ERRATUM TO “FINITE ELEMENT APPROXIMATION OF THE CAHN-HILLIARD-COOK EQUATION”

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Abstract. We prove an additional result on the linearized Cahn-Hilliard-Cook equation to fill in a gap in the main argument in our paper which was published in SIAM J. Numer. Anal. 49 (2011), 2407–2429. The result is a pathwise error estimate, which is proved by an application of the factorization argument for stochastic convolutions.

1. Introduction

The proof of [3, Theorem 5.3] is incomplete and in the present note we provide an additional convergence result to fill in the gap. In order to do so one has to replace [3, Theorem 2.2], which is quoted from [4], by Theorem 2.1 below. Theorem 2.1 provides optimal order of convergence for the linearized Cahn-Hilliard-Cook equation in a stronger topology than the one in [3, Theorem 2.2] in exchange for a slight additional regularity requirement on the covariance operator $Q$. In particular, it implies pathwise convergence with essentially optimal rate for the linearized equation, which is an important ingredient in the proof of the main result in [3].

The note is organized as follows. In Section 2 we state and prove the result which is missing from [3] and in Section 3 we outline what additional small changes one has to make in the arguments of [3] as a consequence.

2. The convergence result

For the explanation of notation we refer to [3].

Theorem 2.1. Let $\epsilon \in (0, \frac{1}{2}]$, $\beta \in [1, 2]$, and $p > \frac{2}{\epsilon}$. Then there is $C = C(p, \epsilon, T)$ such that

$$\left( \mathbb{E} \left( \sup_{t \in [0, T]} \| W_A(t) - W_{A_h}(t) \|^p \right) \right)^{1/p} \leq C h^{\beta} \| A^{(\beta - 2)/4 + \epsilon} Q^{1/2} \|_{\text{HS}}.$$  

Proof. Let $\epsilon$, $\beta$, $p$ be as stated and select $\alpha \in \left( \frac{1}{p}, \frac{\epsilon}{2} \right)$. We denote $E(t) = e^{-tA^2}$, $E_h(t) = e^{-tA_h^2}$, and let $F_h(t) = E(t) - E_h(t)P_h$ be the deterministic error operator. From [1] Lemma 5.2 and a standard interpolation argument we obtain error estimates with smooth and non-smooth data:

1. $\| F_h(t)v \| \leq C h^{\beta} \| A^{\beta/2} v \|$, $t \geq 0$,
2. $\| F_h(t)v \| \leq C h^{\beta} t^{-(\beta - \gamma)/4} \| A^{\gamma/2} v \|$, $t > 0$, $\gamma \in [-1, 1]$.

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Following the factorization method [2, Chapter 5], we write
\[ W_A(t) = c_\alpha \int_0^t E(t-s) \int_{\sigma}^{t-s)(s-\sigma)^{-\alpha} \, ds \, dW(\sigma) \]
\[ = c_\alpha \int_0^t (t-s)^{-1+\alpha} E(t-s) \int_0^s (s-\sigma)^{-\alpha} E(s-\sigma) \, dW(\sigma) \, ds \]
\[ = c_\alpha \int_0^t (t-s)^{-1+\alpha} E(t-s) Y(s) \, ds \]
and, similarly,
\[ W_{Ah}(t) = c_\alpha \int_0^t (t-s)^{-1+\alpha} E_{Ah}(t-s) \int_0^s (s-\sigma)^{-\alpha} E_{Ah}(s-\sigma) \, dW(\sigma) \, ds \]
\[ = c_\alpha \int_0^t (t-s)^{-1+\alpha} E_{Ah}(t-s) Y_h(s) \, ds . \]

Therefore,
\[ W_A(t) - W_{Ah}(t) = c_\alpha \int_0^t (t-s)^{-1+\alpha} F_h(t-s) Y(s) \, ds \]
\[ + c_\alpha \int_0^t (t-s)^{-1+\alpha} E_{Ah}(t-s) [Y(s) - Y_h(s)] \, ds =: I_1(t) + I_2(t). \]

