Partial Differential Equations with Numerical Methods

Stig Larsson and Vidar Thomée, Springer 2003, 2005

List of corrections, October 10, 2006. Page numbers refer to the second corrected printing 2005.

p. 94, l. Problem 6.6: Problem A.15 **should be** Problem A.14

p. 83, l. (6.16): $\Big\|v-\sum_{j=1}^{N}(v,\varphi_j)\varphi_j\Big\| \le C\lambda_{N+1}^{-1/2}$ **should be** $\Big\|v-\sum_{j=1}^{N}(v,\varphi_j)\varphi_j\Big\| \le \lambda_{N+1}^{-1/2}\|\nabla v\|$

p. 40, l. 1-: $\int_\Omega f\,\mathrm{d}x$ **should be** $\frac{1}{|\Omega|}\int_\Omega f\,\mathrm{d}x$

p. 31, l. 2: $L_1(\mathbf{R}^d)$ **should be** $L_1(B)$ in view of (3.14)

p. 31, l. 5-: $\int_{|x|=\epsilon}\varphi\frac{\partial U}{\partial n}\,\mathrm{d}s$ **should be** $-\int_{|x|=\epsilon}\varphi\frac{\partial U}{\partial n}\,\mathrm{d}s$

List of corrections, February 13, 2006. Page numbers refer to the second corrected printing 2005.

p. 88, l. 4: $N_\rho \approx \rho^2 b^2/\pi$ **should be** $N_\rho \approx \rho^2 b^2/(4\pi)$

p. 88, l. 5: $\lambda_n = \lambda_{ml} \approx \rho^2 \approx \pi N_\rho/b^2 \approx \pi n/b^2$ **should be** $\lambda_n = \lambda_{ml} \approx \rho^2 \approx 4\pi N_\rho/b^2 \approx 4\pi n/b^2$

p. 158, l. 4: $n \ge 1$ **should be** $n \ge 0$

p. 236, l. 2: if it **should be** if it is

List of corrections, August 24, 2005.

Most of the following errors have been corrected in the second corrected printing 2005.

p. 3, l. 12-: $\to\infty$ **should be** $t\to\infty$

p. 6, l. 1-: $\Big(\int_\Omega vw\,\mathrm{d}x\Big)^{1/2}$ **should be** $\int_\Omega vw\,\mathrm{d}x$

p. 7, l. 15: we we **should be** we

p. 9, l. 1-: definition of b should be $b = \frac{v_{\mathrm{f}}\sigma_{\mathrm{f}}L}{\lambda_{\mathrm{f}}}\frac{\sigma}{\sigma_{\mathrm{f}}}\frac{v}{v_{\mathrm{f}}}$

p. 10, l. 1.21: $b-\nabla\cdot a$ **should be** $b-\nabla a$

p. 16, l. 2-: for ϵ **should be** for $\epsilon > 0$

p. 23, l. Problem 2.2: where c is a positive constant

p. 27, l. 1: $\le$ **should be** $=$ (in two places)

p. 27, l. 3: $\min_{\bar\Omega} u \le \min\big\{\min_\Gamma u, 0\big\}$ **should be** $\min_{\bar\Omega} u \ge \min\big\{\min_\Gamma u, 0\big\}$

p. 30, l. 5: by parts **should be** by parts twice

p. 30, l. 6: . **should be** ,

p. 31, l. 3-: $\Big|\int_{|x|=\epsilon}\frac{\partial\varphi}{\partial n}U\,\mathrm{d}s\Big| = \Big|\frac{1}{2\pi}\log(\epsilon)\int_{|x|=\epsilon}\frac{\partial\varphi}{\partial n}\,\mathrm{d}s\Big| \le \epsilon|\log(\epsilon)|\|\nabla\varphi\|_{\mathcal{C}} \to 0$

p. 33, l. 14-: formulation (3.23) **should be** formulation (3.20)

p. 35, l. 5-: $\frac{\partial u}{\partial n}$ **should be** $a\frac{\partial u}{\partial n}$

p. 38, l. 14: $m,k=1$ **should be** $j,k=1$

p. 39, l. 11-: Hint: $v(x) = v(y) + \int_{y_1}^{x_1} D_1 v(s,x_2)\,\mathrm{d}s + \int_{y_2}^{x_2} D_2 v(y_1,s)\,\mathrm{d}s$.

