ERRATUM TO "FINITE ELEMENT APPROXIMATION OF THE CAHN-HILLIARD-COOK EQUATION"

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Abstract. We prove an additional result on the linearized Cahn-Hilliard-Cook equation to fill in a gap in the main argument in our paper which was published in SIAM J. Numer. Anal. **49** (2011), 2407–2429. The result is a pathwise error estimate, which is proved by an application of the factorization argument for stochastic convolutions.

1. Introduction

The proof of [3, Theorem 5.3] is incomplete and in the present note we provide an additional convergence result to fill in the gap. In order to do so one has to replace [3, Theorem 2.2], which is quoted from [4], by Theorem 2.1 below. Theorem 2.1 provides optimal order of convergence for the linearized Cahn-Hilliard-Cook equation in a stronger topology than the one in [3, Theorem 2.2] in exchange for a slight additional regularity requirement on the covariance operator Q. In particular, it implies pathwise convergence with essentially optimal rate for the linearized equation, which is an important ingredient in the proof of the main result in [3].

The note is organized as follows. In Section 2 we state and prove the result which is missing from [3] and in Section 3 we outline what additional small changes one has to make in the arguments of [3] as a consequence.

2. The convergence result

For the explanation of notation we refer to [3].

Theorem 2.1. *Let $\epsilon \in (0, \frac{1}{2}]$, $\beta \in [1,2]$, and $p > \frac{2}{\epsilon}$. Then there is $C = C(p, \epsilon, T)$ such that*

$$\left(\mathbf{E}\Big(\sup_{t\in[0,T]} \|W_A(t) - W_{A_h}(t)\|^p \Big) \right)^{1/p} \le Ch^\beta \|A^{(\beta-2)/2+\epsilon} Q^{1/2}\|_{\mathrm{HS}}.$$

Proof. Let ϵ, β, p be as stated and select $\alpha \in (\frac{1}{p}, \frac{\epsilon}{2})$. We denote $E(t) = \mathrm{e}^{-tA^2}$, $E_h(t) = \mathrm{e}^{-tA_h^2}$, and let $F_h(t) = E(t) - E_h(t)P_h$ be the deterministic error operator. From [1, Lemma 5.2] and a standard interpolation argument we obtain error estimates with smooth and non-smooth data:

$$\|F_h(t)v\| \le Ch^\beta \|A^{\beta/2} v\|, \quad t \ge 0, \tag{1}$$

$$\|F_h(t)v\| \le Ch^\beta t^{-(\beta-\gamma)/4} \|A^{\gamma/2} v\|, \quad t > 0,\ \gamma \in [-1,1]. \tag{2}$$

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Following the factorization method [2, Chapter 5], we write

$$\begin{aligned} W_A(t) &= c_\alpha \int_0^t E(t-\sigma) \int_\sigma^t (t-s)^{-1+\alpha}(s-\sigma)^{-\alpha}\,\mathrm{d}s\,\mathrm{d}W(\sigma) \\ &= c_\alpha \int_0^t (t-s)^{-1+\alpha} E(t-s) \int_0^s (s-\sigma)^{-\alpha} E(s-\sigma)\,\mathrm{d}W(\sigma)\,\mathrm{d}s \\ &= c_\alpha \int_0^t (t-s)^{-1+\alpha} E(t-s) Y(s)\,\mathrm{d}s \end{aligned}$$

and, similarly,

$$\begin{aligned} W_{A_h}(t) &= c_\alpha \int_0^t (t-s)^{-1+\alpha} E_h(t-s) \int_0^s (s-\sigma)^{-\alpha} E_h(s-\sigma)\,\mathrm{d}W(\sigma)\,\mathrm{d}s \\ &= c_\alpha \int_0^t (t-s)^{-1+\alpha} E_h(t-s) Y_h(s)\,\mathrm{d}s. \end{aligned}$$

Therefore,

$$\begin{aligned} W_A(t) - W_{A_h}(t) &= c_\alpha \int_0^t (t-s)^{-1+\alpha} F_h(t-s) Y(s)\,\mathrm{d}s \\ &\quad + c_\alpha \int_0^t (t-s)^{-1+\alpha} E_h(t-s)[Y(s)-Y_h(s)]\,\mathrm{d}s =: I_1(t) + I_2(t). \end{aligned}$$

First, by Hölder's inequality and (1),

$$\begin{aligned} &\mathbf{E}\left(\sup_{t\in[0,T]} \|I_1(t)\|^p\right) \\ &\quad \le c_\alpha \left(\int_0^T \left(s^{-1+\alpha}\|F_h(s)A^{-\beta/2}\|\right)^{\frac{p}{p-1}}\,\mathrm{d}s\right)^{p-1} \int_0^T \mathbf{E}\left(\|A^{\beta/2}Y(s)\|^p\right)\mathrm{d}s \\ &\quad \le C_\alpha h^{\beta p} \left(\int_0^T s^{\frac{p}{p-1}(-1+\alpha)}\,\mathrm{d}s\right)^{p-1} \int_0^T \mathbf{E}\left(\|A^{\beta/2}Y(s)\|^p\right)\mathrm{d}s. \end{aligned}$$

