Semilinear parabolic partial differential equations
Theory, approximation, and applications

Stig Larsson
Chalmers University of Technology
Göteborg University

http://www.math.chalmers.se/~stig
Outline

- Semilinear parabolic equation
- Finite element method for elliptic equation
- Finite element method for semilinear parabolic equation
- Application to dynamical systems
- Stochastic parabolic equation
- Computer exercises with the software Puffin
Lecture 1: Semilinear parabolic PDE
Initial-boundary value problem

\[
    u_t - \Delta u = f(u), \quad x \in \Omega, \ t > 0,
\]
\[
    u = 0, \quad x \in \partial \Omega, \ t > 0,
\]
\[
    u(\cdot, 0) = u_0, \quad x \in \Omega,
\]  \hspace{1cm} (1)

\[\Omega \subset \mathbb{R}^d, \ d = 1, 2, 3, \text{ bounded convex polygonal domain}\]

\[u = u(x, t) \in \mathbb{R}, \ u_t = \partial u/\partial t,\]
\[\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}\]

\[f : \mathbb{R} \rightarrow \mathbb{R} \text{ is twice continuously differentiable} \]

\[|f^{(l)}(\xi)| \leq C (1 + |\xi|^{\delta+1-l}), \quad \xi \in \mathbb{R}, \ l = 1, 2, \]  \hspace{1cm} (2)

\[\delta = 2 \text{ if } d = 3, \quad \delta \in [1, \infty) \text{ if } d = 2.\]
Example: Allen-Cahn equation

\[ f(\xi) = -V'(\xi) \]

\[ V(\xi) = \frac{1}{4} \xi^4 - \frac{1}{2} \xi^2 \]

\[ u_t - \Delta u = -(u^3 - u) \]
Sobolev spaces

Hilbert space \( H = L_2(\Omega) \), with standard norm and inner product

\[
\|v\| = \left( \int_\Omega |v|^2 \, dx \right)^{1/2}, \quad (v, w) = \int_\Omega v \cdot w \, dx.
\] (3)

Sobolev spaces \( H^m(\Omega) \), \( m \geq 0 \), norms denoted by

\[
\|v\|_m = \left( \sum_{|\alpha| \leq m} \|D^\alpha v\|^2 \right)^{1/2}.
\] (4)

Hilbert space \( V = H^1_0(\Omega) \), with norm \( \| \cdot \|_1 \), the functions in \( H^1(\Omega) \) that vanish on \( \partial \Omega \).

\( V^* = H^{-1}(\Omega) \) is the dual space of \( V \) with norm

\[
\|v\|_{-1} = \sup_{\chi \in V} \frac{|(v, \chi)|}{\|\chi\|_1}.
\] (5)
Sobolev spaces

$X, Y$ Banach spaces
$L(X, Y)$ denotes the space of bounded linear operators from $X$ into $Y$
$L(X) = L(X, X)$
$B_X(x, R)$ denotes the closed ball in $X$ with center $x$ and radius $R$.
$B_R = B_V(0, R)$ denote the the closed ball of radius $R$ in $V$:

$$B_R = \{ v \in V : \|v\|_1 \leq R \}.$$

We also use the notation

$$\|v\|_{L_\infty([0,T], X)} = \sup_{t \in [0,T]} \|v(t)\|_X.$$
Abstract framework

unbounded operator \( A = -\Delta \) on \( H \)
domain of definition \( \mathcal{D}(A) = H^2(\Omega) \cap H^1_0(\Omega) \)
\( A \) is a closed, densely defined, and self-adjoint positive definite operator in \( H \) with compact inverse.
nonlinear operator \( f : V \to H \) defined by \( f(v)(x) = f(v(x)) \)
The initial-boundary value problem (??) may then be formulated as an initial value problem in \( V \): find \( u(t) \in V \) such that

\[
u' + Au = f(u), \quad t > 0; \quad u(0) = u_0.
\] (6)

Eigenvalue problem:

\[
A \varphi = \lambda \varphi
\]

\( 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \to \infty \)

orthonormal basis of eigenvectors \( \{ \varphi_j \}_{j=1}^{\infty} \)
Abstract framework

The operator $-A$ is the infinitesimal generator of the analytic semigroup $E(t) = \exp(-tA)$ defined by

$$E(t)v = \sum_{j=1}^{\infty} e^{-t\lambda_j} (v, \varphi_j) \varphi_j, \quad v \in H,$$  \hspace{1cm} (7)