p. 44, l. 13: of the **should be** of the absolute values of the

p. 44, l. 12-: $\min_j U_j \le \min\{U_0, U_M, 0\}$ **should be** $\min_j U_j \ge \min\{U_0, U_M, 0\}$

p. 45, l. 2-: **delete** $+b_j(u'(x_j) - \hat{\partial} u(x_j))$

p. 46, l. 12-: inter **should be** interior

p. 49, l. 17: dominant **should be** dominant, i.e., $\sum_{j\neq i} |a_{ij}| \le a_{ii}$

p. 49, l. 17: Hint: assume $a_j \pm \frac{1}{2} h b_j \ge 0$.

p. 54, l. 5: with $\|v\|_{K_j} = \|v\|_{L_2(K_j)}$ and $|v|_{2,K_j} = |v|_{H^2(K_j)}$

p. 54, l. 10: $)^{1/2}$ **should be** $\Big)^{1/2}$

p. 55, l. 9: v **should be** u

p. 56, l. 21: $\le s$ **should be** $\le k$

p. 61, l. 11-: . **should be** ,

p. 65, l. 12: We then find **should be** We then find, for $2 \le s \le r$,

p. 65, l. 13: r **should be** s

p. 65, l. 14: These ... **should be** These estimates thus show a reduced convergence rate $O(h^s)$ if $v \in H^s$ with $s < r$.

p. 73, l. 20-: $\|I_h v - v\|_{\mathcal{C}(K_j)}$ **should be** $\|I_h v - v\|_{\mathcal{C}(K_j)} = \|I_h(v - Q_1 v) + (Q_1 v - v)\|_{\mathcal{C}(K_j)}$

p. 81, l. 11: dimension n **should be** dimension m

p. 87, l. Example 6.2: $\int_0^1$ **should be** $\int_0^b$

p. 88, l. 1: a_0 **should be** $a_0 > 0$

p. 88, l. 9: $a_{j+1/2} U_{j+1} + (a_{j+1/2} + a_{j-1/2}) U_j - a_{j-1/2} U_{j-1}$
should be $a_{j+1/2} U_{j+1} - (a_{j+1/2} + a_{j-1/2}) U_j + a_{j-1/2} U_{j-1}$

p. 93, l. Problem 6.3: Assume that Ω is such that (3.36) holds.

p. 96, l. 3: he **should be** the

p. 97, l. 7-: $g = P^{-1} u$ **should be** $g = P^{-1} f$

p. 112, l. 11: Bu **should be** By

p. 112, l. 18: has **should be** have

p. 115, l. 11: $\hat{v}_j^k \mathrm{e}^{-\lambda_j t}$ **should be** $\hat{v}_i \mathrm{e}^{-\lambda_i t}$

p. 115, l. 3-: C_1 **should be** $\frac{1}{2} C_1$

p. 117, l. 8: t^{-k} **should be** $t^{-m-s/2}$

p. 117, l. 3-: $D_t^m E(t) v(\cdot, t)$ **should be** $D_t^m E(t) v$

p. 117, l. 15: (6.4) **should be** Theorem 6.4

p. 119, l. 4: $D_t e$ **should be** $D_t E$

p. 119, l. (8.27): $=$ **should be** $\le$

p. 123, l. 3: $(\bar{x}, \bar{t})$ **should be** $(\tilde{x}, \tilde{t})$

p. 124, l. 15: $|u(x,t)| \le \mathrm{e}^{c|x|^2}$ **should be** $|u(x,t)| \le M \mathrm{e}^{c|x|^2}$

p. 133, l. 3: $\sum_p a_p \mathrm{e}^{\mathrm{i}(j-p)\xi_0}$ **should be** $\epsilon \sum_p a_p \mathrm{e}^{\mathrm{i}(j-p)\xi_0}$

p. 150, l. 1-: **should be** Since $u_h(t) \in S_h$ we may choose $\chi = u_h(t)$ ...