The first integral is finite because $p > \frac{1}{\alpha}$. To bound the second integral, first notice that $AY(s)$ is a Gaussian random variable for all $s \in [0,T]$ and hence, by [2, Corollary 2.17],

$$\begin{aligned} \mathbf{E}\left(\|A^{\beta/2}Y(s)\|^p\right) &= \mathbf{E}\left(\Big\| \int_0^s (s-\sigma)^{-\alpha} A^{\beta/2} E(s-\sigma)\,\mathrm{d}W(\sigma) \Big\|^p\right) \\ &\le C\Big(\int_0^s \big\|(s-\sigma)^{-\alpha} A^{\beta/2} E(s-\sigma) Q^{\frac12}\big\|_{\mathrm{HS}}^2\,\mathrm{d}\sigma\Big)^{\frac{p}{2}} \\ &= C\Big(\int_0^T s^{-2\alpha} \big\|A^{1-2\alpha} E(s) A^{(\beta-2)/2+2\alpha} Q^{\frac12}\big\|_{\mathrm{HS}}^2\,\mathrm{d}s\Big)^{\frac{p}{2}} \\ &\le C K_\alpha^p \|A^{(\beta-2)/2+2\alpha} Q^{\frac12}\|_{\mathrm{HS}}^p, \end{aligned}$$

where we used that

$$\int_0^T s^{-2\alpha} \big\|A^{1-2\alpha} E(s) v\|^2\,\mathrm{d}s \le K_\alpha^2 \|v\|^2, \quad \text{for } \alpha \in [0, \tfrac12).$$

Therefore, since $2\alpha \leq \epsilon$,

$$\mathbf{E}\left(\sup_{t\in[0,T]} \|I_1(t)\|^p\right) \leq C_{\alpha,p} T h^{\beta p} \|A^{(\beta-2)/2+\epsilon} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^p.$$

To bound I_2, we use Hölder's inequality and $\|E_h(s)\| \leq 1$ to get

$$\begin{aligned}
&\mathbf{E}\left(\sup_{t\in[0,T]} \|I_2(t)\|^p\right) \\
&\leq c_\alpha \left(\int_0^T \left(s^{-1+\alpha}\|E_h(s)\|\right)^{\frac{p}{p-1}} \,\mathrm{d}s\right)^{p-1} \int_0^T \mathbf{E}\left(\|Y(s)-Y_h(s)\|^p\right) \mathrm{d}s \\
&\leq c_\alpha \left(\int_0^T s^{\frac{p}{p-1}(-1+\alpha)} \,\mathrm{d}s\right)^{p-1} \int_0^T \mathbf{E}\left(\|Y(s)-Y_h(s)\|^p\right) \mathrm{d}s.
\end{aligned}$$

Again the first integral is finite because $p > \frac{1}{\alpha}$. To bound the second integral, notice that $Y(s) - Y_h(s) = \int_0^s (s-\sigma)^{-\alpha} F_h(s-\sigma)\,\mathrm{d}W(\sigma)$ and hence it is Gaussian for all $s \in [0,T]$. Therefore, using [2, Corollary 2.17] again, together with (2) with $\gamma = -(2-\beta-2\epsilon) \in [-1,1]$, we get

$$\begin{aligned}
\mathbf{E}\left(\|Y(s)-Y_h(s)\|^p\right) &\leq C\left(\int_0^T s^{-2\alpha}\|F_h(s)Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \,\mathrm{d}s\right)^{\frac{p}{2}} \\
&= C\left(\int_0^T s^{-2\alpha}\|F_h(s)A^{(2-\beta-2\epsilon)/2}A^{(\beta-2)/2+\epsilon}Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \,\mathrm{d}s\right)^{\frac{p}{2}} \\
&\leq Ch^{\beta p}\left(\int_0^T s^{-2\alpha}s^{-1+\epsilon} \,\mathrm{d}s\right)^{\frac{p}{2}} \|A^{(\beta-2)/2+\epsilon}Q^{\frac{1}{2}}\|_{\mathrm{HS}}^p \\
&\leq C_{p,\alpha,\epsilon} h^{\beta p}\|A^{(\beta-2)/2+\epsilon}Q^{\frac{1}{2}}\|_{\mathrm{HS}}^p,
\end{aligned}$$

