Noname manuscript No.
(will be inserted by the editor)

Strong convergence of the finite element method with truncated noise for semilinear parabolic stochastic equations with additive noise

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Received: date / Accepted: date

Abstract We consider a semilinear parabolic PDE driven by additive noise. The equation is discretized in space by a standard piecewise linear finite element method. We show that the orthogonal expansion of the finite-dimensional Wiener process, that appears in the discretized problem, can be truncated severely without losing the asymptotic order of the method, provided that the kernel of the covariance operator of the Wiener process is smooth enough. For example, if the covariance operator is given by the Gauss kernel, then the number of terms to be kept is the quasi-logarithm of the number of terms in the original expansion. Then one can reduce the size of the corresponding linear algebra problem enormously and hence reduce the computational complexity, which is a key issue when stochastic problems are simulated.

Keywords finite element · semilinear parabolic equation · Wiener process · error estimate · stochastic partial differential equation · truncation

M. Kovács was partially supported by the Swedish Research Council (VR). S. Larsson was partially supported by the Swedish Foundation for Strategic Research through GMMC, the Gothenburg Mathematical Modeling Centre and the Swedish Research Council (VR).

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Mathematics Subject Classification (2000) 65N30 · 60H35 · 60H15 · 35R60

1 Introduction

Let $D \subset \mathbf{R}^d$ be a convex polygonal domain, $H := L^2(D)$ with scalar product $\langle \cdot, \cdot \rangle$ and $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space with a normal filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Let $\{W(t)\}_{t \geq 0}$ be an H-valued Wiener process with covariance operator Q with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ (see section 2). Consider the parabolic stochastic partial differential equation written in the abstract form

$$\mathrm{d}X(t) + AX(t)\,\mathrm{d}t = f(X(t))\,\mathrm{d}t + \mathrm{d}W(t),\ t \in (0, T); \quad X(0) = X_0, \tag{1.1}$$

where A is a second order, symmetric, uniformly elliptic partial differential operator considered as an unbounded operator $A : \mathcal{D}(A) \subset H \to H$ with $\mathcal{D}(A) = H^2(D) \cap H_0^1(D)$. It is known that $-A$ is then the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$ on H. We take X_0 to be an $\mathcal{F}_0$-measurable H-valued random variable and assume that $f : H \to H$ satisfies the global Lipschitz condition

$$\|f(x) - f(y)\| \leq L_f \|x - y\|, \quad x, y \in H. \tag{1.2}$$

We say that a process $\{X(t)\}_{t \geq 0}$ is a *mild solution* of (1.1) if it is mean-square continuous, adapted to $\{\mathcal{F}_t\}_{t \geq 0}$ and for all $t \geq 0$ satisfies the integral equation

$$X(t) = T(t)X_0 + \int_0^t T(t-s)f(X(s))\,\mathrm{d}s + \int_0^t T(t-s)\,\mathrm{d}W(s),\ P-a.s., \tag{1.3}$$

where the last integral is an Itô integral (see, for example, [2, Chapters 4 and 7]).

Let $\{S_h\}_{0<h<1}$ be a family of finite-dimensional subspaces of $H_0^1(D)$ consisting of continuous piecewise linear functions with respect to a regular family of triangulations $\{\mathcal{T}_h\}_{0<h<1}$ with maximum mesh size h. Let P_h denote the orthogonal projection of H onto S_h, i.e.,

$$\langle P_h f, \chi \rangle = \langle f, \chi \rangle, \quad \chi \in S_h,\ f \in H.$$

Let $A_h : S_h \to S_h$ denote the "discrete version" of A, i.e.,

$$\langle A_h \eta, \chi \rangle = a(\eta, \chi), \quad \eta, \chi \in S_h,$$

where $a : H_0^1(D) \times H_0^1(D) \to \mathbf{R}$ is the bilinear form corresponding to A. The finite element solution $\{X_h(t)\}_{t \geq 0} \in S_h$ of (1.1) satisfies the stochastic differential equation written in the abstract form

$$\mathrm{d}X_h + A_h X_h\,\mathrm{d}t = P_h f(X_h)\,\mathrm{d}t + P_h\,\mathrm{d}W,\ t \in (0, T); \quad X_h(0) = P_h X_0, \tag{1.4}$$

with its mild solution given by, analogously to (1.3),

$$X_h(t) = T_h(t)X_h(0) + \int_0^t T_h(t-s)P_h f(X_h(s))\,\mathrm{d}s + \int_0^t T_h(t-s)P_h\,\mathrm{d}W(s), \tag{1.5}$$

where $\{T_h(t)\}_{t\geq 0}$ is the analytic semigroup on S_h generated by $-A_h$.