p. 150, l. 1-: $U^n \in S_h$ **should be** $u_h \in S_h$

p. 150, l. 1-: $\chi = u$ **should be** $\chi = u_h$

p. 154, l. 7: 10.1 **should be** 10.3

p. 155, l. 1: $\Big(\int_0^t \|\rho_t\|_2 \, \mathrm{d}s\Big)^{1/2}$ **should be** $\Big(\int_0^t \|\rho_t\|^2 \, \mathrm{d}s\Big)^{1/2}$

p. 155, l. 9-: v **should be** w (four times)

p. 155, l. 3-: v **should be** w

p. 156, l. 12: Φ **should be** Φ_j

p. 158, l. 4-: method **should be** a method

p. 160, l. 2-: and (8.18). **should be** (8.18), and Problem 8.10.

p. 165, l. 9: **delete** which we may assume to be symmetric,

p. 169, l. 1: 11.2 **should be** 11.3

p. 169, l. 10-: bounded **should be** bounded or unbounded

p. 179, l. 16: $+\|f\|\|u\|$ **should be** $+2\|f\|\|u\|$ and $C_1 = 1$

p. 204, l. 5: 13.3 **should be** 13.1

p. 226, l. 6: $w = \lambda v$ **should be** $w = \lambda v$ or $v = \lambda w$

p. 227, l. (A.4): w **should be** u

p. 233, l. 14: for $1 \leq p < \infty$. **should be** for $1 \leq p < \infty$, if Γ is sufficiently smooth.

p. 232, l. 3: The latter **should be** If Ω is bounded, then the latter

p. 233, l. 8: $1 \leq p \leq \infty$, and **should be** $1 \leq p \leq \infty$ if Ω is bounded, and

p. 234, l. 9: C^1 **should be** $\mathcal{C}^1$

p. 235, l. 10: for any l. **should be** for any $l \geq k$, if Γ is sufficiently smooth.

p. 237, l. 14-: $\mathcal{C}(\bar{\Omega}) \subset H^k(\Omega)$ **should be** $H^k(\Omega) \subset \mathcal{C}(\bar{\Omega})$

p. 237, l. 4-: $\mathcal{C}^\ell(\bar{\Omega}) \subset H^k(\Omega)$ **should be** $H^k(\Omega) \subset \mathcal{C}^\ell(\bar{\Omega})$

p. 239, l. 4: $L_2(\mathbf{R})$ **should be** $L_2(\mathbf{R}^d)$

p. 240, l. 3: $\mathrm{e}^{-\mathrm{i}x\cdot\xi}$ **should be** $\mathrm{e}^{-\mathrm{i}z\cdot\xi}$

p. 242, l. 5: $\|v\|_{W_1^2} \leq |\Omega|^{1/2}\|v\|_{H^2}$ **should be** $\|v\|_{W_1^2} \leq C\|v\|_{H^2}$

p. 242, l. 12: $\nabla\hat{v}$ **should be** $\hat{\nabla}\hat{v}$

Here is an improved version of Theorem 6.4.

Theorem 1. *The eigenfunctions* $\{\varphi_j\}_{j=1}^\infty$ *of (6.5) form an orthonormal basis for* L_2. *The series* $\sum_{j=1}^\infty \lambda_j (v,\varphi_j)^2$ *is convergent if and only if* $v \in H_0^1$. *Moreover,*

$$\|\nabla v\|^2 = a(v,v) = \sum_{j=1}^\infty \lambda_j (v,\varphi_j)^2, \quad \textit{for all } v \in H_0^1. \tag{1}$$

Proof. By our above discussion it follows that for the first statement it suffices to show (6.13) for all v in H_0^1, which is a dense subspace of L_2. We shall demonstrate that

$$\Big\| v - \sum_{j=1}^N (v,\varphi_j)\varphi_j \Big\| \le \lambda_{N+1}^{-1/2} \|\nabla v\|, \quad \text{for all } v \in H_0^1, \tag{2}$$

which then implies (6.13) in view of Theorem 6.3.

