THE CONTINUOUS GALERKIN METHOD FOR AN INTEGRO-DIFFERENTIAL EQUATION MODELING DYNAMIC FRACTIONAL ORDER VISCOELASTICITY

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Abstract. We consider a fractional order integro-differential equation with a weakly singular convolution kernel. The equation with homogeneous Dirichlet boundary conditions is reformulated as an abstract Cauchy problem, and well-posedness is verified in the context of linear semigroup theory. Then we formulate a continuous Galerkin method for the problem, and we prove stability estimates. These are then used to prove a priori error estimates. The theory is illustrated by a numerical example.

1. Introduction

Bagley and Torvik [5] have proved that fractional order operators are very suitable for modelling viscoelastic materials. The basic equations of the viscoelastic dynamic problem, with surface loads, can be written in the strong form,

$$
\begin{aligned}
&\rho\ddot{\boldsymbol{u}}(\boldsymbol{x},t) - \nabla\cdot\boldsymbol{\sigma}_0(\boldsymbol{u};\boldsymbol{x},t) \\
&\qquad + \int_0^t b(t-s)\nabla\cdot\boldsymbol{\sigma}_1(\boldsymbol{u};\boldsymbol{x},s)\,ds = \boldsymbol{f}(\boldsymbol{x},t) && \text{in } \Omega\times(0,T),\\
&\boldsymbol{u}(\boldsymbol{x},t) = 0 && \text{on } \Gamma_D\times(0,T),\\
&\boldsymbol{\sigma}(\boldsymbol{u};\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x},t) && \text{on } \Gamma_N\times(0,T),\\
&\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}^0(\boldsymbol{x}) && \text{in } \Omega,\\
&\dot{\boldsymbol{u}}(\boldsymbol{x},0) = \boldsymbol{v}^0(\boldsymbol{x}) && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

(throughout this text we use '$\cdot$' to denote '$\frac{\partial}{\partial t}$') where $\boldsymbol{u}$ is the displacement vector, ρ is the (constant) mass density, $\boldsymbol{f}$ and $\boldsymbol{g}$ represent, respectively, the volume and surface loads, $\boldsymbol{\sigma}_0$ and $\boldsymbol{\sigma}_1$ are the stresses according to

$$
\begin{aligned}
&\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}_0(t) - \int_0^t b(t-s)\boldsymbol{\sigma}_1(s)\,ds, \quad \text{with}\\
&\boldsymbol{\sigma}_0(t) = 2\mu_0\boldsymbol{\epsilon}(t) + \lambda_0\mathrm{Tr}(\boldsymbol{\epsilon}(t))\mathbf{I}, \quad \boldsymbol{\sigma}_1(t) = 2\mu_1\boldsymbol{\epsilon}(t) + \lambda_1\mathrm{Tr}(\boldsymbol{\epsilon}(t))\mathbf{I},
\end{aligned}
\tag{1.2}
$$

where $\lambda_0 > \lambda_1 > 0$ and $\mu_0 > \mu_1 > 0$ are elastic constants of Lamé type, $\boldsymbol{\epsilon}$ is the strain which is defined through the usual linear kinematic relation $\boldsymbol{\epsilon} = \frac{1}{2}(\nabla\boldsymbol{u} +$

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$(\nabla \boldsymbol{u})^T$), and b is the convolution kernel

$$b(t) = -\frac{d}{dt}\Big(\mathrm{E}_\alpha(-(t/\tau)^\alpha)\Big) = \frac{\alpha}{\tau}\Big(\frac{t}{\tau}\Big)^{\alpha-1}\mathrm{E}'_\alpha\Big(-\big(\frac{t}{\tau}\big)^\alpha\Big) \approx Ct^{-1+\alpha},\ t \to 0. \tag{1.3}$$

Here $\tau > 0$ is the relaxation time and $\mathrm{E}_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(1+\alpha k)}$ is the Mittag-Leffler function of order $\alpha \in (0,1)$, $\gamma = \frac{\mu_1}{\mu_0} = \frac{\lambda_1}{\lambda_0} < 1$, so that $\boldsymbol{\sigma}_1 = \gamma \boldsymbol{\sigma}_0$ and we define $\beta(t) = \gamma b(t)$. The convolution kernel is weakly singular and $\beta \in L_1(0,\infty)$ with $\int_0^\infty \beta(t)\, dt = \gamma$. We introduce the function

$$\xi(t) = \gamma - \int_0^t \beta(s)\, ds = \int_t^\infty \beta(s)\, ds, \tag{1.4}$$

which is decreasing with $\xi(0) = \gamma$, $\lim_{t\to\infty} \xi(t) = 0$, so that $0 < \xi(t) \le \gamma$.

We let $\Omega \subset \mathbb{R}^d$, $d = 2,3$, be a bounded polygonal domain with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, where Γ_D and Γ_N are disjoint and $\mathrm{meas}(\Gamma_D) \neq 0$. We introduce the function spaces $H = L_2(\Omega)^d$, $H_{\Gamma_N} = L_2(\Gamma_N)^d$, and $V = \{\boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v}|_{\Gamma_D} = \boldsymbol{0}\}$. We denote the norms in H and H_{Γ_N} by $\|\cdot\|$ and $\|\cdot\|_{\Gamma_N}$, respectively, and we equip V with the inner product $a(\cdot,\cdot)$ and norm $\|\boldsymbol{v}\|_V^2 = a(\boldsymbol{v},\boldsymbol{v})$, where (with the usual summation convention)

$$a(\boldsymbol{v},\boldsymbol{w}) = \int_\Omega \big(2\mu_0 \epsilon_{ij}(\boldsymbol{v})\epsilon_{ij}(\boldsymbol{w}) + \lambda_0 \epsilon_{ii}(\boldsymbol{v})\epsilon_{jj}(\boldsymbol{w})\big)\, dx, \quad \boldsymbol{v},\boldsymbol{w} \in V, \tag{1.5}$$

which is a coercive bilinear form on V. Setting $A\boldsymbol{u} = -\nabla \cdot \boldsymbol{\sigma}_0(\boldsymbol{u})$ with $\mathcal{D}(A) = H^2(\Omega)^d \cap V$ such that $a(\boldsymbol{u},\boldsymbol{v}) = (A\boldsymbol{u},\boldsymbol{v})$ for sufficiently smooth $\boldsymbol{u},\boldsymbol{v} \in V$, we can write the weak form of the equation of motion as: Find $\boldsymbol{u}(t) \in V$ such that $\boldsymbol{u}(0) = \boldsymbol{u}^0$, $\dot{\boldsymbol{u}}(0) = \boldsymbol{v}^0$, and

$$\begin{aligned} \rho(\ddot{\boldsymbol{u}}(t),\boldsymbol{v}) + a(\boldsymbol{u}(t),\boldsymbol{v}) - &\int_0^t \beta(t-s) a(\boldsymbol{u}(s),\boldsymbol{v})\, ds \\ &= (\boldsymbol{f}(t),\boldsymbol{v}) + (\boldsymbol{g}(t),\boldsymbol{v})_{\Gamma_N}, \quad \forall \boldsymbol{v} \in V,\ t \in (0,T), \end{aligned} \tag{1.6}$$

with $(\boldsymbol{g}(t),\boldsymbol{v})_{\Gamma_N} = \int_{\Gamma_N} \boldsymbol{g}(t)\cdot\boldsymbol{v}\, dS$. For more details see [1], [2], [3], [4] and references therein.

Defining $\boldsymbol{u}_1 = \boldsymbol{u}$ and $\boldsymbol{u}_2 = \dot{\boldsymbol{u}}$ we write (1.6) as: Find $\boldsymbol{u}_1(t)$, $\boldsymbol{u}_2(t) \in V$ such that $\boldsymbol{u}_1(0) = \boldsymbol{u}^0$, $\boldsymbol{u}_2(0) = \boldsymbol{v}^0$, and

$$\begin{aligned} &a\big(\dot{\boldsymbol{u}}_1(t),\boldsymbol{v}_1\big) - a\big(\boldsymbol{u}_2(t),\boldsymbol{v}_1\big) = 0, \\ &\rho(\dot{\boldsymbol{u}}_2(t),\boldsymbol{v}_2) + a(\boldsymbol{u}_1(t),\boldsymbol{v}_2) - \int_0^t \beta(t-s) a(\boldsymbol{u}_1(s),\boldsymbol{v}_2)\, ds \\ &\qquad = (\boldsymbol{f}(t),\boldsymbol{v}_2) + (\boldsymbol{g}(t),\boldsymbol{v}_2)_{\Gamma_N}, \quad \forall \boldsymbol{v}_1,\boldsymbol{v}_2 \in V,\ t \in (0,T). \end{aligned} \tag{1.7}$$

In the next section, using (1.6) with $\Gamma_N \neq \varnothing$, $\boldsymbol{g} = 0$ or $\Gamma_N = \varnothing$, we reformulate the problem as an abstract Cauchy problem and prove well-posedness. We also discuss the regularity and obtain some regularity estimates. In §3 we use (1.7) to formulate a continuous Galerkin method based on linear polynomials both in time and space. Then in §4 we show stability estimates for the continuous Galerkin method, and in §5 we use them to prove a priori error estimates that are optimal in $L_\infty(L_2)$ and $L_\infty(H^1)$. Finally, in §6, we illustrate the theory by computing the approximate solutions of (1.1) in a simple but realistic numerical example.

There is an extensive literature on finite element methods for partial differential equations with memory, see, e.g., [1], [7], [8], [9], [10]. The present work extends previous works, e.g., [2], [3], [15], on quasi-static fractional order viscoelasticity ($\rho\ddot{\boldsymbol{u}} \approx 0$) to the dynamic case. The paper [4] also deals with the dynamic case but considers only spatial discretization. A dynamic model for viscoelasticity based on internal variables is studied in [12]. The memory term generates a growing amount of data that has to be stored and used in each time step. This can be dealt with by introducing "sparse quadrature" in the convolution term [16]. For a different approach based on "convolution quadrature", see [13], [14].

The main result in the present work are derived under rather restrictive assumptions, $\Gamma_N = \varnothing$ or $\Gamma_N \neq \varnothing$, $\boldsymbol{g} = 0$, which guarantee the global regularity needed for the a priori error analysis. Also our results do not admit adaptive meshes. In general such global regularity is not present, which calls for adaptive methods based on a posteriori error analysis. We plan to address these issues in future work.

2. **Existence and uniqueness**

In this section, using the theory of linear operator semigroups, we show that there is a unique solution of (1.6), with pure Dirichlet boundary condition, that is, $\Gamma_N = \varnothing$, or with homogeneous mixed Dirichlet-Neumann boundary condition, that is, $\boldsymbol{g} = 0$, $\Gamma_N \neq \varnothing$. The theory presented here does not admit the term $(\boldsymbol{g}, \boldsymbol{v})_{\Gamma_N} \neq 0$ in (1.6). We then investigate the regularity in the case of homogeneous Dirichlet boundary condition, that is, $\Gamma_N = \varnothing$. The techniques are adapted from [6].

We consider the strong form of (1.6), for any fixed $T > 0$, that is,

$$\rho\ddot{\boldsymbol{u}}(t) + A\boldsymbol{u}(t) - \int_0^t \beta(t-s)A\boldsymbol{u}(s)\,ds = \boldsymbol{f}(t), \quad t \in (0,T), \tag{2.1}$$

with the initial conditions

$$\boldsymbol{u}(0) = \boldsymbol{u}^0 \in \mathcal{D}(A), \ \ \dot{\boldsymbol{u}}(0) = \boldsymbol{v}^0 \in V. \tag{2.2}$$

We extend $\boldsymbol{u}$ by $\boldsymbol{u}(t) = \boldsymbol{h}(t)$ for $t < 0$ with $\boldsymbol{h}$ to be chosen. By adding $-\int_{-\infty}^0 \beta(t-s)A\boldsymbol{h}(s)\,ds$ to both sides of (2.1), changing the variables in the convolution terms and defining $\boldsymbol{w}(t,s) = \boldsymbol{u}(t) - \boldsymbol{u}(t-s)$, we get

$$\rho\ddot{\boldsymbol{u}}(t) + \tilde{\gamma}A\boldsymbol{u}(t) + \int_0^\infty \beta(s)A\boldsymbol{w}(t,s)\,ds = \boldsymbol{f}(t) - \int_t^\infty \beta(s)A\boldsymbol{h}(t-s)\,ds, \tag{2.3}$$

where $\tilde{\gamma} = 1 - \gamma = 1 - \int_0^\infty \beta(s)\,ds > 0$.