The semigroup $E(t)$ is the solution operator of the initial value problem for the homogeneous equation,

$$u' + Au = 0, \quad t > 0; \quad u(0) = u_0; \quad u(t) = E(t)u_0$$

By Duhamel's principle: solutions of (7) satisfy

$$u(t) = E(t)u_0 + \int_0^t E(t-s)f(u(s))\,ds, \quad t \geq 0.$$  \hspace{1cm} (8)

Conversely, appropriately defined solutions of the nonlinear integral equation (8) are solutions of the differential equation (7), see below. We shall mainly work with (8) and discretized variants of it.
Fractional powers of $A$

$$\|A^\alpha v\| = \left( \sum_{j=1}^{\infty} \left( \lambda_j^\alpha (v, \varphi_j) \right)^2 \right)^{1/2}, \quad \alpha \in \mathbb{R},$$

(9)

$$\mathcal{D}(A^\alpha) = \left\{ v : \|A^\alpha v\| < \infty \right\}, \quad \alpha \in \mathbb{R}.$$  

elliptic regularity estimate

$$\|v\|_2 \leq C\|Av\|, \quad v \in H^2(\Omega) \cap H^1_0(\Omega),$$

(10)

trace inequality

$$\|v\|_{L^2(\partial\Omega)} \leq C\|v\|_1, \quad v \in H^1(\Omega),$$

$$\mathcal{D}(A^{l/2}) = H^l(\Omega) \cap H^1_0(\Omega), \quad l = 1, 2,$$  

with the equivalence of norms

$$c\|v\|_l \leq \|A^{l/2}v\| \leq C\|v\|_l, \quad v \in \mathcal{D}(A^{l/2}), \quad l = 1, 2;$$

(11)

$$\mathcal{D}(A^{-1/2}) = H^{-1}(\Omega)$$

$$c\|v\|_{-1} \leq \|A^{-1/2}v\| \leq C\|v\|_{-1};$$

(12)
Analytic semigroup

\[ E(t)v = \sum_{j=1}^{\infty} e^{-t\lambda_j} (v, \varphi_j) \varphi_j, \quad v \in H, \]

\[ u(t) = E(t)v \text{ is the solution of} \]

\[ u' + Au = 0; \quad u(0) = v \]

Smoothing property \((D_t = \partial/\partial t)\):

\[ \| D_t^l E(t)v \| = \| A^l E(t)v \| \leq C_l t^{-l} \| v \|, \quad t > 0, \ v \in H, \ l \geq 0. \] (13)

\[ \| D_t^l E(t)v \|_{\beta} \leq C_l t^{-l-(\beta-\alpha)/2} \| v \|_{\alpha}, \quad t > 0, \ v \in D(A^{\alpha/2}), \]

\[ -1 \leq \alpha \leq \beta \leq 2, \ l = 0, 1. \] (14)
Local Lipschitz condition

\( f : V \to H \) nonlinear mapping
\( f' : V \to \mathcal{L}(V, H) \) Fréchet derivative

growth assumption:

\[
|f^{(l)}(\xi)| \leq C(1 + |\xi|^{\delta+1-l}), \quad \xi \in \mathbb{R}, \ l = 1, 2,
\]  \( (15) \)

\( \delta = 2 \) if \( d = 3 \), \( \delta \in [1, \infty) \) if \( d = 2 \).

Sobolev’s inequality

\[
\|v\|_{L^p} \leq C\|v\|_1,
\]  \( (16) \)

\( p = 6 \) if \( d \leq 3 \), \( p < \infty \) if \( d = 2 \), \( p = \infty \) if \( d = 1 \)

Hölder’s inequality

\[
\|v^\delta w\|_{L^r} \leq \|v\|_{L^q}^\delta \|w\|_{L^p}, \quad \frac{\delta}{q} + \frac{1}{p} = \frac{1}{r}, \ \delta > 0.
\]  \( (17) \)
Local Lipschitz condition

\[ f : V \to H \text{ nonlinear mapping} \]
\[ f' : V \to \mathcal{L}(V, H) \text{ Fréchet derivative} \]

**Lemma** For each nonnegative number \( R \) there is a constant \( C(R) \) such that, for all \( u, v \in B_R \subset V \),

\[ \|f'(u)\|_{\mathcal{L}(V,H)} \leq C(R) \tag{18} \]
\[ \|f'(u)\|_{\mathcal{L}(H,V^*)} \leq C(R) \tag{19} \]
\[ \|f(u) - f(v)\| \leq C(R)\|u - v\|_1 \tag{20} \]
\[ \|f(u) - f(v)\|_{-1} \leq C(R)\|u - v\| \tag{21} \]
Proof

By (??) and the Hölder and Sobolev inequalities, for \( z \in V = H_0^1(\Omega) \),

\[
\|f'(u)z\|_{L^2} \leq C\|(1 + |u|^\delta)z\|_{L^2} \leq C(1 + \|u\|_{L_q}^\delta)\|z\|_{L^p}
\]

\[
\leq C(1 + \|u\|_{L^1}^\delta)\|z\|_{L^1},
\]

where \( \frac{1}{p} + \frac{\delta}{q} = \frac{1}{2} \) with \( p = q = 6 \) if \( d = 3 \), and with arbitrary \( p \in (1, \infty) \) if \( d \leq 2 \). This proves (??) and (??) follows.