To prove (2), set $v_N = \sum_{j=1}^N (v,\varphi_j)\varphi_j$ and $r_N = v - v_N$. Then $(r_N, \varphi_j) = 0$ for $j = 1, \dots, N$, so that

$$\frac{\|\nabla r_N\|^2}{\|r_N\|^2} \ge \inf\Big\{ \|\nabla v\|^2 : v \in H_0^1, \|v\| = 1, (v,\varphi_j) = 0, j = 1, \dots, N \Big\} = \lambda_{N+1},$$

and hence

$$\|r_N\| \le \lambda_{N+1}^{-1/2} \|\nabla r_N\|.$$

It now suffices to show that the sequence $\|\nabla r_N\|$ is bounded. We first recall from Theorem 6.1 that $a(\varphi_i, \varphi_j) = 0$ for $i \ne j$, so that $a(r_N, v_N) = 0$. Hence $a(v,v) = a(v_N, v_N) + 2a(v_N, r_N) + a(r_N, r_N) = a(v_N, v_N) + a(r_N, r_N)$ and

$$\|\nabla r_N\|^2 = a(r_N, r_N) = a(v,v) - a(v_N, v_N) \le a(v,v) = \|\nabla v\|^2,$$

which completes the proof of (2).

For the proof of the second statement, we first note that, for $v \in H_0^1$,

$$\sum_{j=1}^N \lambda_j (v,\varphi_j)^2 = a(v_N, v_N) = a(v,v) - a(r_N, r_N) \le a(v,v),$$

and we conclude that $\sum_{j=1}^\infty \lambda_j (v,\varphi_j)^2 < \infty$. Conversely, we assume that $v \in L_2$ and $\sum_{j=1}^\infty \lambda_j (v,\varphi_j)^2 < \infty$. We already know that $v_N \to v$ in L_2 as $N \to \infty$. To obtain convergence in H^1 we note that, with $M > N$,

$$\alpha \|v_N - v_M\|_1^2 \le \|\nabla(v_N - v_M)\|^2 = \sum_{j=N+1}^M \lambda_j (v,\varphi_j)^2 \to 0 \text{ as } N \to \infty.$$

Hence, v_N is a Cauchy sequence in H^1 and converges to a limit in H^1. Clearly, this limit is the same as v. By the trace theorem (Theorem A.4) v_N is also a Cauchy sequence in $L_2(\Gamma)$, and since $v_N = 0$ on Γ we conclude that $v = 0$ on Γ. Hence, $v \in H_0^1$. Finally, (1) is obtained by letting $N \to \infty$ in $a(v_N, v_N) = \sum_{j=1}^N \lambda_j (v,\varphi_j)^2$. □

Here is an improved version of Theorem 13.1.

Theorem 2. *Let u_h and u be the solutions of (13.2) and (13.1). Then we have, for $t \geq 0$,*

$$\begin{aligned}
\|u_{h,t}(t) - u_t(t)\| &\leq C\Big(|v_h - R_h v|_1 + \|w_h - R_h w\|\Big) \\
&\quad + Ch^2\Big(\|u_t(t)\|_2 + \int_0^t \|u_{tt}\|_2 \, ds\Big), \\
\|u_h(t) - u(t)\| &\leq C\Big(|v_h - R_h v|_1 + \|w_h - R_h w\|\Big) \\
&\quad + Ch^2\Big(\|u(t)\|_2 + \int_0^t \|u_{tt}\|_2 \, ds\Big), \\
|u_h(t) - u(t)|_1 &\leq C\Big(|v_h - R_h v|_1 + \|w_h - R_h w\|\Big) \\
&\quad + Ch\Big(\|u(t)\|_2 + \int_0^t \|u_{tt}\|_1 \, ds\Big).
\end{aligned}$$