In practice one also applies a time stepping when solving (1.4). This requires the computation of the increments of the Wiener process $\{W(t)\}_{t\geq 0}$, which is given by an orthogonal series

$$W(t) = \sum_{k=1}^{\infty} \gamma_k^{1/2}\beta_k(t)e_k,$$

where (γ_k, e_k) are the eigenpairs of the covariance operator Q and β_k are mutually independent standard real-valued Brownian motions. The problem that arises then is that the eigenvectors Q are usually not known and, even when they are, one has to compute an infinite series. Yan [12] showed that when the eigenvalues and eigenvectors are explicitly known, it is enough to take $N_h = \dim(S_h)$ terms in the orthogonal expansion of W and use the truncated series in (1.4) and still preserve the order of convergence of the finite element method. However, the eigenfunctions and eigenvectors of Q are rarely available explicitly. Therefore, instead, observe that (1.4) and (1.5) are equivalent to

$$\mathrm{d}X_h + A_h X_h \mathrm{d}t = P_h f(X_h)\mathrm{d}t + \mathrm{d}W_h,\ t \in (0,T); \quad X_h(0) = P_h X_0, \tag{1.6}$$

and

$$X_h(t) = T_h(t)X_h(0) + \int_0^t T_h(t-s)P_h f(X_h(s))\,\mathrm{d}s + \int_0^t T_h(t-s)\,\mathrm{d}W_h(s), \tag{1.7}$$

respectively, where $\{W_h(t)\}_{t\geq 0}$ is a P_hQP_h-Wiener process on S_h. Then we may write

$$W_h(t) = \sum_{k=1}^{N_h} \gamma_{h,k}^{1/2}\beta_k(t)e_{h,k}, \tag{1.8}$$

where the β_k are mutually independent standard real-valued Brownian motions and $(\gamma_{h,k}, e_{h,k})$ are the eigenpairs of $Q_h := P_hQP_h$. In other words, $(\gamma_{h,k}, e_{h,k})$ is the finite element solution of the eigenvalue problem $Qu = \gamma u$. Usually Q is given as an integral operator and therefore the finite element solution of the eigenvalue problem can be very expensive. However, in [7] (in the context of random fields) it is shown that, e.g., for stationary kernels analytic at 0, such as the Gauss kernel, this in fact can be done very efficiently, i.e., in log-linear complexity. Moreover, the number of terms used in (1.8) can be dramatically reduced; by say taking $M \leq N_h$ terms, (M depends on the regularity of the kernel and on h), and defining

$$W_h^M(t) = \sum_{k=1}^{M} \gamma_{h,k}^{1/2}\beta_k(t)e_{h,k},$$

the strong error estimate; that is, the error estimate in the $\|\cdot\|_{L_2(\Omega,H)} = \big(\mathbf{E}(\|\cdot\|^2)\big)^{1/2}$-norm, for $W(t)-W_h(t)$ carries over to $W(t)-W_h^M(t)$.

The main goal of the paper is to show that a similar statement holds in the context of parabolic equations (1.1) for piecewise linear finite elements; i.e., denoting the solution of

$$\mathrm{d}X_h^M + A_h X_h^M \mathrm{d}t = P_h f(X_h)\,\mathrm{d}t + \mathrm{d}\,W_h^M,\ t\in(0,T);\quad X_h^M(0) = P_h X_0, \tag{1.9}$$

by $\{X_h^M(t)\}_{t\geq 0}$, we show that the strong error estimate for $X(t)-X_h(t)$ remains valid for $X(t)-X_h^M(t)$, where $M = M(Q,h) \leq N_h$.