because $\epsilon > 2\alpha$. Thus,

$$\mathbf{E}\left(\sup_{t\in[0,T]} \|I_2(t)\|^p\right) \leq C_{\alpha,p,\epsilon} T h^{\beta p} \|A^{(\beta-1)/2+\epsilon} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^p$$

and the proof is complete. □

3. The necessary changes

The main gap in [3] occurs when deriving the last inequality on page 2426 using [3, Theorem 2.2]. Indeed, one could then only conclude the existence of a set $\Omega_\epsilon = \Omega_{\epsilon,h,t}$ such that the inequality holds. The dependence on t of the set then compromises the rest of the proof of [3, Theorem 5.3] and hence also the proof of [3, Theorem 5.4]. This can be avoided by using Theorem 2.1 instead. The dependence on h does not cause a problem but it should appear explicitly.

First, [3, Corollary 3.2] has to be modified as follows.

Corollary 3.1. *Assume that $\|A^{\gamma/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\gamma > 1$ and that X_0 is $\mathcal{F}_0$-measurable with values in H^1 satisfying*

$$\|X_0\|_{L_2(\Omega,H^1)}^2 + \|X_0\|_{L_4(\Omega,L_4)}^4 \leq \rho$$

for some $\rho \geq 0$*. If* X *is a weak solution of* (3.3) *and* X_h *is the solution of* (3.6)*, then*

$$\mathbf{E}\Big[\sup_{t\in[0,T]}\big(\|\nabla X(t)\|^2 + \|X(t)\|_{L_4}^4\big)\Big] \leq K_T,$$
$$\mathbf{E}\Big[\sup_{t\in[0,T]}\big(\|\nabla X_h(t)\|^2 + \|X_h(t)\|_{L_4}^4\big)\Big] \leq K_T,$$

where K_T *depends on* ρ, K_Q, T*. Moreover, for every* $\epsilon \in (0,1)$ *and* $h > 0$*, there is* $\Omega_{\epsilon,h} \subset \Omega$ *with* $\mathbf{P}(\Omega_{\epsilon,h}) \geq 1-\epsilon$ *and*

$$\begin{aligned}
\|\nabla X(t)\|^2 + \|X(t)\|_{L_4}^4 &\leq \epsilon^{-1}K_T \quad \text{on } \Omega_{\epsilon,h},\ t\in[0,T],\\
\|\nabla X_h(t)\|^2 + \|X_h(t)\|_{L_4}^4 &\leq \epsilon^{-1}K_T \quad \text{on } \Omega_{\epsilon,h},\ t\in[0,T],\\
\|X(t)\|_1^2 + \|X_h(t)\|_1^2 &\leq \epsilon^{-1}K_T \quad \text{on } \Omega_{\epsilon,h},\ t\in[0,T],\\
\|W_A(t)\|_3^2 &\leq \epsilon^{-1}K_T \quad \text{on } \Omega_{\epsilon,h},\ t\in[0,T],
\end{aligned}$$

$$\|W_A(t) - W_{A_h}(t)\| \leq \epsilon^{-1}K_T h^2 \quad \text{on } \Omega_{\epsilon,h},\ t\in[0,T]. \tag{3}$$

The novelty in Corollary 3.1 compared to [3, Corollary 3.2] is the explicit dependence on h in $\Omega_{\epsilon,h}$ instead of Ω_ϵ and the additional inequality (3). The latter is a consequence of Theorem 2.1 with $\beta = 2$, proved by using Chebychev's inequality and noting that $\|A^{\gamma/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\gamma > 1$ implies that $\|A^{\epsilon}Q^{1/2}\|_{\mathrm{HS}} < \infty$ for all $0 < \epsilon \leq \frac{1}{2}$.

Next, in [3, Theorem 5.3] and in its proof, the set Ω_ϵ has to be replaced by $\Omega_{\epsilon,h}$. Furthermore, the proof of the last inequality on page 2426, where the main gap appears, is now included in the new Corollary 3.1. Finally, in the proof of [3, Theorem 5.4], the set Ω_ϵ has to be replaced by $\Omega_{\epsilon,h}$.

References

[1] C. M. Elliott and S. Larsson, Error estimates with smooth and nonsmooth data for a finite element method for the Cahn-Hilliard equation, Math. Comp. **58** (1992), 603–630, S33–S36.

[2] G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions, xviii+454 pp. Cambridge University Press, Cambridge (1992).

[3] M. Kovács, S. Larsson, and A. Mesforoush, Finite element approximation of the Cahn-Hilliard-Cook equation, SIAM J. Numer. Anal. **49** (2011), 2407–2429.

[4] S. Larsson and A. Mesforush, Finite element approximation of the linearized Cahn-Hilliard-Cook equation, IMA J. Numer. Anal. **31** (2011), 1315–1333.

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