2.1. **An abstract Cauchy problem.**

We choose $\boldsymbol{h}(t) = \boldsymbol{u}^0$ in (2.3), so that

$$\rho\ddot{\boldsymbol{u}}(t) + \tilde{\gamma}A\boldsymbol{u}(t) + \int_0^\infty \beta(s)A\boldsymbol{w}(t,s)\,ds = \tilde{\boldsymbol{f}}(t), \tag{2.4}$$

where,

$$\boldsymbol{w}(t,s) = \begin{cases} \boldsymbol{u}(t) - \boldsymbol{u}(t-s), & s \in [0,t], \\ \boldsymbol{u}(t) - \boldsymbol{u}^0, & s \in [t,\infty), \end{cases}$$

and, in view of (1.4),

$$\tilde{\boldsymbol{f}}(t) = \boldsymbol{f}(t) - \xi(t)A\boldsymbol{u}^0. \tag{2.5}$$

Then we reformulate (2.4) as an abstract Cauchy problem and prove well-posedness.

We set $\boldsymbol{v} = \rho\dot{\boldsymbol{u}}$ and define the Hilbert spaces

$$W = L_{2,\beta}\big((0,\infty);V\big) = \Big\{\boldsymbol{w} : \|\boldsymbol{w}\|_W^2 = \rho\int_0^\infty \beta(s)\|\boldsymbol{w}(s)\|_V^2\,ds < \infty\Big\},$$

$$Z = V\times H\times W = \Big\{\boldsymbol{z} = (\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) : \|\boldsymbol{z}\|_Z^2 = \tilde{\gamma}\rho\|\boldsymbol{u}\|_V^2 + \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|_W^2 < \infty\Big\}.$$

We also define the linear operator $\mathcal{A}$ on Z such that, for $\boldsymbol{z} = (\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$,

$$\mathcal{A}\boldsymbol{z} = \Big(\frac{1}{\rho}\boldsymbol{v}\,,\, -A\Big(\tilde{\gamma}\boldsymbol{u} + \int_0^\infty \beta(s)\boldsymbol{w}(s)\,ds\Big)\,,\, \frac{1}{\rho}\boldsymbol{v} - D\boldsymbol{w}\Big),$$

with domain of definition

$$\mathcal{D}(\mathcal{A}) = \Big\{(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\in Z : \boldsymbol{v}\in V,\ \tilde{\gamma}\boldsymbol{u} + \int_0^\infty \beta(s)\boldsymbol{w}(s)\,ds \in \mathcal{D}(A), \boldsymbol{w}\in\mathcal{D}(D)\Big\},$$

where

$$D\boldsymbol{w} = \frac{d}{ds}\boldsymbol{w} \quad \text{with} \quad \mathcal{D}(D) = \{\boldsymbol{w}\in W : D\boldsymbol{w}\in W \text{ and } \boldsymbol{w}(0) = 0\}.$$

Therefore, a solution of (2.1) with (2.2) satisfies the abstract Cauchy problem

$$\begin{aligned} &\dot{\boldsymbol{z}}(t) = \mathcal{A}\boldsymbol{z}(t) + F(t), \quad 0 < t < T,\\ &\boldsymbol{z}(0) = \boldsymbol{z}^0, \end{aligned} \tag{2.6}$$

where $F(t) = (0\,,\,\tilde{\boldsymbol{f}}(t)\,,\,0)$ and $\boldsymbol{z}^0 = \big(\boldsymbol{u}^0,\,\rho\boldsymbol{v}^0,\,0\big)$, since

$$\boldsymbol{w}(0,s) = \boldsymbol{u}(0) - \boldsymbol{u}(-s) = \boldsymbol{u}(0) - h(-s) = \boldsymbol{u}^0 - \boldsymbol{u}^0 = 0. \tag{2.7}$$

We also note that $\boldsymbol{w}(t,0) = \boldsymbol{u}(t) - \boldsymbol{u}(t) = 0$, so that $\boldsymbol{w}(t,\cdot)\in\mathcal{D}(D)$.

A function $\boldsymbol{z}$ which is differentiable a.e. on $[0,T]$ with $\dot{\boldsymbol{z}}\in L_1((0,T);Z)$ is called a *strong solution* of the initial value problem (2.6) if $\boldsymbol{z}(0) = \boldsymbol{z}^0$, $\boldsymbol{z}(t)\in\mathcal{D}(\mathcal{A})$, and $\dot{\boldsymbol{z}}(t) = \mathcal{A}\boldsymbol{z}(t) + F(t)$ a.e. on $[0,T]$.

Lemma 1. *Let $\boldsymbol{z} = (\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$ be a strong solution of* (2.6) *with $\boldsymbol{z}^0 = \big(\boldsymbol{u}^0,\,\rho\boldsymbol{v}^0,\,0\big)$. Then $\boldsymbol{u}$ is a solution of* (2.1) *with initial conditions* (2.2)*.*

Proof. For the components of the strong solution $\boldsymbol{z}$ of (2.6), we have

$$\begin{aligned} &\dot{\boldsymbol{u}}(t) = \frac{1}{\rho}\boldsymbol{v}(t), \quad t\in(0,T),\\ &\dot{\boldsymbol{v}}(t) = -A\Big(\tilde{\gamma}\boldsymbol{u}(t) + \int_0^\infty \beta(s)\boldsymbol{w}(t,s)\,ds\Big) + \tilde{\boldsymbol{f}}(t), \quad t\in(0,T),\\ &\dot{\boldsymbol{w}}(t,s) = \frac{1}{\rho}\boldsymbol{v}(t) - D\boldsymbol{w}(t,s), \quad s\in(0,\infty),\ t\in(0,T). \end{aligned}$$

The first equation and $\boldsymbol{z}^0 = \big(\boldsymbol{u}^0,\,\rho\boldsymbol{v}^0,\,0\big)$ imply the initial conditions (2.2). The first and third equations mean that $\boldsymbol{w}$ satisfies the first order PDE

$$\frac{\partial}{\partial t}\boldsymbol{w} + \frac{\partial}{\partial s}\boldsymbol{w} = \frac{\partial}{\partial t}\boldsymbol{u}.$$

Besides, since $\boldsymbol{w}(0,\cdot) = 0$ and $\boldsymbol{w}(t,\cdot)\in\mathcal{D}(D)$ we have the boundary conditions $\boldsymbol{w}(0,s) = 0$ and $\boldsymbol{w}(t,0) = 0$. Hence $\boldsymbol{w}(t,s) = \boldsymbol{u}(t) - \boldsymbol{u}(t-s)$, $0\le s\le t$, and $\boldsymbol{w}(t,s) = \boldsymbol{u}(t) - \boldsymbol{u}^0 = \boldsymbol{u}(t) - \boldsymbol{u}(t-s)$, $0\le t\le s$. This and the fact that (2.4) is obtained from the first two equations, imply that $\boldsymbol{u}$ satisfies (2.1) a.e. on $[0,T]$ by backward calculations from (2.3). □

Theorem 1. *Assume that* $\Gamma_N = \varnothing$ *or* $\Gamma_N \neq \varnothing$ *and* $\boldsymbol{g} = 0$. *There is a unique solution* $\boldsymbol{u} = \boldsymbol{u}(t)$ *of* (2.1)–(2.2) *for all* $\boldsymbol{u}^0 \in \mathcal{D}(A)$ *and* $\boldsymbol{v}^0 \in V$, *if* $\boldsymbol{f} : [0,T] \to H$ *is Lipschitz continuous. Moreover, for some* $C = C(\tilde{\gamma}, \rho, T)$, *we have the regularity estimate*

$$\|\boldsymbol{u}(t)\|_V + \|\dot{\boldsymbol{u}}(t)\| \leq C\Big(\|\boldsymbol{u}^0\|_{H^2} + \|\boldsymbol{v}^0\| + \int_0^t \|\boldsymbol{f}\|\,ds\Big), \quad t \in [0,T]. \tag{2.8}$$

Proof. For any $\boldsymbol{u}^0 \in \mathcal{D}(A)$ and $\boldsymbol{v}^0 \in V$, we have $\boldsymbol{z}^0 = \big(\boldsymbol{u}^0, \boldsymbol{v}^0, 0\big) \in \mathcal{D}(\mathcal{A})$. We first show that F in (2.6) is differentiable a.e. on $[0,T]$ and $\dot{F} \in L_1([0,T];Z)$. We then show that the linear operator $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup $e^{t\mathcal{A}}$ on Z. These prove that there is a unique strong solution of (2.6) by [11, Corollary 4.2.10], and the proof of the first part is then complete by Lemma 1. Finally we prove (2.8).

1. By assumption $\boldsymbol{f}$ is Lipschitz continuous on $[0,T]$. Hence $\boldsymbol{f}$ is differentiable a.e. on $[0,T]$ and $\dot{\boldsymbol{f}} \in L_1((0,T);H)$, since H is a Hilbert space. Since $\dot{\xi}(t) = -\beta(t)$ by (1.4), from (2.5) we get

$$\dot{\tilde{\boldsymbol{f}}}(t) = \dot{\boldsymbol{f}}(t) + A\boldsymbol{u}^0\beta(t),$$

which shows that $\tilde{\boldsymbol{f}}$ is differentiable a.e. on $[0,T]$. Thus F is differentiable a.e. on $[0,T]$ and $\dot{F} \in L_1((0,T);Z)$.

2. We use the Lumer-Philips Theorem [11] to show that $\mathcal{A}$ generates a C_0 semigroup of contractions on Z. To this end we first show that $\mathcal{A}$ is dissipative. For $\boldsymbol{z} = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \mathcal{D}(\mathcal{A})$ we have

$$\begin{aligned}\langle \mathcal{A}\boldsymbol{z}, \boldsymbol{z}\rangle_Z &= \tilde{\gamma}a(\boldsymbol{v}, \boldsymbol{u}) - \Big(A\big(\tilde{\gamma}\boldsymbol{u} + \int_0^\infty \beta(s)\boldsymbol{w}(s)\,ds\big), \boldsymbol{v}\Big) + \Big(\frac{1}{\rho}\boldsymbol{v} - D\boldsymbol{w}, \boldsymbol{w}\Big)_W \\ &= -\rho\int_0^\infty \beta(s)a(D\boldsymbol{w}(s), \boldsymbol{w}(s))\,ds = -\frac{1}{2}\rho\int_0^\infty \beta(s)D\|\boldsymbol{w}(s)\|_V^2\,ds.\end{aligned}$$

To prove that the last term is non-positive, and hence $\mathcal{A}$ is dissipative, we consider for $\epsilon > 0$,

$$\begin{aligned}\int_\epsilon^\infty \beta(s)D\|\boldsymbol{w}(s)\|_V^2\,ds &= \lim_{M\to\infty}\int_\epsilon^M \beta(s)D\|\boldsymbol{w}(s)\|_V^2\,ds \\ &= \lim_{M\to\infty}\Big(\beta(M)\|\boldsymbol{w}(M)\|_V^2 - \beta(\epsilon)\|\boldsymbol{w}(\epsilon)\|_V^2 - \int_\epsilon^M \beta'(s)\|\boldsymbol{w}(s)\|_V^2\,ds\Big) \\ &\geq -\beta(\epsilon)\|\boldsymbol{w}(\epsilon)\|_V^2,\end{aligned}$$

because $\beta'(s) < 0$ and $\lim_{M\to\infty}\beta(M)\|\boldsymbol{w}(M)\|_V^2 = 0$, since $\int_0^\infty \beta(s)\|\boldsymbol{w}(s)\|_V^2\,ds < \infty$. Since $\boldsymbol{w}(\epsilon) = \int_0^\epsilon D\boldsymbol{w}(s)\,ds$, by the Cauchy-Schwarz inequality we have

$$\|\boldsymbol{w}(\epsilon)\|_V^2 \leq \Big(\int_0^\epsilon \|D\boldsymbol{w}(s)\|_V\,ds\Big)^2 \leq \int_0^\epsilon \frac{1}{\beta(s)}\,ds\int_0^\epsilon \beta(s)\|D\boldsymbol{w}(s)\|_V^2\,ds,$$

and consequently we get

$$\beta(\epsilon)\|\boldsymbol{w}(\epsilon)\|_V^2 \leq \int_0^\epsilon \frac{\beta(\epsilon)}{\beta(s)}\,ds\int_0^\epsilon \beta(s)\|D\boldsymbol{w}(s)\|_V^2\,ds \leq \epsilon\frac{1}{\rho}\|D\boldsymbol{w}\|_W^2,$$

since $\beta(\epsilon) \leq \beta(s)$ and $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \mathcal{D}(\mathcal{A})$ implies $D\boldsymbol{w} \in W$. Therefore

$$\langle \mathcal{A}\boldsymbol{z}, \boldsymbol{z}\rangle_Z \leq \lim_{\epsilon\to 0^+}\frac{\epsilon}{2}\|D\boldsymbol{w}\|_W^2 = 0,$$

and $\mathcal{A}$ is dissipative.