The paper is organized as follows. In Section 2 we discuss some basic notions of infinite dimensional stochastic analysis and review some standard material form operator semigroups and finite element analysis. In Section 3 we first extend, in Proposition 3.3, the strong convergence error estimate for linear problems, [11], to the case of the semilinear finite element problem (1.4). More importantly, we also show, in Theorem 3.4 and Corollary 3.5, that for covariance operators with fast decaying eigenvalues the order of the method is preserved when the expansion of the semidiscrete Wiener process $\{W_h(t)\}_{t\geq 0}$ is truncated. As in the context of random fields [7], for piecewise analytic kernels we may keep only $M = c(\ln N_h)^d$ terms instead of N_h terms and retain the order of convergence of the original approximation.

2 Preliminaries

First we discuss H-valued Wiener processes. We say that $\{W(t)\}_{t\geq 0}$ is a *H-valued Wiener process with covariance operator Q with respect to* $\{\mathcal{F}_t\}_{t\geq 0}$ if

(i) $W(0)=0$,
(ii) W has continuous trajectories (almost surely),
(iii) W has independent increments,
(iv) $W(t)-W(s)$, $0\leq s\leq t$, is a H-valued Gaussian random variable with zero mean and covariance operator $(t-s)Q$,

and

(v) $\{W(t)\}_{t\geq 0}$ is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$; that is, $W(t)$ is $\mathcal{F}_t$ measurable for all $t\geq 0$;
(vi) the random variable $W(t)-W(s)$ is independent of $\mathcal{F}_s$ for all fixed $s\in[0,t]$.

Condition (iv) implies that the trace of Q is finite as the covariance operator of a Gaussian random variable is necessarily of trace class, see [2, Proposition 2.15]. Then Q has a decreasing sequence $\gamma_k \to 0$ as $k\to\infty$ of eigenvalues with corresponding orthonormal eigenvectors e_k. In this case $\{W(t)\}_{t\geq 0}$ can be written as an orthogonal series

$$W(t) = \sum_{k=1}^{\infty} \gamma_k^{1/2}\beta_k(t)e_k, \tag{2.1}$$

where (γ_k, e_k) are the eigenpairs of Q and β_k are mutually independent standard real-valued Brownian motions. Furthermore, the series converges in the space $L_2\big(\Omega, \mathcal{F}, P; C\big([0,T], H\big)\big)$ (see, for example, [2, Chapter 4] and [6, Chapter 2]). We remark that one is often interested in covariance operators which are not of trace class, such as $Q = I$. In this case one starts with the expansion (2.1), which only converges in $L_2\big(\Omega, \mathcal{F}, P; C\big([0,T], H_1\big)\big)$, where H_1 is a suitably chosen Hilbert space (usually larger than H), and obtains an H_1-valued process which we still call an H-valued (cylindrical) Wiener process even if it is strictly speaking not H-valued. Nevertheless one can still define Itô-type stochastic integrals of $\mathcal{B}(H)$-valued processes, where $\mathcal{B}(H)$ denote the Banach algebra of bounded linear operators on H with the usual norm, with respect to these Wiener processes as well. Since the main results of the paper requires smooth noise; that is, we look at covariance operators with fast decaying eigenvalues, we refer the interested reader to [2, Chapter 4.3] and [6, Chapter 2.5] for furher reading on cylindrical process and integrals with respect to them.

Next we briefly recall some standard results from deterministic finite element and operator semigroup theory. Note that in the rest of the paper the letter C is used to denote various positive constants that need not be the same at each time. It is well known that under our assumption on A we have

$$\begin{aligned} \int_0^t \|T(s)v\|^2 \,\mathrm{d}s &\le \tfrac{1}{2}\|A^{-1/2}v\|^2, \quad \text{for all } v \in H,\, t \ge 0, \\ \int_0^t \|T_h(s)P_h v\|^2 \,\mathrm{d}s &\le \tfrac{1}{2}\|A_h^{-1/2}P_h v\|^2, \quad \text{for all } v \in H,\, t \ge 0. \end{aligned} \tag{2.2}$$