Moreover, for any \( z, \chi \in V \),

\[
(f'(u)z, \chi) \leq C(1 + \|u\|_{L_q}^\delta)\|z\|_{L^2}\|\chi\|_{L^p}
\]

\[
\leq C(1 + \|u\|_{L^1}^\delta)\|z\|\|\chi\|_{L^1},
\]

where \( \frac{\delta}{q} + \frac{1}{2} + \frac{1}{p} = 1 \), i.e., with the same \( p \) and \( q \) as before. This proves (??) and (??) follows.
Local existence

Theorem. For any $R_0 > 0$ there is $\tau = \tau(R_0)$ such that (??) has a unique solution $u \in C([0, \tau], V)$ for any initial value $u_0 \in V$ with $\|u_0\|_1 \leq R_0$. Moreover, there is $c$ such that $\|u\|_{L_\infty([0, \tau], V)} \leq cR_0$. 
Proof

Let \( u_0 \in B_{R_0} \), define

\[
S(u)(t) = E(t)u_0 + \int_0^t E(t-s)f(u(s)) \, ds,
\]

and note that (??) is a fixed point equation \( u = S(u) \). We shall choose \( \tau \) and \( R \) such that we can apply Banach’s fixed point theorem (the contraction mapping theorem) in the closed ball

\[
B = \{ u \in C([0, \tau], V) : \| u \|_{L_\infty([0,\tau], V)} \leq R \}
\]

in the Banach space \( C([0, \tau], V) \).

We must show (i) that \( S \) maps \( B \) into itself, (ii) that \( S \) is a contraction on \( B \). In order to prove (i) we take \( u \in B \) and first note that the Lipschitz condition (??) implies that

\[
\| f(u(t)) \| \leq \| f(0) \| + \| f(u(t)) - f(0) \|
\leq \| f(0) \| + C(R)\| u(t) \|_1
\leq \| f(0) \| + C(R)R, \quad 0 \leq t \leq \tau.
\]
Proof, cont’d

Hence, using also (??), we get

\[ \|S(u)(t)\|_1 \leq \|E(t)u_0\|_1 + \int_0^t \|E(t-s)f(u(s))\|_1 \, ds \]
\[ \leq c_0\|u_0\|_1 + c_1 \int_0^t (t-s)^{-1/2}\|f(u(s))\| \, ds \]
\[ \leq c_0R_0 + 2c_1\tau^{1/2}(\|f(0)\| + C(R)R), \quad 0 \leq t \leq \tau. \]

This implies

\[ \|S(u)\|_{L_\infty([0,\tau],V)} \leq c_0R_0 + 2c_1\tau^{1/2}(\|f(0)\| + C(R)R). \]

Choose \( R = 2c_0R_0 \) and \( \tau = \tau(R_0) \) so small that

\[ 2c_1\tau^{1/2}(\|f(0)\| + C(R)R) \leq \frac{1}{2}R. \quad (23) \]

Then \( \|S(u)\|_{L_\infty([0,\tau],V)} \leq R \) and we conclude that \( S \) maps \( \mathcal{B} \) into itself.
Proof, cont’d

To show (ii) we take \( u, v \in \mathcal{B} \) and note that

\[
\| f(u(t)) - f(v(t)) \| \leq C(R) \| u - v \|_{L_\infty([0,\tau],V)}, \quad 0 \leq t \leq \tau.
\]

Hence

\[
\| S(u)(t) - S(v)(t) \|_1 \leq \int_0^t \| E(t - s)(f(u(s)) - f(v(s))) \|_1 \, ds
\]

\[
\leq c_1 \int_0^t (t - s)^{-1/2} \| f(u(s)) - f(v(s)) \| \, ds
\]

\[
\leq 2c_1 \tau^{1/2} C(R) \| u - v \|_{L_\infty([0,\tau],V)}, \quad 0 \leq t \leq \tau,
\]

so that

\[
\| S(u) - S(v) \|_{L_\infty([0,\tau],V)} \leq 2c_1 \tau^{1/2} C(R) \| u - v \|_{L_\infty([0,\tau],V)}.
\]

It follows from (??) that \( 2c_1 \tau^{1/2} C(R) \leq \frac{1}{2} \) and we conclude that \( S \) is a contraction on \( \mathcal{B} \).