Next we show that $R(I-\mathcal{A})=Z$. To see this, for arbitrary $(\boldsymbol{\phi},\boldsymbol{\psi},\boldsymbol{\theta})\in Z$ we must find a unique $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\in\mathcal{D}(\mathcal{A})$ such that $(I-\mathcal{A})(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=(\boldsymbol{\phi},\boldsymbol{\psi},\boldsymbol{\theta})$, that is,

$$\begin{aligned}
&\boldsymbol{u}-\frac{1}{\rho}\boldsymbol{v}=\boldsymbol{\phi},\\
&\boldsymbol{v}+A\Big(\tilde{\gamma}\boldsymbol{u}+\int_0^\infty\beta(s)\boldsymbol{w}(s)\,ds\Big)=\boldsymbol{\psi},\\
&\boldsymbol{w}-\frac{1}{\rho}\boldsymbol{v}+D\boldsymbol{w}=\boldsymbol{\theta},\quad \boldsymbol{w}(0)=0.
\end{aligned}\tag{2.9}$$

From the first and third equations and $\boldsymbol{w}(0)=0$ we get

$$\boldsymbol{v}=\rho(\boldsymbol{u}-\boldsymbol{\phi}),\quad \boldsymbol{w}(s)=\int_0^s e^{r-s}\big(\frac{1}{\rho}\boldsymbol{v}+\boldsymbol{\theta}(r)\big)\,dr.$$

Substituting these into the second equation of (2.9), we get

$$\rho(\boldsymbol{u}-\boldsymbol{\phi})+A\Big(\tilde{\gamma}\boldsymbol{u}+\int_0^\infty\beta(s)\int_0^s e^{r-s}\big(\boldsymbol{u}-\boldsymbol{\phi}+\boldsymbol{\theta}(r)\big)\,dr\,ds\Big)=\boldsymbol{\psi},$$

and hence

$$\boldsymbol{u}+\kappa A\boldsymbol{u}=\boldsymbol{\phi}+\frac{1}{\rho}\Big(\boldsymbol{\psi}+\int_0^\infty\beta(s)e^{-s}\int_0^s e^{r}A\big(\boldsymbol{\phi}-\boldsymbol{\theta}(r)\big)\,dr\,ds\Big),\tag{2.10}$$

where $\kappa=\frac{1}{\rho}\big(1-\int_0^\infty\beta(s)e^{-s}ds\big)>0$. Now we need to show that this equation has a solution. The weak form is to find $\boldsymbol{u}\in V$ such that

$$b(\boldsymbol{u},\boldsymbol{v})=L(\boldsymbol{v})\quad\forall\boldsymbol{v}\in V,$$

with the bilinear form

$$b(\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{u},\boldsymbol{v})+\kappa a(\boldsymbol{u},\boldsymbol{v})\quad\text{for }\boldsymbol{u},\boldsymbol{v}\in V,$$

and the linear form

$$L(\boldsymbol{v})=(\boldsymbol{\phi},\boldsymbol{v})+\frac{1}{\rho}(\boldsymbol{\psi},\boldsymbol{v})+\frac{1}{\rho}\int_0^\infty\beta(s)e^{-s}\int_0^s e^{r}a\big(\boldsymbol{\phi}-\boldsymbol{\theta}(r),\boldsymbol{v}\big)\,dr\,ds.$$

Clearly $b(\cdot,\cdot)$ is bounded and coercive on V, and L is bounded on V. Therefore by the Riesz representation theorem, there is a unique solution, hence $R(I-\mathcal{A})=Z$.

Since Z is a Hilbert space, it follows from [11, Theorem 1.4.6], that $\overline{\mathcal{D}(\mathcal{A})}=Z$. So we have verified all the hypotheses of the Lumer-Philips theorem to complete the first part of the proof.

3. The unique strong solution of (2.6), is given by

$$\boldsymbol{z}(t)=e^{t\mathcal{A}}\boldsymbol{z}^0+\int_0^t e^{(t-s)\mathcal{A}}F(s)\,ds,$$

and $\|e^{t\mathcal{A}}\|_Z\le 1$, since $\mathcal{A}$ generates a C_0 semigroup of contractions. Therefore

$$\|\boldsymbol{z}(t)\|_Z\le\|\boldsymbol{z}^0\|_Z+\int_0^t\|F(s)\|_Z\,ds.$$

Since $\boldsymbol{v} = \rho\dot{\boldsymbol{u}}$, $\boldsymbol{z}^0 = (\boldsymbol{u}^0,\, \rho\boldsymbol{v}^0,\, 0)$ and $\|F(s)\|_Z = \|\tilde{\boldsymbol{f}}(s)\| = \|\boldsymbol{f}(s) - \xi(s)A\boldsymbol{u}^0\|$, we have

$$\Big(\tilde{\gamma}\rho\|\boldsymbol{u}(t)\|_V^2 + \rho^2\|\dot{\boldsymbol{u}}(t)\|^2 + \rho\int_0^\infty \beta(s)\|\boldsymbol{w}(t)\|_V^2\,ds\Big)^{1/2}$$
$$\le \big(\tilde{\gamma}\rho\|\boldsymbol{u}^0\|_V^2 + \rho^2\|\boldsymbol{v}^0\|^2\big)^{1/2} + \int_0^t \big(\|\boldsymbol{f}(s)\| + \xi(s)\|A\boldsymbol{u}^0\|\big)\,ds.$$

Consequently, we have the estimate (2.8) with $C = C(\tilde{\gamma}, \rho, T)$. □

2.2. **Regularity.** In order to prove higher regularity we specialize to the homogeneous Dirichlet boundary condition, that is, $\Gamma_N = \varnothing$, and assume that the polygonal domain Ω is convex. This guarantees that we have the elliptic regularity estimate,

$$\|\boldsymbol{u}\|_{H^2} \le C\|A\boldsymbol{u}\|, \quad \boldsymbol{u} \in H^2(\Omega) \cap V. \tag{2.11}$$

We first choose $\boldsymbol{h}(t) = \boldsymbol{u}^0 + t\boldsymbol{v}^0$ in (2.3), so that

$$\rho\ddot{\boldsymbol{u}}(t) + \tilde{\gamma}A\boldsymbol{u}(t) + \int_0^\infty \beta(s)A\boldsymbol{w}(t,s)\,ds = \breve{\boldsymbol{f}}(t), \tag{2.12}$$

where

$$\boldsymbol{w}(t,s) = \begin{cases} \boldsymbol{u}(t) - \boldsymbol{u}(t-s), & s \in [0,t], \\ \boldsymbol{u}(t) - \boldsymbol{u}^0 - (t-s)\boldsymbol{v}^0, & s \in [t,\infty), \end{cases}$$

and, in view of (1.4),

$$\breve{\boldsymbol{f}}(t) = \boldsymbol{f}(t) - A\boldsymbol{v}^0\int_t^\infty (t-s)\beta(s)\,ds - \xi(t)A\boldsymbol{u}^0. \tag{2.13}$$

Then differentiating the equation (2.12) in time we get

$$\rho\,\dddot{\boldsymbol{u}}(t) + \tilde{\gamma}A\dot{\boldsymbol{u}}(t) + \int_0^\infty \beta(s)A\dot{\boldsymbol{w}}(t,s)\,ds = \dot{\breve{\boldsymbol{f}}}(t), \tag{2.14}$$

which, with an underline instead of one time derivative, can be written as

$$\rho\ddot{\underline{\boldsymbol{u}}}(t) + \tilde{\gamma}A\underline{\boldsymbol{u}}(t) + \int_0^\infty \beta(s)A\underline{\boldsymbol{w}}(t,s)\,ds = \underline{\breve{\boldsymbol{f}}}(t), \tag{2.15}$$

with the initial values

$$\underline{\boldsymbol{u}}(0) = \underline{\boldsymbol{u}}^0 = \boldsymbol{v}^0\,,\ \ \dot{\underline{\boldsymbol{u}}}(0) = \underline{\boldsymbol{v}}^0 = \frac{1}{\rho}\big(\boldsymbol{f}(0) - A\boldsymbol{u}^0\big), \tag{2.16}$$

and

$$\underline{\breve{\boldsymbol{f}}}(t) = \dot{\breve{\boldsymbol{f}}}(t) = \dot{\boldsymbol{f}}(t) - \xi(t)A\boldsymbol{v}^0 + \beta(t)A\boldsymbol{u}^0, \tag{2.17}$$

and

$$\underline{\boldsymbol{w}}(t,s) = \begin{cases} \dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}(t-s), & s \in [0,t], \\ \dot{\boldsymbol{u}}(t) - \boldsymbol{v}^0, & s \in [t,\infty), \end{cases}$$

so that $\underline{\boldsymbol{w}}(t,0) = 0$. We note that $\underline{\boldsymbol{w}}$ is continuous and $\underline{\boldsymbol{w}}(t,\cdot) \in \mathcal{D}(D)$ for $t \ge 0$.