We introduce spaces and norms of fractional order $\beta \in \mathbf{R}$:

$$\dot H^\beta = \mathcal{D}(A^{\beta/2}), \quad \|v\|_{\dot H^\beta} = \|A^{\beta/2}v\| = \Big(\sum_{k=1}^\infty \lambda_k^\beta \langle v, \phi_k\rangle^2\Big)^{1/2},$$

where (λ_k, ϕ_k) denote the eigenpairs of A. Under the assumptions made in the introduction we have the following error estimate for the deterministic parabolic finite element problem

$$\|F_h(t)v\| \le Ch^\beta t^{-\beta/2}\|v\|, \quad 0 \le \beta \le 2; \quad \text{with } F_h(t) = T(t) - T_h(t)P_h, \tag{2.3}$$

see [8, Theorem 3.5], and the stability estimate for the elliptic problem

$$\|A_h^{-1}P_h v\| \le C\|A^{-1}v\|, \quad v \in H,\ h > 0. \tag{2.4}$$

In particular, it follows from (2.3) that

$$\|T_h(t)P_h\| \le C \text{ for all } t \ge 0 \text{ and } h > 0. \tag{2.5}$$

Finally, an operator $T \in \mathcal{B}(H)$ is *Hilbert-Schmidt* if

$$\|T\|_{\mathrm{HS}} := \sum_{k=1}^\infty \|Te_k\|^2 < \infty,$$

where $\{e_k\}_{k=1}^\infty$ is an orthonormal basis of H (see, e.g., [10]). The sum is then independent of the choice of the orthonormal basis.

3 Convergence Results

Before the error analysis of the finite element method we need an a priori bound on the solution of (1.1).

Lemma 3.1 *If $\|A^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$, $X_0 \in L_2(\Omega, H)$ and* (1.2) *holds, then there is a unique mild solution $\{X(t)\}_{t\geq 0}$. In particular, $\|X(t)\|_{L_2(\Omega,H)} \leq K$ for $0 \leq t \leq T$, where $K = K(T, L_f)$.*

Proof By our assumption we have $\|A^{-1/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$, which by (2.2) implies that $\int_0^t \|T(s)Q^{1/2}\|^2_{\mathrm{HS}}\,\mathrm{d}s < \infty$ (see also [11]) and therefore the statements follow from [1, Theorem 3.2] and [1, Proposition 3.3].

In the linear case, $f = 0$, we have the following error estimate proved in [11].

Proposition 3.2 *If $f = 0$, $\|A^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [0, 2]$, and $X_0 \in L_2(\Omega, \dot{H}^\beta)$, then*

$$\|X_h(t) - X(t)\|_{L_2(\Omega,H)} \leq Ch^\beta \Big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + \|A^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} \Big).$$

Before proceeding to the semilinear case, note that in [3] and [4] a very general framework for spatial (and also some temporal) approximation of semilinear stochastic equations is presented. However, for more specific equations and more specific methods, as in the present work, stronger results can be obtained under sharp regularity assumptions involving A and Q. This is done in [11] (c.f., Proposition 3.2 above) for linear equations with additive noise and in [12] for equations with multiplicative noise. Using Proposition 3.2 together with finite element techniques for deterministic semilinear parabolic problems (see, for example, [5] and [8]) we obtain the following result for the semilinear stochastic problem, which shows that the order of convergence from the linear case is preserved under the same condition on A and Q.

Proposition 3.3 *Let $\{X(t)\}_{t\geq 0}$ and $\{X_h(t)\}_{t\geq 0}$ be the mild solutions of* (1.1) *and* (1.4)*, respectively. If $\|A^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [0, 2]$ and $X_0 \in L_2(\Omega, \dot{H}^\beta)$, then there is $K = K(X_0, T, \beta)$ such that, for $0 \leq \beta < 2$,*

$$\|X(t) - X_h(t)\|_{L_2(\Omega,H)} \leq Kh^\beta, \quad 0 \leq t \leq T.$$

If $\beta = 2$, then

$$\|X(t) - X_h(t)\|_{L_2(\Omega,H)} \leq Kh^2(1 + \max(0, \ln(t/h^2))), \quad 0 < t \leq T.$$