First, by Hölder’s inequality and \([1]\),
\[
\mathbf{E} \left( \sup_{t \in [0,T]} \| I_1(t) \|^p \right) \\
\leq c_\alpha \left( \int_0^T \left( s^{-1+\alpha} \| F_h(s) A^{-\beta/2} \| \right)^{\frac{p}{p+1}} \, ds \right)^{1+\alpha} \int_0^T \mathbf{E} \left( \| A^{\beta/2} Y(s) \|^p \right) \, ds \\
\leq C_\alpha h^{\beta p} \left( \int_0^T s^{-\alpha} (1+\alpha) \, ds \right)^{p-1} \int_0^T \mathbf{E} \left( \| A^{\beta/2} Y(s) \|^p \right) \, ds .
\]

The first integral is finite because \( p > \frac{1}{\alpha} \). To bound the second integral, first notice that \( AY(s) \) is a Gaussian random variable for all \( s \in [0,T] \) and hence, by [2, Corollary 2.17],
\[
\mathbf{E} \left( \| A^{\beta/2} Y(s) \|^p \right) = \mathbf{E} \left( \left\| \int_0^s (s-\sigma)^{-\alpha} A^{\beta/2} E(s-\sigma) \, dW(\sigma) \right\|^p \right) \\
\leq C \left( \int_0^T s^{-\alpha} \left\| A^{\beta/2} E(s-\sigma) Q_{\text{HS}}^2 \right\| ds \right)^{p/2} \\
= C \left( \int_0^T s^{-2\alpha} \left\| A^{(\beta-2)/2+2\alpha} Q_{\text{HS}}^2 \right\| ds \right)^{p} \\
\leq C K_\alpha^p \left\| A^{(\beta-2)/2+2\alpha} Q_{\text{HS}}^2 \right\| ,
\]
where we used that
\[
\int_0^T s^{-2\alpha} \left\| A^{1-2\alpha} E(s) v \right\|^2 \, ds \leq K_\alpha^2 \| v \|^2 , \quad \text{for } \alpha \in \left[ 0, \frac{1}{2} \right).
\]
Therefore, since $2\alpha \leq \epsilon$,
\[
E \left( \sup_{t \in [0,T]} \|I_1(t)\|^p \right) \leq C_{\alpha,p} T h^{\beta p} \|A^{(\beta-2)/2+\epsilon} Q^\frac{1}{2}\|_{H^S}^p.
\]

To bound $I_2$, we use Hölder’s inequality and $\|E_h(s)\| \leq 1$ to get
\[
E \left( \sup_{t \in [0,T]} \|I_2(t)\|^p \right) \\
\leq c_\alpha \left( \int_0^T (s^{-1+\alpha}\|E_h(s)\|)^\frac{p}{2} \, ds \right)^{p-1} \int_0^T E(\|Y(s) - Y_h(s)\|^p) \, ds \\
\leq c_\alpha \left( \int_0^T s^{-\frac{p}{2}(1-\alpha)} \, ds \right)^{p-1} \int_0^T E(\|Y(s) - Y_h(s)\|^p) \, ds.
\]

Again the first integral is finite because $p > \frac{1}{\alpha}$. To bound the second integral, notice that $Y(s) - Y_h(s) = \int_0^s (s - \sigma)^{\alpha} F_h(s - \sigma) \, dW(\sigma)$ and hence it is Gaussian for all $s \in [0,T]$. Therefore, using [2 Corollary 2.17] again, together with [2] with $\gamma = -(2 - \beta - 2\epsilon) \in [-1,1]$, we get
\[
E(\|Y(s) - Y_h(s)\|^p) \leq C \left( \int_0^T s^{-2\alpha}\|F_h(s)\|_{H^S}^2 \, ds \right)^{\frac{p}{2}} \\
= C \left( \int_0^T \|F_h(s)A^{(2-\beta-2\epsilon)/2}A^{(\beta-2)/2+\epsilon}Q^\frac{1}{2}\|_{H^S}^2 \, ds \right)^{\frac{p}{2}} \\
\leq C h^{\beta p} \left( \int_0^T s^{-2\alpha} s^{-1+\epsilon} \, ds \right)^{\frac{p}{2}} \|A^{(\beta-2)/2+\epsilon} Q^\frac{1}{2}\|_{H^S}^p \\
\leq C_{p,\alpha,\epsilon} h^{\beta p} \|A^{(\beta-2)/2+\epsilon} Q^\frac{1}{2}\|_{H^S}^p,
\]
because $\epsilon > 2\alpha$. Thus,
\[
E \left( \sup_{t \in [0,T]} \|I_2(t)\|^p \right) \leq C_{\alpha,p,\epsilon} T h^{\beta p} \|A^{(\beta-1)/2+\epsilon} Q^\frac{1}{2}\|_{H^S}^p
\]
and the proof is complete. \qed