Then, in the same way as in §2.1 with $\underline{\boldsymbol{v}} = \rho\dot{\underline{\boldsymbol{u}}}$, we can reformulate (2.15)–(2.16) as the abstract Cauchy problem

$$\begin{aligned} &\dot{\underline{\boldsymbol{z}}}(t) = \mathcal{A}\,\underline{\boldsymbol{z}}(t) + \underline{\breve{F}}(t), \quad 0 < t < T, \\ &\underline{\boldsymbol{z}}(0) = \underline{\boldsymbol{z}}^0, \end{aligned} \tag{2.18}$$

where $\underline{\breve{F}}(t) = (0\,,\, \underline{\breve{\boldsymbol{f}}}(t)\,,\, 0)$ and $\underline{\boldsymbol{z}}^0 = \big(\underline{\boldsymbol{u}}^0,\, \rho\underline{\boldsymbol{v}}^0,\, 0\big)$, since

$$\underline{\boldsymbol{w}}(0,s) = \dot{\boldsymbol{u}}(0) - \boldsymbol{v}^0 = \boldsymbol{v}^0 - \boldsymbol{v}^0 = 0. \tag{2.19}$$

Lemma 2. *Let $\underline{\boldsymbol{z}} = (\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}})$ be a strong solution of (2.18) with $\underline{\boldsymbol{z}}^0 = \big(\underline{\boldsymbol{u}}^0,\, \rho\underline{\boldsymbol{v}}^0,\, 0\big)$. Then $\boldsymbol{u}(t) = \boldsymbol{u}^0 + \int_0^t \underline{\boldsymbol{u}}(s)\,ds$ is a solution of (2.1) with initial conditions (2.2).*

Proof. Clearly $\boldsymbol{u}(0) = \boldsymbol{u}^0$ and $\dot{\boldsymbol{u}} = \underline{\boldsymbol{u}}$. Hence $\underline{\boldsymbol{z}}^0 = \big(\underline{\boldsymbol{u}}^0,\, \underline{\boldsymbol{v}}^0,\, 0\big)$ implies $\dot{\boldsymbol{u}}(0) = \underline{\boldsymbol{u}}(0) = \underline{\boldsymbol{u}}^0 = \boldsymbol{v}^0$, so that (2.2) holds. Then since $\dot{\underline{\boldsymbol{z}}}(t) = \mathcal{A}\,\underline{\boldsymbol{z}}(t) + \underline{\breve{F}}(t)$ a.e. on $[0,T]$, we have,

$$\begin{aligned}
\dot{\underline{\boldsymbol{u}}}(t) &= \frac{1}{\rho}\underline{\boldsymbol{v}}(t), \quad t \in (0,T),\\
\dot{\underline{\boldsymbol{v}}}(t) &= -A\Big(\tilde{\gamma}\underline{\boldsymbol{u}}(t) + \int_0^\infty \beta(s)\underline{\boldsymbol{w}}(t,s)\,ds\Big) + \underline{\breve{\boldsymbol{f}}}(t), \quad t \in (0,T),\\
\dot{\underline{\boldsymbol{w}}}(t,s) &= \frac{1}{\rho}\underline{\boldsymbol{v}}(t) - D\underline{\boldsymbol{w}}(t,s), \quad s \in (0,\infty),\, t \in (0,T).
\end{aligned}$$

The first and the third equation with $\underline{\boldsymbol{w}}(t,0) = 0$, $\underline{\boldsymbol{w}}(0,s) = 0$ has the unique solution $\underline{\boldsymbol{w}}(t,s) = \underline{\boldsymbol{u}}(t) - \underline{\boldsymbol{u}}(t-s)$ that implies, by integration with respect to t, $\boldsymbol{w}(t,s) := \boldsymbol{u}(t) - \boldsymbol{u}(t-s) = \int_0^t \underline{\boldsymbol{w}}(\tau,s)\,d\tau$. By the first equation we have $\ddot{\boldsymbol{u}} = \dot{\underline{\boldsymbol{u}}} = \frac{1}{\rho}\underline{\boldsymbol{v}}$, so that the second equation is (2.14). The proof is completed by backward calculation from (2.14). □

In the next theorem we find the circumstances under which there is a unique solution of (2.1) with more regularity.

Theorem 2. *Assume that $\Gamma_N = \varnothing$ and that Ω is a convex polygonal domain. There is a unique solution $\boldsymbol{u} = \boldsymbol{u}(t)$ of (2.1)–(2.2) if $\boldsymbol{v}^0 \in \mathcal{D}(A)$, $\boldsymbol{f}(0) - A\boldsymbol{u}^0 \in V$, and $\dot{\boldsymbol{f}} : [0,T] \to H$ is Lipschitz continuous. Moreover, for some $C = C(\tilde{\gamma}, \rho, T)$, we have the regularity estimate*

$$\begin{aligned}
&\|\dot{\boldsymbol{u}}(t)\|_V + \|\ddot{\boldsymbol{u}}(t)\| + \|\boldsymbol{u}(t)\|_{H^2}\\
&\qquad \le C\Big(\|\boldsymbol{f}(0) - A\boldsymbol{u}^0\| + \|\boldsymbol{v}^0\|_{H^2} + \int_0^t \|\dot{\boldsymbol{f}}\|\,ds\Big), \quad t \in [0,T].
\end{aligned} \tag{2.20}$$

Proof. 1. From the assumptions on $\boldsymbol{u}^0$, $\boldsymbol{v}^0$ and $\boldsymbol{f}(0)$ and recalling (2.16) and (2.19), we have $\underline{\boldsymbol{z}}^0 = \big(\underline{\boldsymbol{u}}^0, \rho\underline{\boldsymbol{v}}^0, \underline{\boldsymbol{w}}^0(\cdot)\big) \in \mathcal{D}(\mathcal{A})$. We split the load term $\underline{\breve{F}}$ in (2.18) as

$$\underline{\breve{F}} = \underline{\breve{F}}_1 + \underline{\breve{F}}_2,$$

where $\underline{\breve{F}}_1(t) = (0\,,\, \underline{\breve{\boldsymbol{f}}}_1(t)\,,\, 0) = (0\,,\, \dot{\boldsymbol{f}}(t) - \xi(t)A\boldsymbol{v}^0\,,\, 0)$ and $\underline{\breve{F}}_2(t) = (0\,,\, \underline{\breve{\boldsymbol{f}}}_2(t)\,,\, 0) = (0\,,\, \beta(t)A\boldsymbol{u}^0\,,\, 0)$. We show that each one of the abstract Cauchy problems, for $i = 1,\, 2$,

$$\begin{aligned}
&\dot{\underline{\boldsymbol{z}}}(t) = \mathcal{A}\,\underline{\boldsymbol{z}}(t) + \underline{\breve{F}}_i(t), \quad 0 < t < T,\\
&\underline{\boldsymbol{z}}(0) = \underline{\boldsymbol{z}}^0,
\end{aligned} \tag{2.21}$$

has a unique strong solution, and consequently there is a unique strong solution of (2.18). We recall that the linear operator $\mathcal{A}$ is an infinitesimal generator of a C_0-semigroup of contractions $e^{t\mathcal{A}}$ on Z by the proof of Theorem 1.

Considering

$$\dot{\underline{\breve{\boldsymbol{f}}}}_1(t) = \ddot{\boldsymbol{f}}(t) + \beta(t)A\boldsymbol{v}^0,$$

and the assumptions on $\boldsymbol{v}^0$ and $\dot{\boldsymbol{f}}$, $\breve{\underline{F}}_1$ is differentiable a.e. on $[0,T]$ and $\dot{\breve{\underline{F}}}_1 \in L_1((0,T);Z)$. By [11, Corollary 4.2.10], there is a unique strong solution of (2.21) for $i=1$. On the other hand $\breve{\underline{F}}_2(t)$ is continuous on $(0,T)$, $\breve{\underline{F}}_2(t) \in \mathcal{D}(\mathcal{A})$, $t \in (0,T)$, and $\mathcal{A}\breve{\underline{F}}_2 \in L_1((0,T);Z)$, since $\beta(t)$ is continuous on $(0,T)$ and $A\boldsymbol{u}^0 \in V$. Therefore, by [11, Corollary 4.2.6], there is a unique classical solution of (2.21) with $i=2$. Since any classical solution is a strong solution, the proof of existence and uniqueness is completed by Lemma 2.

2. We have the unique strong solution of (2.18), i.e.

$$\underline{\boldsymbol{z}}(t) = e^{t\mathcal{A}}\underline{\boldsymbol{z}}^0 + \int_0^t e^{(t-s)\mathcal{A}}\breve{\underline{F}}(s)\,ds,$$

with $\|e^{t\mathcal{A}}\|_Z \leq 1$. Following step 3 of Theorem 1, using (2.11), we get (2.20). □

3. The continuous Galerkin method

Recalling the function spaces $H = L_2(\Omega)^d$, $H_{\Gamma_N} = L_2(\Gamma_N)^d$ and $V = \{\boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v}|_{\Gamma_D} = 0\}$ $(d=2,3)$, we provide some definitions which will be used in the forthcoming discussions.

Let $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n < \cdots < t_N = T$ be a partition of the time interval $I=[0,T]$. To each time subinterval $I_n = (t_{n-1}, t_n)$ of length $k_n = t_n - t_{n-1}$, we associate a triangulation $\mathcal{T}_n$ of Ω with piecewise constant mesh function h_n defined by

$$h_n(x) = \operatorname{diam}(K)\,, \quad \text{for } x \in K,\ K \in \mathcal{T}_n, \tag{3.1}$$

and the corresponding finite element space V_n of vector-valued continuous piecewise linear polynomials, that vanish on Γ_D (This requires that the mesh is adjusted to fit Γ_D.). We also define the spaces, for $q = 0$, 1,

$$W^{(q)} = \Big\{\boldsymbol{w} : \boldsymbol{w}|_{\Omega\times I_n} = \boldsymbol{w}^n \in W_n^{(q)},\ n = 1,\ldots,N\Big\},$$

where,

$$W_n^{(q)} = \Big\{\boldsymbol{w} : \boldsymbol{w}(x,t) = \sum_{i=0}^{q} \boldsymbol{w}_i(x)t^i,\ \boldsymbol{w}_i \in V_n\Big\}.$$

Note that $\boldsymbol{w} \in W^{(q)}$ may be discontinuous at $t = t_n$, and $w \in W^{(0)}$ is piecewise constant in time.

With $\mathbb{P}_q^d$ denoting the set of all vector-valued polynomials of degree at most q, the orthogonal projections $\mathcal{R}_{h,n} : V \to V_n$, $\mathcal{P}_{h,n} : H \to V_n$ and $\mathcal{P}_{k,n} : L_2(I_n)^d \to \mathbb{P}_{q-1}^d(I_n)$ are defined, respectively, by

$$\begin{aligned} &a(\mathcal{R}_{h,n}\boldsymbol{v} - \boldsymbol{v}, \boldsymbol{\chi}) = 0, && \forall \boldsymbol{v} \in V,\ \boldsymbol{\chi} \in V_n,\\ &(\mathcal{P}_{h,n}\boldsymbol{v} - \boldsymbol{v}, \boldsymbol{\chi}) = 0, && \forall \boldsymbol{v} \in H,\ \boldsymbol{\chi} \in V_n,\\ &\int_{I_n} (\mathcal{P}_{k,n}\boldsymbol{v} - \boldsymbol{v})\cdot\boldsymbol{\psi}\,dt = 0, && \forall \boldsymbol{v} \in L_2(I_n)^d,\ \boldsymbol{\psi} \in \mathbb{P}_{q-1}^d\,. \end{aligned} \tag{3.2}$$

Correspondingly, we define $\mathcal{R}_h\boldsymbol{v}$, $\mathcal{P}_h\boldsymbol{v}$ and $\mathcal{P}_k\boldsymbol{v}$ by $(\mathcal{R}_h\boldsymbol{v})(t) = \mathcal{R}_{h,n}\boldsymbol{v}(t)$, $(\mathcal{P}_h\boldsymbol{v})(t) = \mathcal{P}_{h,n}\boldsymbol{v}(t)$ for $t \in I_n$, and $\mathcal{P}_k\boldsymbol{v} = \mathcal{P}_{k,n}(\boldsymbol{v}|_{I_n})$, $(n = 1,\cdots,N)$. We also define the orthogonal projections, $R_n : L_2(I_n,V) \to W_n^{(q-1)}$ and $P_n : L_2(I_n,H) \to W_n^{(q-1)}$, such that

$$\begin{aligned}
&\int_{I_n} a(R_n\boldsymbol{u}-\boldsymbol{u},\boldsymbol{\psi})\,dt=0, \quad \forall\boldsymbol{\psi}\in W_n^{(q-1)}\,,\ \boldsymbol{u}\in L_2(I_n,V)\,,\\
&\int_{I_n} (P_n\boldsymbol{u}-\boldsymbol{u},\boldsymbol{\psi})\,dt=0, \qquad \forall\boldsymbol{\psi}\in W_n^{(q-1)}\,,\ \boldsymbol{u}\in L_2(I_n,H)\,.
\end{aligned} \tag{3.3}$$

Correspondingly, we define $R : L_2(I,V)\to W^{(0)}$, $P : L_2(I,H)\to W^{(0)}$ in the obvious way.

