligned}\Big\|\int_0^t A_h^2 \mathrm{e}^{-(t-s)A_h^2} R_h \rho_u(s)\,\mathrm{d}s\Big\| &\le \int_0^t \|A_h^2 \mathrm{e}^{-(t-s)A_h^2} P_h(u(s) - R_h u(s))\|\,\mathrm{d}s \\ &\le \int_0^t \|A_h^2 \mathrm{e}^{-(t-s)A_h^2} P_h\|\,\mathrm{d}s \max_{0\le s\le t}\|u(s) - R_h u(s)\|.\end{aligned}$$

Here we use $\|A_h\| \le Ch^{-2}$ from (2.12) and (2.9) to get

$$\begin{aligned}\int_0^t \|A_h^2 \mathrm{e}^{-(t-s)A_h^2} P_h P\|\,\mathrm{d}s &= \int_0^{h^4} \|A_h\|^2 \|\mathrm{e}^{-sA_h^2}\|\,\mathrm{d}s + \int_{h^4}^t \|A_h^2 \mathrm{e}^{-sA_h^2}\|\,\mathrm{d}s \\ &\le Ch^{-4}h^4 + C\int_{h^4}^t s^{-1}\mathrm{e}^{-cs}\,\mathrm{d}s \le C(1 + \log(1/h)) \le C|\log(h)|\end{aligned}$$

for $h \in (0, \frac{1}{2}]$. Hence, by (2.11), we have

$$\Big\|\int_0^t A_h^2 \mathrm{e}^{-(t-s)A_h^2} R_h \rho_u(s)\,\mathrm{d}s\Big\| \le Ch^2 |\log(h)| \max_{0\le s\le t} |u(s)|_2. \tag{5.11}$$

Inserting (5.10) and (5.11) into (5.9) gives

$$\|\theta_u(t)\| \leq Ch^2\Big\{\Big(\int_0^t |v(s)|_2^2\,\mathrm{d}s\Big)^{\frac{1}{2}} + |\log(h)| \max_{0\leq s\leq t} |u(s)|_2\Big\}. \tag{5.12}$$

Finally, by the best approximation property of P_h,

$$\|\rho_u(t)\| = \|P_h u - u\| \leq \|R_h u - u\| \leq Ch^2|u(t)|_2. \tag{5.13}$$

Inserting (5.12) and (5.13) into (5.8) gives the desired result (5.3). □

The following regularity estimate for the linear Cahn-Hilliard equation (5.1) is proved by an elementary energy argument.

Lemma 5.2. *Assume that u, v are weak solutions of* (5.1). *Then*

$$|u(t)|_2^2 + \int_0^t |v(s)|_2^2\,\mathrm{d}s \leq |u_0|_2^2 + \int_0^t |f(s)|_2^2\,\mathrm{d}s.$$

5.2. Error estimate for the stochastic Cahn-Hilliard equation. In the next theorem we prove an error estimate for the nonlinear Cahn-Hilliard-Cook equation.

Theorem 5.3. *Assume that $\|A^{\frac{\gamma}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ for some $\gamma > 1$ and that X_0 is $\mathcal{F}_0$-measurable with values in H^3 satisfying $\|X_0\|^2_{L_2(\Omega,H^1)} + \|X_0\|^4_{L_4(\Omega,L_4)} \leq \rho$ for some $\rho \geq 0$. Let $T > 0$, $\epsilon \in (0,1)$, and let $\Omega_\epsilon \subset \Omega$ and K_T be as in Corollary 3.2 such that also $\|X_0\|_3 \leq C$ on Ω_ϵ. If X is the weak solution of* (3.3) *and X_h is the solution of* (3.6)*, then, for $h \in (0, \frac{1}{2}]$,*

$$\|X_h(t) - X(t)\| \leq C(\|X_0\|_3, \epsilon^{-1}K_T, T)h^2|\log(h)|, \quad \text{on } \Omega_\epsilon,\ t \in [0,T].$$

The constant $C(\|X_0\|_3, \epsilon^{-1}K_T, T)$ grows rapidly with $\epsilon^{-1}K_T$ and T due to the use of Gronwall's lemma in the proof. As noted before in the proof of Theorem 4.3, it is always possible to choose Ω_ϵ in Corollary 3.2 such that also $\|X_0\|_3 \leq C$ on Ω_ϵ.

Proof. Let $\omega \in \Omega_\epsilon$ be fixed. Set

$$X(t) = Y(t) + W_A(t), \tag{5.14}$$

where $W_A(t)$ is the stochastic convolution (2.14) and $Y(t)$ is the weak solution (4.5) of (4.1). Also set

$$X_h(t) = Z_h(t) + W_{A_h}(t), \tag{5.15}$$

where $W_{A_h}(t)$ is the stochastic convolution (2.15) and

$$Z_h(t) = \mathrm{e}^{-tA_h^2}P_hX_0 - \int_0^t \mathrm{e}^{-(t-s)A_h^2}A_hP_hf(X_h(s))\,\mathrm{d}s. \tag{5.16}$$

Finally, let

$$Y_h(t) = \mathrm{e}^{-tA_h^2}P_hX_0 - \int_0^t \mathrm{e}^{-(t-s)A_h^2}A_hP_hf(X(s))\,\mathrm{d}s. \tag{5.17}$$

We subtract (5.14) from (5.15) and take norms,

$$\|X_h - X\| \leq \|W_{A_h} - W_A\| + \|Y_h - Y\| + \|Z_h - Y_h\|. \tag{5.18}$$

We must compute the three norms on the right-hand side.