Proof By (1.3) and (1.5),

$$
\begin{aligned}
&e(t) := \|X(t) - X_h(t)\|_{L_2(\Omega,H)}\\
&\le \Big\| T(t)X_0 + \int_0^t T(t-s)\,\mathrm{d}W(s)\\
&\quad - T_h(t)P_hX_0 - \int_0^t T_h(t-s)P_h\,\mathrm{d}W(s)\Big\|_{L_2(\Omega,H)}\\
&\quad + \Big\| \int_0^t T(t-s)f(X(s))\,\mathrm{d}s - \int_0^t T_h(t-s)P_hf(X_h(s))\,\mathrm{d}s\Big\|_{L_2(\Omega,H)}\\
&=: e_1(t) + e_2(t).
\end{aligned}
$$

By Proposition 3.2,

$$
e_1(t) \le Ch^\beta\Big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + \|A^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}}\Big). \tag{3.1}
$$

Next we bound e_2. By the Lipschitz condition (1.2), the stability bound (2.5), the error estimate (2.3) with $0 \le \beta < 2$, and the a priori bound on $X(t)$ from Lemma 3.1, we have

$$
\begin{aligned}
e_2(t) &\le \int_0^t \|T_h(t-s)P_h(f(X(s)) - f(X_h(s)))\|_{L_2(\Omega,H)}\,\mathrm{d}s\\
&\quad + \int_0^t \|F_h(t-s)f(X(s))\|_{L_2(\Omega,H)}\,\mathrm{d}s\\
&\le C\int_0^t e(s)\,\mathrm{d}s + \int_0^t \|F_h(t-s)f(X(s))\|_{L_2(\Omega,H)}\,\mathrm{d}s \qquad (3.2)\\
&\le C\int_0^t e(s)\,\mathrm{d}s + Ch^\beta\int_0^t (t-s)^{-\beta/2}\|f(X(s))\|_{L_2(\Omega,H)}\,\mathrm{d}s\\
&\le C\int_0^t e(s)\,\mathrm{d}s + Ch^\beta\int_0^t (t-s)^{-\beta/2}\big(1 + \|X(s)\|_{L_2(\Omega,H)}\big)\,\mathrm{d}s\\
&\le C\int_0^t e(s)\,\mathrm{d}s + Ch^\beta. \qquad (3.3)
\end{aligned}
$$

Together with (3.1) and Gronwall's lemma this finishes the proof in this case. If $\beta = 2$, then the bound on e_2 has to be altered. If $0 \le t \le h^2$, then the stability of $X(t)$, by Lemma 3.1, (2.3) with $\beta = 0$ and (3.2) implies that (3.3) holds with $\beta = 2$, which shows the claim in view of (3.1). If $0 \le h^2 < t$, then starting from (3.2), using (2.3) with $\beta = 2$ and Lemma 3.1, we obtain

$$
\begin{aligned}
e_2(t) &\le C\int_0^t e(s)\,\mathrm{d}s + \Big(\int_0^{t-h^2} + \int_{t-h^2}^t\Big)\|F_h(t-s)f(X(s))\|_{L_2(\Omega,H)}\,\mathrm{d}s\\
&\le C\int_0^t e(s)\,\mathrm{d}s + Ch^2\int_0^{t-h^2}(t-s)^{-1}\,\mathrm{d}s + Ch^2\\
&\le Ch^2(1 + \ln(t/h^2)) + C\int_0^t e(s)\,\mathrm{d}s,
\end{aligned}
$$

which finishes the proof by (3.1) and Gronwall's lemma.

The next theorem shows that the order of convergence obtained in Proposition 3.3 may remain valid after truncation of the expansion of $W(t)$, if the eigenvalues of the covariance operator Q decay fast enough.