One can easily show that

$$R=\mathcal{R}_h\mathcal{P}_k=\mathcal{P}_k\mathcal{R}_h\,,\quad P=\mathcal{P}_h\mathcal{P}_k=\mathcal{P}_k\mathcal{P}_h\,, \tag{3.4}$$

and for $\boldsymbol{u}\in W_n^{(1)}\,,\ \boldsymbol{v}\in W_n^{(0)}$,

$$\int_{I_n}(\boldsymbol{u},\boldsymbol{v})\,dt=\int_{I_n}(\mathcal{P}_{k,n}\boldsymbol{u},\boldsymbol{v})\,dt,\quad \int_{I_n}a(\boldsymbol{u},\boldsymbol{v})\,dt=\int_{I_n}a(\mathcal{P}_{k,n}\boldsymbol{u},\boldsymbol{v})\,dt. \tag{3.5}$$

We introduce the linear operator $A_{n,r}:V_r\to V_n$ by

$$a(\boldsymbol{v}_r,\boldsymbol{w}_n)=(A_{n,r}\boldsymbol{v}_r,\boldsymbol{w}_n)\qquad \forall\boldsymbol{v}_r\in V_r,\ \boldsymbol{w}_n\in V_n.$$

We set $A_n=A_{n,n}$, with discrete norms

$$|\boldsymbol{v}_n|_{h,l}=\|A_n^{l/2}\boldsymbol{v}_n\|=\sqrt{(\boldsymbol{v}_n,A_n^l\boldsymbol{v}_n)}\,,\quad \boldsymbol{v}_n\in V_n \text{ and } l\in\mathbb{R}\,,$$

and A_h so that $A_h\boldsymbol{v}|_{I_n}=A_n\boldsymbol{v}$ for $\boldsymbol{v}\in V_n$. For later use in our error analysis we note that $\mathcal{P}_hA=A_h\mathcal{R}_h$.

We define the bilinear and linear forms $B:\mathcal{W}\times\mathcal{W}\to\mathbb{R}$ and $L:\mathcal{W}\to\mathbb{R}$ by

$$\begin{aligned}
B(\boldsymbol{u},\boldsymbol{v})&=\sum_{n=1}^N\int_{I_n}\Big(-a(\boldsymbol{u}_2,\boldsymbol{v}_1)+a(\dot{\boldsymbol{u}}_1,\boldsymbol{v}_1)+\rho(\dot{\boldsymbol{u}}_2,\boldsymbol{v}_2)+a(\boldsymbol{u}_1,\boldsymbol{v}_2)\Big)\,dt\\
&\quad-\sum_{n=1}^N\int_{I_n}\int_0^t\beta(t-s)a\big(\boldsymbol{u}_1(s),\boldsymbol{v}_2(t)\big)ds\,dt,\\
L(\boldsymbol{w})&=\sum_{n=1}^N\int_{I_n}(\boldsymbol{f},\boldsymbol{w}_2)+(\boldsymbol{g},\boldsymbol{w}_2)_{\Gamma_N}\,dt,
\end{aligned}$$

where $\mathcal{W}$ is the space of pairs of vector-valued functions $\boldsymbol{u}(t)=(\boldsymbol{u}_1(t),\,\boldsymbol{u}_2(t))\in V$ that are piecewise smooth with respect to the temporal mesh. We may note that $(W^{(q)})^2\subset\mathcal{W}$ for $q\geq 0$.

The continuous Galerkin method of degree $(1,1)$ is based on the variational formulation (1.7) and reads: Find $U=(U_1,U_2)\in\big(W^{(1)}\big)^2$ such that, $U_{1,0}^-=\boldsymbol{u}^0$, $U_{2,0}^-=\boldsymbol{v}^0$, and, for $n=1,\,\dots,\,N$,

$$\begin{aligned}
&\int_{I_n}a(\dot U_1,V_1)-a(U_2,V_1)\,dt=0,\\
&\int_{I_n}\Big(\rho(\dot U_2,V_2)+a(U_1,V_2)-\int_0^t\beta(t-s)a\big(U_1(s),V_2(t)\big)\,ds\Big)\,dt\\
&\qquad\qquad=\int_{I_n}(\boldsymbol{f},V_2)\,dt+\int_{I_n}(\boldsymbol{g},V_2)_{\Gamma_N}\,dt,\quad\forall(V_1,V_2)\in\big(W_n^{(0)}\big)^2,\\
&U_{1,n-1}^+=\mathcal{R}_{h,n}U_{1,n-1}^-\,,\quad U_{2,n-1}^+=\mathcal{P}_{h,n}U_{2,n-1}^-\,,
\end{aligned} \tag{3.6}$$

where $U_{i,n}^{\pm} = \lim_{s\to 0^{\pm}} U_i(t_n+s)$, $i=1,\,2$. Hence $U \in \big(W^{(1)}\big)^2$ defined in (3.6) satisfies:

$$\begin{aligned}
&B(U,\mathcal{P}_kV) = L(\mathcal{P}_kV), \qquad\quad \forall\, V \in \big(W^{(1)}\big)^2,\\
&U_{1,n-1}^{+} = \mathcal{R}_{h,n}U_{1,n-1}^{-}\,, \quad U_{2,n-1}^{+} = \mathcal{P}_{h,n}U_{2,n-1}^{-}\,,\\
&U_{1,0}^{-} = \boldsymbol{u}^0, \quad U_{2,0}^{-} = \boldsymbol{v}^0,
\end{aligned}$$

where $\mathcal{P}_kV = (\mathcal{P}_kV_1, \mathcal{P}_kV_2)$.

Since the variational equation (1.7) can be written as: Find $\boldsymbol{u} \in \mathcal{W}$ such that

$$B(\boldsymbol{u},\boldsymbol{v}) = L(\boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \mathcal{W},$$

we may, for later reference, note the Galerkin orthogonality

$$B(U-\boldsymbol{u},\mathcal{P}_kV) = 0, \quad \forall\, V \in \big(W^{(1)}\big)^2. \tag{3.7}$$

Considering the fact that functions in $W_n^{(0)}$ are constant with respect to time, we can write (3.6) as

$$\begin{aligned}
&A_n(U_{1,n}^{-} - U_{1,n-1}^{+}) - \frac{k_n}{2}A_n(U_{2,n}^{-} + U_{2,n-1}^{+}) = 0,\\
&A_n\Big((\frac{k_n}{2} - \gamma\omega_{n,n}^{-})U_{1,n}^{-} + (\frac{k_n}{2} - \gamma\omega_{n,n-1}^{+})U_{1,n-1}^{+}\Big) + \rho\,(U_{2,n}^{-} - U_{2,n-1}^{+}) = H_n + b_n,
\end{aligned}$$

where

$$\begin{aligned}
&b_n = k_n(P_n\boldsymbol{f} + P_n^{\Gamma_N}\boldsymbol{g}),\\
&H_n = \gamma\sum_{r=1}^{n-1} k_rA_{n,r}(\omega_{n,r}^{-}U_{1,r}^{-} + \omega_{n,r-1}^{+}U_{1,r-1}^{+}),\\
&\omega_{n,r}^{-} = \int_{I_n}\int_{t_{r-1}}^{t_r\wedge t}\beta(t-s)\psi_r^{-}(s)\,ds\,dt\,, \qquad t_r\wedge t = \min(t_r,t),\\
&\omega_{n,r-1}^{+} = \int_{I_n}\int_{t_{r-1}}^{t_r\wedge t}\beta(t-s)\psi_{r-1}^{+}(s)\,ds\,dt,
\end{aligned}$$

and ψ_n^{-}, ψ_{n-1}^{+} are the linear Lagrange basis functions on I_n, so that, for $i=1,\,2$,

$$U_i(x,t)\,|_{\Omega\times I_n} = \psi_{n-1}^{+}(t)U_{i,n-1}^{+}(x) + \psi_n^{-}(t)U_{i,n}^{-}(x).$$

From now on we assume, for simplicity, that $\mathcal{T}_{n-1} \subset \mathcal{T}_n$, $n=2,\,\dots,\,N$, which means that the spatial mesh is allowed to be refined (or unchanged) at t_{n-1}. Then $V_{n-1} \subset V_n\,(n=2,\,\dots,\,N)$, $U_{i,n-1}^{+} = U_{i,n-1}^{-}\,(n=1,\,\dots,\,N,\ i=1,\,2)$, and the initial conditions in (3.6) reduce to $U_1(\cdot,0) = \boldsymbol{u}_h^0 = \mathcal{R}_{h,1}\boldsymbol{u}^0$ and $U_2(\cdot,0) = \boldsymbol{v}_h^0 = \mathcal{P}_{h,1}\boldsymbol{v}^0$. In this case U is continuous with respect to t.

4. **Stability estimates**

We consider a modified problem by adding an extra load function, say $\boldsymbol{f}_1 = \boldsymbol{f}_1(t)$, to the first equation of (3.6). This kind of problem will occur in our error analysis below. Moreover, in the error equations the term corresponding to the surface load is zero, i.e., $\boldsymbol{g} = 0$. In this section we therefore consider the problem:

Find $U \in \big(W^{(1)}\big)^2$ such that, for $n = 1, \dots, N$,

$$
\begin{aligned}
&\int_{I_n} a(\dot U_1, V_1) - a(U_2, V_1)\,dt = \int_{I_n} a(\boldsymbol{f}_1, V_1)\,dt,\\
&\int_{I_n} \Big(\rho(\dot U_2, V_2) + a(U_1, V_2) - \int_0^t \beta(t-s) a\big(U_1(s), V_2(t)\big) ds\Big)\,dt\\
&\qquad\qquad = \int_{I_n} (\boldsymbol{f}_2, V_2)\,dt\,, \quad \forall (V_1, V_2) \in \big(W_n^{(0)}\big)^2,\\
&U_1,\, U_2 \ \text{ continuous at } t_{n-1},\\
&U_1(\cdot, 0) = \boldsymbol{u}_h^0\,, \quad U_2(\cdot, 0) = \boldsymbol{v}_h^0\,.
\end{aligned}
\tag{4.1}
$$

Then U satisfies:

$$
\begin{aligned}
&B(U, \mathcal{P}_k V) = \hat L(\mathcal{P}_k V), \quad \forall\, V \in \big(W^{(1)}\big)^2,\\
&U_1,\, U_2 \ \text{ continuous at } t_{n-1},\\
&U_1(\cdot, 0) = \boldsymbol{u}_h^0\,, \quad U_2(\cdot, 0) = \boldsymbol{v}_h^0\,,
\end{aligned}
\tag{4.2}
$$

where the linear form $\hat L : \mathcal{W} \to \mathbb{R}$ is defined by,

$$
\hat L(\boldsymbol{w}) = \sum_{n=1}^N \int_{I_n} \Big(a(\boldsymbol{f}_1, \boldsymbol{w}_1) + (\boldsymbol{f}_2, \boldsymbol{w}_2)\Big)\,dt.
$$

In the next theorem we prove an energy identity for problem (4.1) which will be used for proving the error estimates in the next section.

Theorem 3. *Let $U = (U_1, U_2)$ be a solution of (4.1). Then for any $l \in \mathbb{R}$, $T > 0$, we have the equality*

$$
\begin{aligned}
\rho |U_{2,N}|_{h,l}^2 &+ \tilde\xi(T) |U_{1,N}|_{h,l+1}^2 + \int_0^T \beta |U_1|_{h,l+1}^2\,dt\\
&+ \int_0^T \int_0^t \beta(t-s) D_t |W_1(t,s)|_{h,l+1}^2\,ds\,dt\\
&= \rho |\boldsymbol{v}_h^0|_{h,l}^2 + |\boldsymbol{u}_h^0|_{h,l+1}^2\\
&+ 2\int_0^T (P\boldsymbol{f}_2, A_h^l U_2)\,dt + 2\int_0^T \tilde\xi a(R\boldsymbol{f}_1, A_h^l U_1)\,dt\\
&+ 2\int_0^T \int_0^t \beta(t-s) a\big(R\boldsymbol{f}_1(t), A_h^l W_1(t,s)\big)\,ds\,dt,
\end{aligned}
\tag{4.3}
$$

where $W_1(t,s) = U_1(t) - U_1(s)$ and, recalling (1.4),

$$
\tilde\xi(t) = \xi(t) + 1 - \gamma = 1 - \int_0^t \beta(s)\,ds,
\tag{4.4}
$$

with $0 < 1 - \gamma < \tilde\xi(t) \le 1$. All terms on the left side of (4.3) are non-negative.