Proof. Writing as usual

$$u_h - u = (u_h - R_h u) + (R_h u - u) = \theta + \rho,$$

we may bound ρ and ρ_t as in the proof of Theorem 10.1 by

$$\|\rho(t)\| + h|\rho(t)|_1 \leq Ch^2\|u(t)\|_2, \quad \|\rho_t(t)\| \leq Ch^2\|u_t(t)\|_2. \tag{3}$$

For $\theta(t)$ we have, after a calculation analogous to that in (10.14),

$$(\theta_{tt}, \chi) + a(\theta, \chi) = -(\rho_{tt}, \chi), \quad \forall \chi \in S_h, \quad \text{for } t > 0. \tag{4}$$

Imitating the proof of Lemma 13.1, we choose $\chi = \theta_t$:

$$\tfrac{1}{2}\frac{d}{dt}\big(\|\theta_t\|^2 + |\theta|_1^2\big) \leq \|\rho_{tt}\| \, \|\theta_t\|.$$

After integration in t we obtain

$$\begin{aligned}
\|\theta_t(t)\|^2 + |\theta(t)|_1^2 &\leq \|\theta_t(0)\|^2 + |\theta(0)|_1^2 + 2\int_0^t \|\rho_{tt}\| \, \|\theta_t\| \, \mathrm{d}s \\
&\leq \|\theta_t(0)\|^2 + |\theta(0)|_1^2 + 2\int_0^t \|\rho_{tt}\| \, \mathrm{d}s \max_{s\in[0,t]} \|\theta_t\| \\
&\leq \|\theta_t(0)\|^2 + |\theta(0)|_1^2 + 2\Big(\int_0^T \|\rho_{tt}\| \, \mathrm{d}s\Big)^2 + \tfrac{1}{2}\Big(\max_{s\in[0,T]} \|\theta_t\|\Big)^2,
\end{aligned}$$

for $t \in [0, T]$. This implies

$$\tfrac{1}{2}\Big(\max_{s\in[0,T]} \|\theta_t\|\Big)^2 \leq \|\theta_t(0)\|^2 + |\theta(0)|_1^2 + 2\Big(\int_0^T \|\rho_{tt}\| \, \mathrm{d}s\Big)^2$$

and hence

$$\|\theta_t(t)\|^2 + |\theta(t)|_1^2 \leq 2\|\theta_t(0)\|^2 + 2|\theta(0)|_1^2 + 4\Big(\int_0^T \|\rho_{tt}\| \, \mathrm{d}s\Big)^2,$$

for $t \in [0, T]$. In particular this holds with $t = T$ where T is arbitrary. Using also bounds for ρ_{tt} similar to (3), we obtain

$$\begin{aligned}
\|\theta_t(t)\| + \|\theta(t)\| &\leq C\Big(\|\theta_t(t)\| + |\theta(t)|_1\Big) \\
&\leq C\Big(\|w_h - R_h w\| + |v_h - R_h v|_1\Big) + Ch^2 \int_0^t \|u_{tt}\|_2 \, \mathrm{d}s,
\end{aligned}$$

and

$$|\theta(t)|_1 \le C\Big(\|w_h - R_h w\| + |v_h - R_h v|_1\Big) + Ch\int_0^t \|u_{tt}\|_1 \,\mathrm{d}s.$$

Together with the bounds in (3) this completes the proof. □

We remark that the choices $v_h = R_h v$ and $w_h = R_h w$ in Theorem 2 give optimal order error estimates for all the three quantities considered, but that other optimal choices of v_h could cause a loss of one power of h, because of the gradient in the first term on the right. This can be avoided by a more refined argument. The regularity requirement on the exact solution can also be reduced.

`/stig`