Theorem 3.4 *Assume that Q is a compact operator on H with an orthonormal basis of eigenvectors $\{e_k\}_{k=1}^{\infty}$ and with a corresponding decreasing sequence of eigenvalues $\{\gamma_k\}_{k=1}^{\infty}$. Let $\{e_{k,h}\}_{k=1}^{N_h}$ be orthonormal eigenvectors of $Q_h = P_h Q P_h$ with corresponding decreasing sequence of eigenvalues $\{\gamma_{k,h}\}_{k=1}^{N_h}$. Let $\{X(t)\}_{t\geq 0}$ and $\{X_h^M(t)\}_{t\geq 0}$ be the mild solutions of (1.1) and (1.9), respectively, with $M \leq N_h$. If $\|A^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [0,2]$ and $X_0 \in L_2(\Omega, \dot{H}^\beta)$, then there is $K = K(X_0, T, \beta)$ such that, for $0 \leq \beta < 2$,*

$$\|X(t) - X_h^M(t)\|_{L_2(\Omega,H)} \leq K \left(h^\beta + \|A^{-\frac{1}{2}}\| \Big(\sum_{k=M}^{N_h} \gamma_k \Big)^{\frac{1}{2}} \right), \quad 0 \leq t \leq T.$$

If $\beta = 2$, then

$$\|X(t) - X_h^M(t)\|_{L_2(\Omega,H)} \leq K \left(h^2(1 + \max(0, \ln(\tfrac{t}{h^2})) + \|A^{-\frac{1}{2}}\| \Big(\sum_{k=M}^{N_h} \gamma_k \Big)^{\frac{1}{2}} \right), \quad 0 < t \leq T.$$

Proof We use the solution $\{X_h(t)\}_{t\geq 0}$ of (1.4) to split the error as follows:

$$\begin{aligned} \|X(t) - X_h^M(t)\|_{L_2(\Omega,H)} &\leq \|X(t) - X_h(t)\|_{L_2(\Omega,H)} \\ &\quad + \|X_h(t) - X_h^M(t)\|_{L_2(\Omega,H)} =: e_1(t) + e_2(t). \end{aligned} \tag{3.4}$$

The first term $e_1(t)$ is bounded by Ch^β for $0 \leq \beta < 2$, and for $\beta = 2$ by $Ch^2(1 + \max(0, \ln(\frac{t}{h^2}))$ according to Proposition 3.3. To bound e_2 we first use the observation that $\{X_h(t)\}_{t\geq 0}$ is the solution of (1.6) and therefore

$$\begin{aligned} e_2(t) &\leq \Big\| \int_0^t T_h(t-s) P_h(f(X_h(s)) - f(X_h^M(s)))\,\mathrm{d}s \Big\|_{L_2(\Omega,H)} \\ &\quad + \Big(\mathbf{E} \Big\| \sum_{k=M}^{N_h} \gamma_{h,k}^{1/2} \int_0^t T_h(t-s) e_{h,k}\,\mathrm{d}\beta_k(s) \Big\|^2 \Big)^{1/2}. \end{aligned} \tag{3.5}$$

Using the global Lipschitz condition and (2.5), the first term can be estimated as

$$\Big\| \int_0^t T_h(t-s) P_h(f(X_h(s)) - f(X_h^M(s)))\,\mathrm{d}s \Big\|_{L_2(\Omega,H)} \leq C \int_0^t e_2(t)\,\mathrm{d}t. \tag{3.6}$$

Using the independence of the Brownian motions $\{\beta_k\}$, Itô's isometry, and (2.2), we have for the second term in (3.5),

$$\begin{aligned}\mathbf{E}\Big\| \sum_{k=M}^{N_h} \gamma_{h,k}^{1/2} \int_0^t T_h(t-s)e_{h,k}\,\mathrm{d}\beta_k(s)\Big\|^2 &= \sum_{k=M}^{N_h} \gamma_{h,k} \int_0^t \|T_h(s)e_{h,k}\|^2\,\mathrm{d}s \\ &\le \sum_{k=M}^{N_h} \gamma_{h,k} \|A_h^{-\frac{1}{2}} e_{h,k}\|^2.\end{aligned}$$

It is well known that $\gamma_{h,k} \le \gamma_k$ for all $k \in \mathbf{N}$ (see, for example, [9, Proposition 1.2]) and thus, using (2.4) and the self-adjointness of A^{-1},

$$\begin{aligned}\mathbf{E}\Big\| \sum_{k=M}^{N_h} \gamma_{h,k}^{1/2} \int_0^t T_h(t-s)e_{h,k}\,\mathrm{d}\beta_k(s)\Big\|^2 &\le C\|A^{-1}\| \sum_{k=M}^{N_h} \gamma_k \\ &= C\|A^{-\frac{1}{2}}\|^2 \sum_{k=M}^{N_h} \gamma_k.\end{aligned} \tag{3.7}$$

Finally, by using Proposition 3.3, (3.6), (3.7) in (3.4) together with Gronwall's lemma we complete the proof.