Proof. Throughout the proof we note that U_i $(i = 1, 2)$ are continuous, hence piecewise differentiable with respect to t. We organize our proof in 6 steps.

1. Expressing U_2 in terms of U_1. The first equation of (4.1) may be written as

$$
\int_{I_n} a(\mathcal{P}_{k,n} U_2, V_1)\,dt = \int_{I_n} a(\dot U_1 - R_n \boldsymbol{f}_1, V_1)\,dt, \quad \forall\, V_1 \in W_n^{(0)}.
$$

Therefore, we get

$$\mathcal{P}_k U_2 = \dot{U}_1 - R\boldsymbol{f}_1 \in W^{(0)}. \tag{4.5}$$

2. Using the calculation

$$\begin{aligned} U_1(t) - \int_0^t \beta(t-s)U_1(s)\,ds &= U_1(t) + \int_0^t \beta(t-s)\big(U_1(t) - U_1(s)\big)\,ds \\ &\quad - \int_0^t \beta(s)\,ds\,U_1(t) \\ &= \tilde{\xi}(t)U_1(t) + \int_0^t \beta(t-s)W_1(t,s)\,ds, \end{aligned}$$

and recalling the definitions of the P and P^{Γ_N} in (3.3) and the functions W_1 and $\tilde{\xi}$, we can write the second equation of (4.1) in the form

$$\begin{aligned} \int_0^T \Big(\rho(\dot{U}_2, V_2) + \tilde{\xi}a(U_1, V_2) + \int_0^t \beta(t-s)a\big(W_1(t,s), V_2(t)\big)\,ds\Big)\,dt \\ = \int_0^T (P_n\boldsymbol{f}_2, V_2)\,dt, \quad \forall V_2 \in W^{(0)}. \end{aligned}$$

Choosing $V_2 = A_h^l \mathcal{P}_k U_2$ gives us

$$\begin{aligned} \int_0^T \rho(\dot{U}_2, A_h^l\mathcal{P}_k U_2)\,dt + \int_0^T \tilde{\xi}a(U_1, A_h^l\mathcal{P}_k U_2)\,dt \\ + \int_0^T\int_0^t \beta(t-s)a\big(W_1(t,s), A_h^l\mathcal{P}_k U_2(t)\big)\,ds\,dt \\ = \int_0^T (P\boldsymbol{f}_2, A_h^l U_2)\,dt. \end{aligned} \tag{4.6}$$

We study the three terms in the left side of the above equation.

3. Using $\dot{U}_2 \in W^{(0)}$, by (3.5), we can write the first term of the left side of (4.6) as

$$\begin{aligned} \int_0^T \rho\big(\dot{U}_2, A_h^l\mathcal{P}_k U_2\big)\,dt &= \rho\int_0^T \big(\dot{U}_2, A_h^l U_2\big)\,dt = \frac{\rho}{2}\int_0^T D_t|U_2|_{h,l}^2\,dt \\ &= \frac{\rho}{2}\sum_{n=1}^N\Big(|U_{2,n}|_{h,l}^2 - |U_{2,n-1}|_{h,l}^2\Big) = \frac{\rho}{2}\Big(|U_{2,N}|_{h,l}^2 - |\boldsymbol{v}_h^0|_{h,l}^2\Big), \end{aligned}$$

where in the last equality we used the continuity of U_2.

4. With (4.5), we can write the second term as

$$\int_0^T \tilde{\xi}a\big(U_1, A_h^l\mathcal{P}_k U_2\big)\,dt = \frac{1}{2}\sum_{n=1}^N\int_{I_n}\tilde{\xi}D_t|U_1|_{h,l+1}^2\,dt - \int_0^T \tilde{\xi}a(U_1, A_h^l R\boldsymbol{f}_1)\,dt.$$

Then we integrate by parts in the first term of the right hand side and use the facts that $\dot{\tilde{\xi}}(t) = -\beta(t)$ and $\tilde{\xi}(0) = 1$, to get

$$\int_0^T \tilde{\xi} a\big(U_1, A_h^l \mathcal{P}_k U_2\big)\, dt = \frac{1}{2}\sum_{n=1}^N \Big(\tilde{\xi}(t_n)|U_{1,n}|_{h,l+1}^2 - \tilde{\xi}(t_{n-1})|U_{1,n-1}|_{h,l+1}^2\Big)$$
$$- \frac{1}{2}\sum_{n=1}^N \int_{I_n} \dot{\tilde{\xi}} |U_1|_{h,l+1}^2\, dt - \int_0^T \tilde{\xi} a(U_1, A_h^l R\boldsymbol{f}_1)\, dt$$
$$= \frac{1}{2}\Big(\tilde{\xi}(T)|U_{1,N}|_{h,l+1}^2 - |\boldsymbol{u}_h^0|_{h,l+1}^2\Big) + \frac{1}{2}\int_0^T \beta |U_1|_{h,l+1}^2\, dt - \int_0^T \tilde{\xi} a(R\boldsymbol{f}_1, A_h^l U_1)\, dt\,,$$

where we used the continuity of U_1.

5. Consider now the third term in the left hand side of (4.6). Using (4.5) and the fact that $\dot{U}_1(t) = D_t W(t,s)$ we have

$$\int_0^T\int_0^t \beta(t-s) a\big(W_1(t,s), A_h^l \mathcal{P}_k U_2\big)\, ds\, dt = \frac{1}{2}\int_0^T\int_0^t \beta(t-s) D_t |W_1(t,s)|_{h,l+1}^2 ds\, dt$$
$$- \int_0^T\int_0^t \beta(t-s) a\big(A_h^l W_1(t,s), R\boldsymbol{f}_1(t)\big)\, ds\, dt.$$

The first term on the right hand side is non-negative. To prove this, take $\epsilon \in (0,T)$. Then

$$\begin{aligned}
&\int_\epsilon^T\int_0^{t-\epsilon} \beta(t-s) D_t |W_1(t,s)|_{h,l+1}^2\, ds\, dt\\
&\quad= \int_0^{T-\epsilon}\int_{s+\epsilon}^T \beta(t-s) D_t |W_1(t,s)|_{h,l+1}^2 dt\, ds\\
&\quad= \int_0^{T-\epsilon} \beta(T-s)|W_1(T,s)|_{h,l+1}^2\, ds - \int_0^{T-\epsilon} \beta(\epsilon)|W_1(s+\epsilon,s)|_{h,l+1}^2\, ds\\
&\qquad - \int_0^{T-\epsilon}\int_{s+\epsilon}^T \dot{\beta}(t-s)|W_1(t,s)|_{h,l+1}^2 dt\, ds\\
&\quad\geq -\beta(\epsilon)\int_0^{T-\epsilon} |W_1(s+\epsilon,s)|_{h,l+1}^2\, ds\,,
\end{aligned} \tag{4.7}$$

where we used the facts that $\dot{\beta}(t) \leq 0$ and $\beta(t) \geq 0$ for the last inequality. Then using $W_1(s+\epsilon,s) = U_1(s+\epsilon) - U_1(s) = \int_s^{s+\epsilon} \dot{U}_1(t)\, dt$ we get

$$|W_1(s+\epsilon,s)|_{h,l+1}^2 \leq \Big(\int_s^{s+\epsilon} |\dot{U}_1(t)|_{h,l+1}\, dt\Big)^2 \leq \epsilon^2 \max_{0\leq t\leq T} |\dot{U}_1(t)|_{h,l+1}^2.$$

So (4.7) yields

$$\int_\epsilon^T\int_0^{t-\epsilon} \beta(t-s) D_t |W_1(t,s)|_{h,l+1}^2\, ds\, dt \geq -\epsilon^2 \beta(\epsilon) \max_{0\leq t\leq T} |\dot{U}_1(t)|_{h,l+1}^2.$$

Therefore, for a fixed mesh, we let $\epsilon \to 0$ and conclude

$$\int_0^T\int_0^t \beta(t-s) D_t |W_1(t,s)|_{h,l+1}^2\, ds\, dt \geq 0\,.$$

6. Putting the results from steps 3, 4, and 5 into (4.6) completes the proof. □

Remark 1. In [4] the auxiliary function $\boldsymbol{w}(t,s)=\boldsymbol{u}(t)-\boldsymbol{u}(t-s)$ was used in the same way as in our §2, to obtain stability estimates for the spatially semidiscrete finite element method. This does not work here because $s\mapsto U_1(t)-U_1(t-s)$ does not belong to $W^{(1)}$ if the temporal mesh is non-uniform.

We use (4.3) to obtain a stability estimate to be used in the error analysis. To this end, from (4.3), we have

$$\begin{aligned}\rho|U_{2,N}|_{h,l}^2+\tilde{\xi}(T)|U_{1,N}|_{h,l+1}^2 &\le \rho|\boldsymbol{v}_h^0|_{h,l}^2+|\boldsymbol{u}_h^0|_{h,l+1}^2+2\int_0^T(A_h^lP\boldsymbol{f}_2,U_2)\,dt\\ &\quad+2\int_0^T a(A_h^lR\boldsymbol{f}_1,U_1)\,dt\\ &\quad+2\int_0^T\int_0^t\beta(t-s)a\big(A_h^lR\boldsymbol{f}_1(t),W_1(t,s)\big)\,ds\,dt.\end{aligned}$$

Therefore using (3.4), $1-\gamma<\tilde{\xi}(t)\le 1$ and $\int_0^t\beta(s)\,ds\le\gamma$, we get

$$\begin{aligned}&\rho|U_{2,N}|_{h,l}^2+(1-\gamma)|U_{1,N}|_{h,l+1}^2\\ &\quad\le \rho|\boldsymbol{v}_h^0|_{h,l}^2+|\boldsymbol{u}_h^0|_{h,l+1}^2+2\int_0^T(A_h^{l/2}\mathcal{P}_k\mathcal{P}_h\boldsymbol{f}_2,A_h^{l/2}U_2)\,dt\\ &\qquad+2\int_0^T a(A_h^{l/2}\mathcal{P}_k\mathcal{R}_h\boldsymbol{f}_1,A_h^{l/2}U_1)\,dt\\ &\qquad+2\int_0^T\int_0^t\beta(t-s)a\big(A_h^{l/2}\mathcal{P}_k\mathcal{R}_h\boldsymbol{f}_1(t),A_h^{l/2}W_1(t,s)\big)\,ds\,dt\\ &\quad\le \rho|\boldsymbol{v}_h^0|_{h,l}^2+|\boldsymbol{u}_h^0|_{h,l+1}^2+2\int_0^T|\mathcal{P}_k\mathcal{P}_h\boldsymbol{f}_2|_{h,l}|U_2|_{h,l}\,dt\\ &\qquad+2\int_0^T|\mathcal{P}_k\mathcal{R}_h\boldsymbol{f}_1|_{h,l+1}|U_1|_{h,l+1}\,dt\\ &\qquad+2\gamma\int_0^T|\mathcal{P}_k\mathcal{R}_h\boldsymbol{f}_1(t)|_{h,l+1}\max_{0\le s\le T}|W_1(t,s)|_{h,l+1}\,dt\\ &\quad\le \rho|\boldsymbol{v}_h^0|_{h,l}^2+|\boldsymbol{u}_h^0|_{h,l+1}^2+\frac{1}{2}\rho\max_{[0,T]}|U_2|_{h,l}^2+C\Big(\int_0^T|\mathcal{P}_k\mathcal{P}_h\boldsymbol{f}_2|_{h,l}\,dt\Big)^2\\ &\qquad+\frac{1}{2}(1-\gamma)\max_{[0,T]}|U_1|_{h,l+1}^2+C\Big(\int_0^T|\mathcal{P}_k\mathcal{R}_h\boldsymbol{f}_1|_{h,l+1}\,dt\Big)^2\end{aligned}$$

where $C=C(\rho,\gamma)$. Using that, for piecewise linear functions, we have

$$\max_{[0,T]}|U_i|\le\max_{[0,T]}|U_{i,n}|,$$

and

$$\int_0^T|\mathcal{P}_k\boldsymbol{f}|\,dt\le\int_0^T|\boldsymbol{f}|\,dt,$$

and that the above inequality holds for arbitrary N, we conclude in a standard way

$$\begin{aligned}&|U_{2,N}|_{h,l}+|U_{1,N}|_{h,l+1}\\ &\quad\le C\bigg(|\boldsymbol{v}_h^0|_{h,l}+|\boldsymbol{u}_h^0|_{h,l+1}+\int_0^T\Big(|\mathcal{R}_h\boldsymbol{f}_1|_{h,l+1}+|\mathcal{P}_h\boldsymbol{f}_2|_{h,l}\Big)\,dt\bigg),\end{aligned}\tag{4.8}$$

with $C=C(\rho,\,\gamma)$.