First we compute $\|W_{A_h}(t) - W_A(t)\|$. Since $\|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$, we have that $\|Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$, see (3.19), and hence, by Theorem 2.2 and Chebyshev's inequality, we get

$$\begin{aligned}\|W_{A_h}(t) - W_A(t)\| &\le \epsilon^{-\frac{1}{2}}\big(\mathbf{E}[\|W_{A_h}(t) - W_A(t)\|^2]\big)^{\frac{1}{2}} \\ &\le \epsilon^{-\frac{1}{2}}Ch^2|\log(h)|\|Q^{\frac{1}{2}}\|_{\mathrm{HS}} \le C(\epsilon^{-1}K_Q)^{\frac{1}{2}}h^2|\log(h)|,\end{aligned}$$

where K_Q is as in Theorem 3.1. Since $K_Q \le K_T$, we conclude

$$\|W_{A_h}(t) - W_A(t)\| \le C(\epsilon^{-1}K_T)^{\frac{1}{2}}h^2|\log(h)|. \tag{5.19}$$

Now we consider $\|Y_h(t) - Y(t)\|$ and use Theorem 5.1 to get

$$\|Y_h(t) - Y(t)\| \le Ch^2\Big\{|\log(h)| \max_{0\le s\le t}|Y(s)|_2 + \Big(\int_0^t |V(s)|_2^2\,\mathrm{d}s\Big)^{\frac{1}{2}}\Big\}, \tag{5.20}$$

where $Y(t)$ and $V(t)$ are the solutions of

$$\dot{Y} + AV = 0,\ V = AY + f(X),\ t \in (0,T];\ Y(0) = X_0.$$

By using Lemma 5.2, (2.17), (3.25), and Theorem 4.3 , we get

$$\begin{aligned}\int_0^t |V(s)|_2^2\,\mathrm{d}s &\le |X_0|_2^2 + \int_0^t |f(X(s))|_2^2\,\mathrm{d}s \\ &\le \|X_0\|_2^2 + C\int_0^t (1 + \|X(s)\|_1^2)^2\|X(s)\|_3^2\,\mathrm{d}s \\ &\le C(\|X_0\|_3, \epsilon^{-1}K_T, T).\end{aligned} \tag{5.21}$$

Now we bound $|Y(t)|_2$. By Theorem 4.3 we have

$$|Y(t)|_2 \le \|Y(t)\|_3 \le C(\|X_0\|_3, \epsilon^{-1}K_T, T). \tag{5.22}$$

Using (5.21) and (5.22) in (5.20) gives

$$\|Y_h(t) - Y(t)\| \le C(\|X_0\|_3, \epsilon^{-1}K_T, T)h^2|\log(h)|. \tag{5.23}$$

Finally we compute $\|e_h(t)\| = \|Z_h(t) - Y_h(t)\|$. By subtraction of (5.16) and (5.17), we obtain

$$\begin{aligned}\|e_h(t)\| &\le \int_0^t \|\mathrm{e}^{-(t-s)A_h^2}A_hP_hP(f(X_h(s)) - f(X(s)))\|\,\mathrm{d}s \\ &\le \int_0^t \|A_h^{\frac{3}{2}}\mathrm{e}^{-(t-s)A_h^2}\|\|A_h^{-\frac{1}{2}}P(f(X_h(s)) - f(X(s)))\|\,\mathrm{d}s,\end{aligned}$$

since the constant eigenmodes cancel (cf. (2.16)). Using (2.18) and (2.9) gives

$$\|e_h(t)\| \le C\int_0^t (t-s)^{-\frac{3}{4}}\big(1 + \|X_h(s)\|_1^2 + \|X(s)\|_1^2\big)\|X_h(s) - X(s)\|\,\mathrm{d}s.$$

By Corollary 3.2 we have

$$\begin{aligned}
\|e_h(t)\| &\le C\int_0^t (t-s)^{-\frac34}\big(1+\epsilon^{-1}K_T\big)\big(\|W_{A_h}(s)-W_A(s)\| \\
&\qquad + \|Y_h(s)-Y(s)\| + \|e_h(s)\|\big)\,\mathrm{d}s \\
&\le C\big(1+\epsilon^{-1}K_T\big)T^{\frac14}\max_{0\le s\le T}\big(\|W_{A_h}(s)-W_A(s)\|+\|Y_h(s)-Y(s)\|\big) \\
&\qquad + C\big(1+\epsilon^{-1}K_T\big)\int_0^t (t-s)^{-\frac34}\|e_h(s)\|\,\mathrm{d}s.
\end{aligned}$$