Assume now that Q is given as an integral operator

$$(Qf)(x) := \int_{\mathcal{D}} q(x,y)f(y)\,\mathrm{d}y. \tag{3.8}$$

The kernel q in (3.8) is called *piecewise analytic*, *piecewise smooth*, or *piecewise* $H^{p,r}$ (with $p, r \in [0,\infty)$), if there is a partition $\mathfrak{D} = \{D_j\}_{j=1}^J$ of the polygonal domain D into a finite set of simplices D_j and a finite set $\mathfrak{G} = \{G_j\}_{j=1}^J$ of open subsets of $\mathbf{R}^d$, such that $\overline{D} = \cup_{j=1}^J \overline{D}_j$, $\overline{D}_j \subset G_j$ for all $j = 1, ..., J$, and such that $q|_{D_j \times D_{j'}}$ has an extension to $G_j \times G_{j'}$, which is analytic in $G_j \times G_{j'}$, is smooth in $G_j \times G_{j'}$, or is in $H^p(G_j) \otimes H^r(G_{j'})$ for all pairs (j, j'). (Here H^p denotes the standard Sobolev space with $H^0 = L_2$). We denote the corresponding regularity spaces by $\mathcal{A}_{\mathfrak{D},\mathfrak{G}}(\mathcal{D}^2)$, $\mathcal{C}^\infty_{\mathfrak{D},\mathfrak{G}}(\mathcal{D}^2)$, and $H^{p,r}_{\mathfrak{D},\mathfrak{G}}(\mathcal{D}^2)$, respectively.

It turns out that the eigenvalue decay rate for integral operators is determined by the regularity of their kernels as defined above (see, e.g., [7] and[9]) and therefore the regularity determines how much we may truncate the noise in Theorem 3.4. Recall that the eigenvalues are ordered in a decreasing manner.

Corollary 3.5 *Assume the conditions of Theorem 3.4 and let Q be given by* (3.8)*. Assume also that the mesh family is quasi-uniform. Then there are $c = c(\beta, d, q)$ and $C = C(X_0, T, \beta, d, q)$, or $C = C(X_0, T, \beta, q, s)$ in case (ii) below, such that, for $0 \le \beta < 2$,*

$$\|X(t) - X_h^M(t)\|_{L_2(\Omega,H)} \le Ch^\beta, \quad 0 \le t \le T,$$

and for $\beta = 2$,

$$\|X(t) - X_h^M(t)\|_{L_2(\Omega,H)} \le Ch^2(1 + \max(0, \ln(t/h^2))), \quad 0 < t \le T,$$

provided that M *is chosen as follows.*

(i) *If* $q \in \mathcal{A}_{\mathfrak{D},\mathfrak{G}}(\mathcal{D}^2)$, *then set* $M = c(\ln N_h)^d$.

(ii) *If* $q \in \mathcal{C}^\infty_{\mathfrak{D},\mathfrak{G}}(\mathcal{D}^2)$, *then set* $M = N_h^{\frac{2\beta}{sd}}$, *where* $s > 1$ *is arbitrary with* $sd > 2\beta$.

(iii) *If* $q \in H^{p,0}_{\mathfrak{D},\mathfrak{G}}(\mathcal{D}^2)$, *then set* $M = N_h^{\frac{2\beta}{p}}$ *if* $p > \max(d, 2\beta)$ *and* $M = N_h$ *otherwise.*