5. A priori error estimates

To simplify the notation we denote the Sobolev norms $\|\cdot\|_{H^i(\Omega)}$ by $\|\cdot\|_i$. We define the standard interpolant $I_k\boldsymbol{v} \in W^{(1)}$ by

$$I_k\boldsymbol{v}(t_n) = \boldsymbol{v}(t_n), \quad n = 0, 1, \cdots, N. \tag{5.1}$$

By standard arguments in approximation theory we see that, for $q = 0, 1$,

$$\int_0^T \|I_k\boldsymbol{v} - \boldsymbol{v}\|_i \, dt \le Ck^{q+1} \int_0^T \|D_t^{q+1}\boldsymbol{v}\|_i \, dt, \quad \text{for } i = 0, 1, 2, \tag{5.2}$$

where $k = \max_{1\le n\le N} k_n$.

We assume the elliptic regularity estimate $\|\boldsymbol{v}\|_2 \le C\|A\boldsymbol{v}\|, \ \forall\, \boldsymbol{v} \in \mathcal{D}(A)$, so that the following error estimates for the Ritz projection (3.2), hold true

$$\|\mathcal{R}_h v - v\| \le Ch^s\|v\|_s, \quad \forall \boldsymbol{v} \in H^s \cap V, \quad s = 1, 2. \tag{5.3}$$

Hence, as in §2.2, we must specialize to the pure Dirichlet boundary condition and a convex polygonal domain. We note that the energy norm $\|\cdot\|_V$ is equinalent to $\|\cdot\|_1$ on V.

Theorem 4. *Assume that $\Gamma_N = \varnothing$, Ω is a convex polygonal domain, and $V_{n-1} \subset V_n$, $n = 2, \dots, N$. Let $\boldsymbol{u}$ and U be the solutions of* (1.7) *and* (4.1)*. Then, with $\boldsymbol{e} = U - \boldsymbol{u}$ and $C = C(\rho, \gamma)$, we have*

$$\begin{aligned}
\|\boldsymbol{e}_{2,N}\| &\le Ch^2\Big(\|\boldsymbol{v}^0\|_2 + \|\boldsymbol{u}_{2,N}\|_2 + \int_0^T \|\dot{\boldsymbol{u}}_2\|_2 \, dt\Big) + Ck^2\int_0^T \big(\|\ddot{\boldsymbol{u}}_2\|_1 + \|\ddot{\boldsymbol{u}}_1\|_2\big)\, dt,\\
\|\boldsymbol{e}_{1,N}\|_1 &\le Ch\Big(\|\boldsymbol{u}_{1,N}\|_2 + \|\boldsymbol{v}^0\|_1 + \int_0^T \|\dot{\boldsymbol{u}}_2\|_1 \, dt\Big) + Ck^2\int_0^T \big(\|\ddot{\boldsymbol{u}}_2\|_1 + \|\ddot{\boldsymbol{u}}_1\|_2\big)\, dt,\\
\|\boldsymbol{e}_{1,N}\| &\le Ch^2\Big(\|\boldsymbol{u}_{1,N}\|_2 + \int_0^T \|\boldsymbol{u}_2\|_2 \, dt\Big) + Ck^2\int_0^T \big(\|\ddot{\boldsymbol{u}}_2\| + \|\ddot{\boldsymbol{u}}_1\|_1\big)\, dt.
\end{aligned}$$

Proof. We set

$$\boldsymbol{e} = \boldsymbol{\theta} + \boldsymbol{\eta} + \boldsymbol{\rho} = (U - \pi\boldsymbol{u}) + (\pi\boldsymbol{u} - J\boldsymbol{u}) + (J\boldsymbol{u} - \boldsymbol{u}), \tag{5.4}$$

for some suitable operators π and J which will be specified in terms of the time interpolant I_k in (5.1) and projectors $\mathcal{R}_h$ and $\mathcal{P}_h$ in (3.2), so that $\pi\boldsymbol{u} \in W^{(1)}$ and $\boldsymbol{\eta}$ and $\boldsymbol{\rho}$ will correspond to the temporal and spatial errors, respectively. Due to (5.2)–(5.3) we just need to estimate $\boldsymbol{\theta}$. To this end, using the Galerkin orthogonality

(3.7) and the definition of $\boldsymbol{\theta}$, we get

$$\begin{aligned}
B(\boldsymbol{\theta},\mathcal{P}_kV) &= -B(\boldsymbol{\eta},\mathcal{P}_kV) - B(\boldsymbol{\rho},\mathcal{P}_kV)\\
&= \int_0^T a(\boldsymbol{\eta}_2,\mathcal{P}_kV_1) - a(\dot{\boldsymbol{\eta}}_1,\mathcal{P}_kV_1) - \rho(\dot{\boldsymbol{\eta}}_2,\mathcal{P}_kV_2) - a(\boldsymbol{\eta}_1,\mathcal{P}_kV_2)\,dt\\
&\quad + \int_0^T\int_0^t \beta(t-s)a\big(\boldsymbol{\eta}_1(s),\mathcal{P}_kV_2(t)\big)\,ds\,dt\\
&\quad + \int_0^T a(\boldsymbol{\rho}_2,\mathcal{P}_kV_1) - a(\dot{\boldsymbol{\rho}}_1,\mathcal{P}_kV_1) - \rho(\dot{\boldsymbol{\rho}}_2,\mathcal{P}_kV_2) - a(\boldsymbol{\rho}_1,\mathcal{P}_kV_2)\,dt\\
&\quad + \int_0^T\int_0^t \beta(t-s)a\big(\boldsymbol{\rho}_1(s),\mathcal{P}_kV_2(t)\big)\,ds\,dt\\
&= \sum_{j=1}^{10} E_j, \quad \forall V \in \big(W^{(1)}\big)^2.
\end{aligned} \tag{5.5}$$

We consider two different choices of the operators π and J. In order to prove the first two error estimates we set, for $i = 1, 2$,

$$\boldsymbol{\theta}_i = U_i - I_k\mathcal{R}_h\boldsymbol{u}_i, \quad \boldsymbol{\eta}_i = (I_k - I)\mathcal{R}_h\boldsymbol{u}_i, \quad \boldsymbol{\rho}_i = (\mathcal{R}_h - I)\boldsymbol{u}_i.$$

Integrating by parts in E_2 and E_3 with respect to time and using (5.1) we have for both cases

$$E_2 = E_3 = 0. \tag{5.6}$$

Moreover, by the definitions of $\boldsymbol{\eta}$ and $\boldsymbol{\rho}$, we have

$$E_6 = E_7 = E_9 = E_{10} = 0.$$

Therefore,

$$\begin{aligned}
B(\boldsymbol{\theta},\mathcal{P}_kV) &= \int_0^T a(\boldsymbol{\eta}_2,\mathcal{P}_kV_1)\,dt\\
&\quad + \int_0^T \Big(a\big(-\boldsymbol{\eta}_1 + \int_0^t \beta(t-s)\boldsymbol{\eta}_1(s)ds, \mathcal{P}_kV_2\big) - \rho\big(\dot{\boldsymbol{\rho}}_2,\mathcal{P}_kV_2\big)\Big)\,dt\\
&= \hat{L}(\mathcal{P}_kV), \quad \forall V \in \big(W^{(1)}\big)^2,
\end{aligned}$$

which is of the form (4.2) with $\boldsymbol{f}_1 = \boldsymbol{\eta}_2$, $\boldsymbol{f}_2 = A_h\big(-\boldsymbol{\eta}_1 + \int_0^t \beta(t-s)\boldsymbol{\eta}_1(s)\,ds\big) - \rho\dot{\boldsymbol{\rho}}_2$.

Applying the stability inequality (4.8) with $l = 0$, and considering the fact that $|\cdot|_{0,h} = \|\cdot\|$ and $|\cdot|_{h,1} = \|\cdot\|_1$, we have

$$\begin{aligned}
\|\boldsymbol{\theta}_{2,N}\| + \|\boldsymbol{\theta}_{1,N}\|_1 &\le C\Big(\|\boldsymbol{\theta}_2(0)\| + \|\boldsymbol{\theta}_1(0)\|_1\Big) + C\int_0^T \|\mathcal{R}_h\boldsymbol{\eta}_2\|_1\,dt\\
&\quad + C\int_0^T\Big(\|\mathcal{P}_hA_h\boldsymbol{\eta}_1\| + \Big\|\mathcal{P}_hA_h\int_0^t \beta(t-s)\boldsymbol{\eta}_1(s)\,ds\Big\| + \rho\|\mathcal{P}_h\dot{\boldsymbol{\rho}}_2\|\Big)\,dt,
\end{aligned}$$

where $\boldsymbol{\theta}_1(0) = 0$, since $U_1(0) = \mathcal{R}_{h,1}\boldsymbol{u}^0$. Since $\|\mathcal{R}_h\boldsymbol{v}\|_1 \le C\|\boldsymbol{v}\|_1$, $\|\mathcal{P}_h\boldsymbol{v}\| \le \|\boldsymbol{v}\|$, $\forall \boldsymbol{v} \in V$ and $A_h\mathcal{R}_h = \mathcal{P}_hA$, we have

$$\begin{aligned}
&\|\mathcal{R}_h\boldsymbol{\eta}_2\|_1 = \|(I_k - I)\mathcal{R}_h\boldsymbol{u}_2\|_1 \le C\|(I_k - I)\boldsymbol{u}_2\|_1,\\
&\|\mathcal{P}_hA_h\boldsymbol{\eta}_1\| = \|A_h\boldsymbol{\eta}_1\| = \|(I_k - I)A_h\mathcal{R}_h\boldsymbol{u}_1\| = \|(I_k - I)\mathcal{P}_hA\boldsymbol{u}_1\|\\
&\qquad \le \|(I_k - I)A\boldsymbol{u}_1\| \le C\|(I_k - I)\boldsymbol{u}_1\|_2,
\end{aligned}$$

and

$$\begin{aligned}\int_0^T \Big\| \mathcal{P}_h A_h \int_0^t \beta(t-s) \boldsymbol{\eta}_1(s)\,ds \Big\|\,dt &\le \int_0^T \Big\| A_h \int_0^t \beta(t-s)\boldsymbol{\eta}_1(s)\,ds \Big\|\,dt \\ &\le C \int_0^T \int_0^t \beta(t-s) \|(I_k - I)\boldsymbol{u}_1(s)\|_2\,ds\,dt \\ &\le C\gamma \int_0^T \|(I_k - I)\boldsymbol{u}_1\|_2\,dt.\end{aligned}$$

Therefore by $\boldsymbol{\theta} = \boldsymbol{e} - \boldsymbol{\eta} - \boldsymbol{\rho}$, $\boldsymbol{\eta}(t_n) = 0$ and $\boldsymbol{\theta}_1(0) = 0$, we get

$$\begin{aligned}\|\boldsymbol{e}_{2,N}\| &\le \|\boldsymbol{\rho}_{2,N}\| + C\|\boldsymbol{\theta}_2(0)\| \\ &\quad + C \int_0^T \Big(\|(I_k - I)\boldsymbol{u}_2\|_1 + \|(I_k - I)\boldsymbol{u}_1\|_2 + \|(\mathcal{R}_h - I)\dot{\boldsymbol{u}}_2\| \Big)\,dt, \\ \|\boldsymbol{e}_{1,N}\|_1 &\le \|\boldsymbol{\rho}_{1,N}\|_1 + C\|\boldsymbol{\theta}_2(0)\| \\ &\quad + C \int_0^T \Big(\|(I_k - I)\boldsymbol{u}_2\|_1 + \|(I_k - I)\boldsymbol{u}_1\|_2 + \|(\mathcal{R}_h - I)\dot{\boldsymbol{u}}_2\| \Big)\,dt,\end{aligned}$$

which implies the first two estimates by (5.2) and (5.3).