We apply Gronwall's Lemma 2.3 with $\alpha=1$, $\beta=\frac14$ and

$$\begin{aligned}
A &= C\big(1+\epsilon^{-1}K_T\big)T^{\frac14}\max_{0\le s\le T}\big(\|W_{A_h}(s)-W_A(s)\|+\|Y_h(s)-Y(s)\|\big), \\
B &= C\big(1+\epsilon^{-1}K_T\big),
\end{aligned}$$

to get

$$\|Z_h(t)-Y_h(t)\| = \|e_h(t)\| \le AC(B,T), \quad t\in[0,T]. \tag{5.24}$$

But we already obtained bounds for $\|W_{A_h}(t)-W_A(t)\|$ and $\|Y_h(t)-Y(t)\|$ in (5.19) and (5.23). By inserting these and (5.24) into (5.18) we get the desired result. $\square$

Since we have regularity of order 3 on Ω_ϵ, it would be possible to prove convergence of order 3 for piecewise quadratic finite elements. We do not find this worth the extra effort since our main result does not show a rate of convergence anyway.

We finally show that X_h converges strongly to X.

THEOREM 5.4. *Assume that $\|A^{\frac{\gamma}{2}}Q^{\frac12}\|_{\mathrm{HS}}<\infty$ for some $\gamma>1$ and that X_0 is $\mathcal{F}_0$-measurable with values in H^3 satisfying $\|X_0\|^2_{L_2(\Omega,H^1)}+\|X_0\|^4_{L_4(\Omega,L_4)}\le\rho$ for some $\rho\ge0$. If X is the weak solution of* (3.3) *and X_h is the solution of* (3.6)*, then*

$$\Big(\mathbf{E}\Big[\sup_{t\in[0,T]}\|X_h(t)-X(t)\|^2\Big]\Big)^{\frac12}\to 0 \quad \text{as } h\to 0.$$

Proof. From Corollary 3.2 it follows that

$$\mathbf{E}\Big[\sup_{t\in[0,T]}\|X(t)\|^4_{L_4}\Big]\le K_T,\quad \mathbf{E}\Big[\sup_{t\in[0,T]}\|X_h(t)\|^4_{L_4}\Big]\le K_T,\quad t\in[0,T], \tag{5.25}$$

with K_T as in Corollary 3.2. Let $\epsilon\in(0,1)$ and let Ω_ϵ be as in Corollary 3.2 such that also $\|X_0\|_3\le C$ on Ω_ϵ. Then

$$\begin{aligned}
\mathbf{E}\Big[\sup_{t\in[0,T]}\|X_h(t)-X(t)\|^2\Big] &\le \int_{\Omega_\epsilon}\sup_{t\in[0,T]}\|X_h(t)-X(t)\|^2\,\mathrm{d}\mathbf{P} \\
&\quad + 2\int_{\Omega_\epsilon^c}\Big(\sup_{t\in[0,T]}\|X_h(t)\|^2+\sup_{t\in[0,T]}\|X(t)\|^2\Big)\,\mathrm{d}\mathbf{P}.
\end{aligned}$$

Here, by Hölder's inequality and (5.25), we have

$$\begin{aligned}
\int_{\Omega_\epsilon^c}\sup_{t\in[0,T]}\|X(t)\|^2\,\mathrm{d}\mathbf{P} &\le \Big(\int_{\Omega_\epsilon^c}1^2\,\mathrm{d}\mathbf{P}\Big)^{\frac12}\Big(\int_{\Omega_\epsilon^c}\sup_{t\in[0,T]}\|X(t)\|^4_{L_4}\,\mathrm{d}\mathbf{P}\Big)^{\frac12} \\
&\le \epsilon^{\frac12}\Big(\mathbf{E}\Big[\sup_{t\in[0,T]}\|X(t)\|^4_{L_4}\Big]\Big)^{\frac12}\le\epsilon^{\frac12}K_T^{\frac12}
\end{aligned}$$

and similarly for X_h. Therefore, by Theorem 5.3,

$$\Big(\mathbf{E}\Big[\sup_{t\in[0,T]}\|X_h(t)-X(t)\|^2\Big]\Big)^{\frac{1}{2}} \le C(\epsilon^{-1}K_T,T)h^2|\log(h)| + CK_T^{\frac{1}{4}}\epsilon^{\frac{1}{4}}.$$

Since $\frac{\epsilon^{\frac{1}{4}}}{C(\epsilon^{-1}K_T,T)} \to 0$ monotonically as $\epsilon \to 0$, we may choose ϵ depending on h, such that the two terms are equal. □

Since $C(\epsilon^{-1}K_T,T)$ grows rapidly with ϵ^{-1}, is not possible to obtain a rate of convergence from this proof.

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