Proof If the kernel q belongs to $\mathcal{A}_{\mathfrak{D},\mathfrak{G}}(\mathcal{D}^2)$, $\mathcal{C}^\infty_{\mathfrak{D},\mathfrak{G}}(\mathcal{D}^2)$, or $H^{p,0}_{\mathfrak{D},\mathfrak{G}}(\mathcal{D}^2)$, then the eigenvalues $\{\gamma_k\}_{k=1}^\infty$ of Q are bounded by $ae^{-bk^{1/d}}$, $c_s k^{-s}$ for any $s > 0$, or $ck^{-p/d}$, respectively, see [7] and[9]. Since the $h \sim N_h^{-1/d}$ for quasi-uniform triangulations, Theorem 3.4 yields the desired result. For example, in the analytic case we have

$$\Big(\sum_{k=M}^{N_h} \gamma_k\Big)^{\frac{1}{2}} \sim \Big(\sum_{k=c(\ln N_h)^d}^{\infty} ae^{-bk^{\frac{1}{d}}}\Big)^{\frac{1}{2}} \sim e^{-\frac{1}{4}2^{\frac{1}{d}}c^{\frac{1}{d}}b\ln N_h}$$

$$\sim N_h^{-\frac{1}{4}b(2c)^{\frac{1}{d}}} \sim h^{\frac{1}{4}b(2c)^{\frac{1}{d}}d} \sim h^\beta$$

for some $c = c(\beta, d, q)$.

Note that for piecewise analytic kernels it is enough to keep $c(\ln N_h)^d$ terms in the noise expansion instead of N_h terms, which is a significant computational advantage. For kernels with low finite regularity, truncation does not reduce the computational cost severely.

4 A Numerical Example

In this section we present some numerical results to illustrate Corollary 3.5. Let $\mathcal{D}$ be the unit square in $\mathbb{R}^2$, $A = -\Delta$, $f \equiv 0$ and let Q be the integral operator corresponding to the Gauss kernel $q(x,y) := \frac{1000}{2\pi\sigma^2} e^{-|x-y|^2/2\sigma^2}$. In this case the kernel is analytic, $\|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{HS} < \infty$ and therefore the order of strong convergence from Corollary 3.5 is expected to be almost 2 even with taking only $M \sim (\ln N_h)^2$ (instead of N_h) terms in the noise expansion in (1.9).

The computations were performed using Matlab with σ taking the values 10, 1 and 0.1. Since an exact solution is not available to compare with, we take instead a finite element approximation on a mesh as fine as possible which, in our case, corresponds to $h = 2.2 \cdot 10^{-2}$. Time discretization is done by the backward Euler method with time-step $\Delta t = 10^{-4}$, performing 10^3 steps. The increments of the semidiscrete Wiener process have been simulated by multiplying $\gamma_{h,k}^{1/2} e_{h,k}$ by a $N(0, \Delta t)$-distributed random variable ξ_k^n in its expansion at time-step n. To be able to compare spatial discretizations on

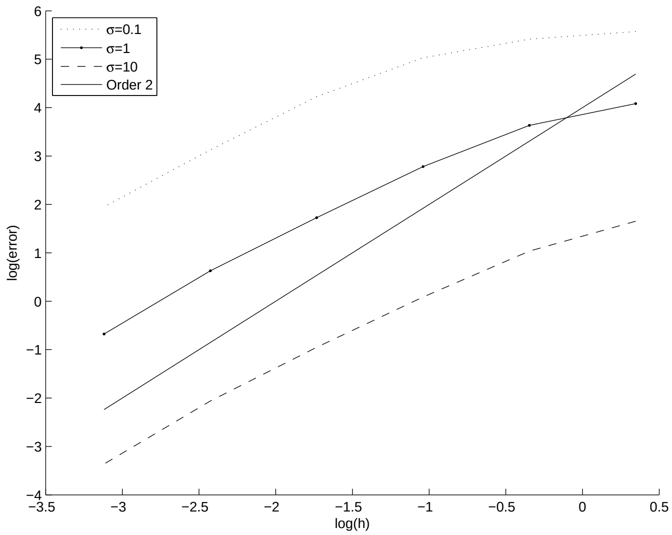

Fig. 1

different levels the same random number ξ_k^n is used at all levels. The number of simulations performed for each σ is 10000. The results are presented in Figure 1. In all three cases when h gets small the order of convergence shows good agreement with the predicted order of almost 2. Furthermore, as expected, the error increases when σ decreases.

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