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Partial Differential Equations with Numerical Methods

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Preface

Our purpose in this book is to give an elementary, relatively short, and hopefully readable account of the basic types of linear partial differential equations and their properties, together with the most commonly used methods for their numerical solution. Our approach is to integrate the mathematical analysis of the differential equations with the corresponding numerical analysis. For the mathematician interested in partial differential equations or the person using such equations in the modelling of physical problems, it is important to realize that numerical methods are normally needed to find actual values of the solutions, and for the numerical analyst it is essential to be aware that numerical methods can only be designed, analyzed, and understood with sufficient knowledge of the theory of the differential equations, using discrete analogues of properties of these.

In our presentation we study the three major types of linear partial differential equations, namely elliptic, parabolic, and hyperbolic equations, and for each of these types of equations the text contains three chapters. In the first of these we introduce basic mathematical properties of the differential equation, and discuss existence, uniqueness, stability, and regularity of solutions of the various boundary value problems, and the remaining two chapters are devoted to the most important and widely used classes of numerical methods, namely finite difference methods and finite element methods.

Historically, finite difference methods were the first to be developed and applied. These are normally defined by looking for an approximate solution on a uniform mesh of points and by replacing the derivatives in the differential equation by difference quotients at the mesh-points. Finite element methods are based instead on variational formulations of the differential equations and determine approximate solutions that are piecewise polynomials on some partition of the domain under consideration. The former method is somewhat restricted by the difficulty of adapting the mesh to a general domain whereas the latter is more naturally suited for a general geometry. Finite element methods have become most popular for elliptic and also for parabolic problems, whereas for hyperbolic equations the finite difference method continues to dominate. In spite of the somewhat different philosophy underlying the two classes it is more reasonable in our view to consider the latter as further developments of the former rather than as competitors, and we feel that the practitioner of differential equations should be familiar with both.

To make the presentation more easily accessible, the elliptic chapters are preceded by a chapter about the two-point boundary value problem for a second order ordinary differential equation, and those on parabolic and hyperbolic evolution equations by a short chapter about the initial value problem for a system of ordinary differential equations. We also include a chapter about eigenvalue problems and eigenfunction expansion, which is an important tool in the analysis of partial differential equations. There we also give some simple examples of numerical solution of eigenvalue problems.

The last chapter provides a short survey of other classes of numerical methods of importance, namely collocation methods, finite volume methods, spectral methods, and boundary element methods.

The presentation does not presume a deep knowledge of mathematical and functional analysis. In an appendix we collect some of the basic material that we need in these areas, mostly without proofs, such as elements of abstract linear spaces and function spaces, in particular Sobolev spaces, together with basic facts about Fourier transforms. In the implementation of numerical methods it will normally be necessary to solve large systems of linear algebraic equations, and these generally have to be solved by iterative methods. In a second appendix we therefore include an orientation about such methods.

Our purpose has thus been to cover a rather wide variety of topics, notions, and ideas, rather than to expound on the most general and far-reaching results or to go deeply into any one type of application. In the problem sections, which end the various chapters, we sometimes ask the reader to prove some results which are only stated in the text, and also to further develop some of the ideas presented. In some problems we propose testing some of the numerical methods on the computer, assuming that MATLAB or some similar software is available. At the end of the book we list a number of standard references where more material and more detail can be found, including issues concerned with implementation of the numerical methods.

This book has developed from courses that we have given over a rather long period of time at Chalmers University of Technology and Göteborg University originally for third year engineering students but later also in beginning graduate courses for applied mathematics students. We would like to thank the many students in these courses for the opportunities for us to test our ideas.

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