3. THE NECESSARY CHANGES

The main gap in [3] occurs when deriving the last inequality on page 2426 using [3 Theorem 2.2]. Indeed, one could then only conclude the existence of a set $\Omega_t = \Omega_{\epsilon,h,t}$ such that the inequality holds. The dependence on $t$ of the set then compromises the rest of the proof of [3 Theorem 5.3] and hence also the proof of [3 Theorem 5.4]. This can be avoided by using Theorem [2,1] instead. The dependence on $h$ does not cause a problem but it should appear explicitly.

First, [3 Corollary 3.2] has to be modified as follows.

**Corollary 3.1.** Assume that $\|A^{\gamma/2} Q^{1/2}\|_{H^S} < \infty$ for some $\gamma > 1$ and that $X_0$ is $F_0$-measurable with values in $H^1$ satisfying
\[
\|X_0\|_{L^2(\Omega, H^1)}^2 + \|X_0\|_{L^4(\Omega, L^4)}^4 \leq \rho
\]
for some $\rho \geq 0$. If $X$ is a weak solution of (3.3) and $X_h$ is the solution of (3.6), then
\[
E \left[ \sup_{t \in [0,T]} \left( \| \nabla X(t) \|^2 + \| X(t) \|^4_{L^4} \right) \right] \leq K_T,
\]
\[
E \left[ \sup_{t \in [0,T]} \left( \| \nabla X_h(t) \|^2 + \| X_h(t) \|^4_{L^4} \right) \right] \leq K_T,
\]
where $K_T$ depends on $\rho, K_Q, T$. Moreover, for every $\epsilon \in (0,1)$ and $h > 0$, there is
\[
\Omega_{\epsilon,h} \subset \Omega \text{ with } P(\Omega_{\epsilon,h}) \geq 1 - \epsilon \text{ and }
\]
\[
\| \nabla X(t) \|^2 + \| X(t) \|^4_{L^4} \leq \epsilon^{-1} K_T \text{ on } \Omega_{\epsilon,h}, t \in [0,T],
\]
\[
\| \nabla X_h(t) \|^2 + \| X_h(t) \|^4_{L^4} \leq \epsilon^{-1} K_T \text{ on } \Omega_{\epsilon,h}, t \in [0,T],
\]
\[
\| X(t) \|^2 + \| X_h(t) \|^2 \leq \epsilon^{-1} K_T \text{ on } \Omega_{\epsilon,h}, t \in [0,T],
\]
\[
\| W_A(t) \|^2 \leq \epsilon^{-1} K_T \text{ on } \Omega_{\epsilon,h}, t \in [0,T],
\]
(3) \[
\| W_A(t) - W_A(t) \| \leq \epsilon^{-1} K_T h^2 \text{ on } \Omega_{\epsilon,h}, t \in [0,T].
\]

The novelty in Corollary 3.1 compared to [3, Corollary 3.2] is the explicit dependence on $h$ in $\Omega_{\epsilon,h}$ instead of $\Omega_{\epsilon}$ and the additional inequality (3). The latter is a consequence of Theorem 2.1 with $\beta = 2$, proved by using Chebychev’s inequality and noting that $\| A^{\gamma/2} Q^{1/2} \|_{\text{HS}} < \infty$ for some $\gamma > 1$ implies that $\| A^\epsilon Q^{1/2} \|_{\text{HS}} < \infty$ for all $0 < \epsilon \leq 1/4$.

Next, in [3, Theorem 5.3] and in its proof, the set $\Omega_{\epsilon}$ has to be replaced by $\Omega_{\epsilon,h}$. Furthermore, the proof of the last inequality on page 2426, where the main gap appears, is now included in the new Corollary 3.1. Finally, in the proof of [3, Theorem 5.4], the set $\Omega_{\epsilon}$ has to be replaced by $\Omega_{\epsilon,h}$.

REFERENCES


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