Finally, we choose

$$\begin{aligned}\boldsymbol{\theta}_1 &= U_1 - I_k \mathcal{R}_h \boldsymbol{u}_1, \quad \boldsymbol{\eta}_1 = (I_k - I)\mathcal{R}_h \boldsymbol{u}_1, \quad \boldsymbol{\rho}_1 = (\mathcal{R}_h - I)\boldsymbol{u}_1, \\ \boldsymbol{\theta}_2 &= U_2 - I_k \mathcal{P}_h \boldsymbol{u}_2, \quad \boldsymbol{\eta}_2 = (I_k - I)\mathcal{P}_h \boldsymbol{u}_2, \quad \boldsymbol{\rho}_2 = (\mathcal{P}_h - I)\boldsymbol{u}_2.\end{aligned}$$

By the definitions of $\mathcal{R}_h$ and $\mathcal{P}_h$ in (3.2), this implies

$$E_7 = E_8 = E_9 = E_{10} = 0,$$

and we still have (5.6). Therefore, (5.5) becomes

$$\begin{aligned}B(\boldsymbol{\theta}, \mathcal{P}_k V) &= \int_0^T a(\boldsymbol{\eta}_2 + \boldsymbol{\rho}_2, \mathcal{P}_k V_1)\,dt \\ &\quad + \int_0^T a\big(-\boldsymbol{\eta}_1 + \int_0^t \beta(t-s)\boldsymbol{\eta}_1(s) ds, \mathcal{P}_k V_2 \big)\,dt \\ &= \hat{L}(\mathcal{P}_k V), \quad \forall V \in \big(W^{(1)}\big)^2,\end{aligned}$$

which is of the form (4.2) with $\boldsymbol{f}_1 = \boldsymbol{\eta}_2 + \boldsymbol{\rho}_2$, $\boldsymbol{f}_2 = A_h\big(-\boldsymbol{\eta}_1 + \int_0^t \beta(t-s)\boldsymbol{\eta}_1(s)\,ds \big)$.

Again applying the stability inequality (4.8), this time with $l = -1$, and using $|\cdot|_{h,0} = \|\cdot\|$, we have

$$\begin{aligned}\|\boldsymbol{\theta}_{1,N}\| &\le C \int_0^T \Big(\|\mathcal{R}_h \boldsymbol{\eta}_2\| + \|\mathcal{R}_h \boldsymbol{\rho}_2\| \Big)\,dt \\ &\quad + C \int_0^T \Big(|\mathcal{P}_h A_h \boldsymbol{\eta}_1|_{h,-1} + |\mathcal{P}_h A_h \int_0^t \beta(t-s)\boldsymbol{\eta}_1(s) ds|_{h,-1} \Big)\,dt,\end{aligned}$$

where we used that $\boldsymbol{\theta}(0) = 0$, since $U_1(0) = \mathcal{R}_{h,1}\boldsymbol{u}^0$ and $U_2(0) = \mathcal{P}_{h,1}\boldsymbol{v}^0$. Then, since

$$\begin{aligned}&\|\mathcal{R}_h \boldsymbol{\eta}_2\| = \|(I_k - I)\mathcal{P}_h \boldsymbol{u}_2\| \le \|(I_k - I)\boldsymbol{u}_2\|, \\ &\|\mathcal{R}_h \boldsymbol{\rho}_2\| = \|\mathcal{P}_h (I - \mathcal{R}_h)\boldsymbol{u}_2\| \le \|(\mathcal{R}_h - I)\boldsymbol{u}_2\|, \\ &|\mathcal{P}_h A_h \boldsymbol{\eta}_1|_{h,-1} = |A_h \mathcal{R}_h (I_k - I)\boldsymbol{u}_1|_{h,-1} = |\mathcal{R}_h (I_k - I)\boldsymbol{u}_1|_{h,1} \le C\|(I_k - I)\boldsymbol{u}_1\|_1\,,\end{aligned}$$

and

$$\int_0^T |\mathcal{P}_h A_h \int_0^t \beta(t-s)\boldsymbol{\eta}_1(s)\,ds|_{h,-1}\,dt \leq C\int_0^T\int_0^t \beta(t-s)\|(I_k-I)\boldsymbol{u}_1(s)\|_1\,ds\,dt$$
$$\leq C\,\gamma\int_0^T \|(I_k-I)\boldsymbol{u}_1\|_1\,dt,$$

we conclude

$$\|\boldsymbol{e}_{1,N}\| \leq \|\boldsymbol{\rho}_{1,N}\| + C\int_0^T \Big(\|(I_k-I)\boldsymbol{u}_2\| + \|(\mathcal{R}_h-I)\boldsymbol{u}_2\| + \|(I_k-I)\boldsymbol{u}_1\|_1\Big)\,dt,$$

which implies the last estimate by (5.2) and (5.3). □

6. Numerical example

In this section we illustrate the numerical method by solving a simple but realistic example for a two dimensional structure, see Figure 1 (a), using piecewise linear polynomials. This shows that the model captures the mechanical behaviour of the material.

We consider the initial conditions: $\boldsymbol{u}(x,0)=0\,\mathrm{m}$, $\dot{\boldsymbol{u}}(x,0)=0\,\mathrm{m/s}$, the boundary conditions: $\boldsymbol{u}=0$ at $x=0$, $\boldsymbol{g}=(0,-1)\,\mathrm{Pa}$ at $x=1.5$ and zero on the rest of the boundary. The volume load is assumed to be $\boldsymbol{f}=0\,\mathrm{N/m^3}$. And the model parameters are: $\gamma=0.5$, $\tau=0.25$, $\nu=0.3$, $E=5\mathrm{MPa}$ and $\rho=7000\,\mathrm{kg/m^3}$. The deformed mesh at $t/\tau=9$ for $\alpha=1/2$ is displayed in Figure 1 (b), with the displacement magnified by the factor 10^5, and the computed vertical displacement at the point $(1.5,1.5)$ for different α is shown in Figure 2. We note that for small α there is less damping, that is what we expect, since in the limit $\alpha=0$ there is no convolution term in the model. While at the other limit $\alpha=1$ we expect strong damping, since the kernel β with $\alpha=1$ is an exponential function, see (1.3).

We also verify numerically the temporal rate of convergence $O(k^2)$ for $\|e_{1,N}\|$. In the lack of an explicit solution we compare with a numerical solution with fine mesh sizes h, k. Here we consider $h=0.0223$, $k_{\min}=0.0266$, $\alpha=1/2$, $\tau=1/4$. The result is displayed in Figure 3.

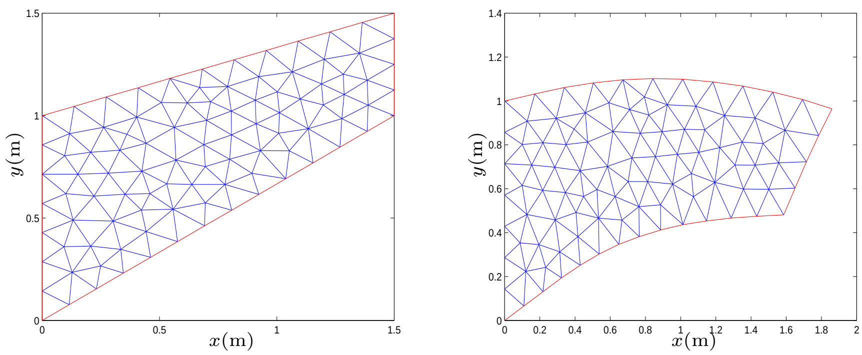

FIGURE 1. (a) Undeformed mesh. (b) Deformed mesh at $t/\tau=9$ for $\alpha=1/2$.

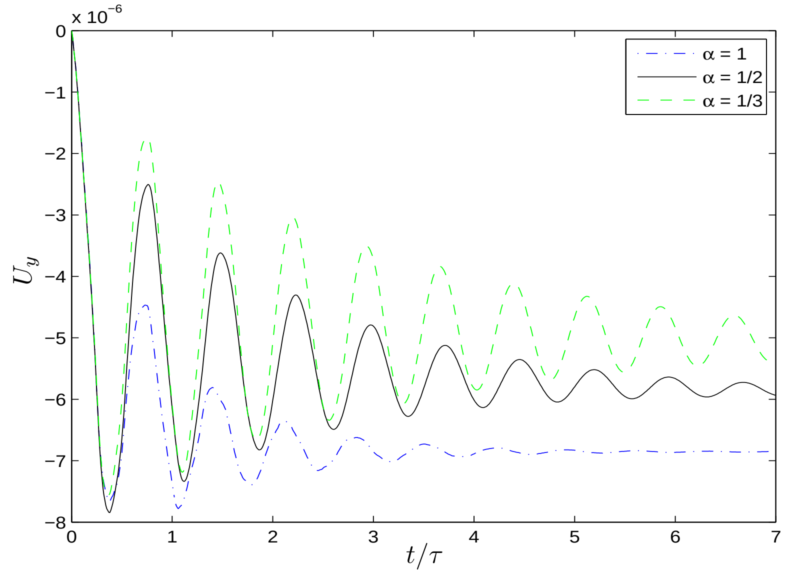

FIGURE 2. Vertical displacement for different α.

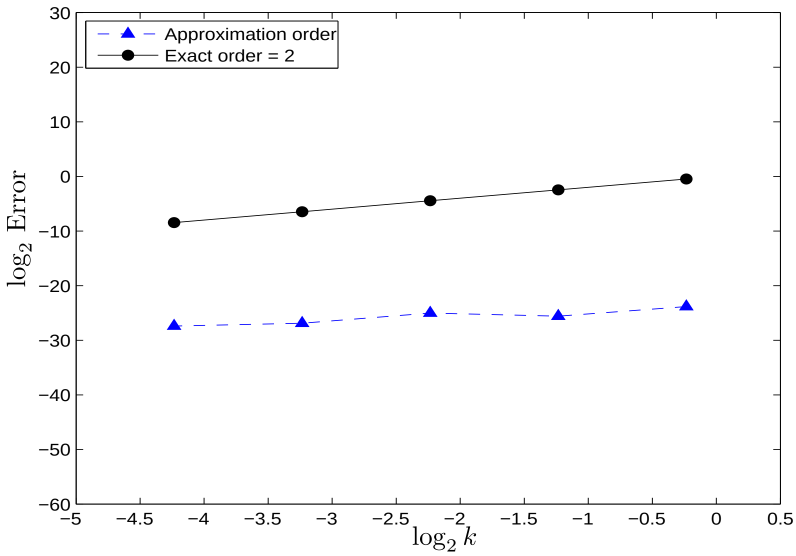

FIGURE 3. Convergence order for time discretization.

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