Hence \( S \) has a unique fixed point \( u \in \mathcal{B} \).
Nonlinear semigroup

The initial value problem thus has a unique local solution for any initial datum $u_0 \in V$. We denote by

$$S(t, \cdot) : V \rightarrow V$$

the corresponding solution operator, so that $u(t) = S(t, u_0)$ is the solution of

$$u(t) = E(t)u_0 + \int_0^t E(t-s)f(u(s)) \, ds, \quad t \geq 0.$$  \hfill (24)
Regularity

This will be used in our error analysis, but the theorem also shows that \( u'(t) \in H \) and \( Au(t) \in H \) for \( t > 0 \), so that any solution of the integral equation (??) is also a solution of the differential equation (??).

**THEOREM** Let \( R \geq 0 \) and \( \tau > 0 \) be given and let \( u \in C([0, \tau], V) \) be a solution. If \( \|u(t)\|_1 \leq R \) for \( t \in [0, \tau] \), then

\[
\|u(t)\|_2 \leq C(R, \tau)t^{-1/2}, \quad t \in (0, \tau],
\]

\[
\|u_t(t)\|_s \leq C(R, \tau)t^{-1-(s-1)/2}, \quad t \in (0, \tau], \ s = 0, 1, 2.
\]

Proof will be provided in the notes.
Global existence

Assume we can provide a global a priori bound: there is $R$ such that if $u \in C([0, T], V)$ is a solution then

$$\|u(t)\|_1 \leq R, \quad t \in [0, T]$$

Repeated application of the local existence theorem with $\tau = \tau(R)$ then proves existence for $t \in [0, T]$. 
Example: Allen-Cahn equation

\[ u_t - \Delta u = -(u^3 - u) = -V'(u) \]

\[ V(\xi) = \frac{1}{4} \xi^4 - \frac{1}{2} \xi^2 \text{ bounded from below: } V(\xi) \geq -K \]

\[ (u_t, u_t) - (\Delta u, u_t) = -(V'(u), u_t) \]

\[ (u_t, u_t) + (\nabla u, \nabla u_t) = -(V'(u), u_t) \]

\[ \|u_t\|^2 + \frac{1}{2} D_t \|\nabla u\|^2 = -D_t \int_{\Omega} V(u) \, dx \]

\[ \int_0^t \|u_t\|^2 \, ds + \frac{1}{2} \|\nabla u(t)\|^2 = \frac{1}{2} \|\nabla u_0\|^2 - \int_{\Omega} V(u(t)) \, dx + \int_{\Omega} V(u_0) \, dx \leq C \]

We conclude

\[ \|u(t)\|_1 \leq R, \quad t \in [0, \infty) \]
Lecture 2: Finite element method
Elliptic equation

Let $\Omega$ be a convex polygonal domain in $\mathbb{R}^2$:

$$Au := -\nabla \cdot (a \nabla u) = f \quad x \in \Omega$$

$$u = 0 \quad x \in \partial \Omega$$

$a = a(x)$ is smooth with $a(x) \geq a_0 > 0$ in $\overline{\Omega}$ and $f \in L_2$

Weak formulation: find $u \in V = H^1_0$ such that

$$a(u, v) = (f, v), \quad \forall v \in V = H^1_0$$

where

$$a(v, w) = \int_{\Omega} a \nabla v \cdot \nabla w \, dx \quad \text{and} \quad (f, v) = \int_{\Omega} f v \, dx$$
Elliptic equation

Poincaré’s inequality

\[ \|v\| \leq C\|\nabla v\| \quad \forall v \in V \]

implies

\[ c\|v\|_1^2 \leq a_0\|\nabla v\|^2 \leq a(v, v) \leq C\|v\|_1^2 \quad \forall v \in V \]

so \( a(\cdot, \cdot) \) is a scalar product and the norm \( \|v\|_a = \sqrt{a(v, v)} \) is equivalent to the standard norm on \( V \). Hence there is a unique solution in \( u \in V \).

\( \Omega \) is convex: \( u \in H^2 \) and

\[ \|u\|_2 \leq C\|f\|. \]
Finite element

\{K\} a set of closed triangles \(K\), a triangulation, such that

\[\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K, \quad h_K = \text{diam}(K), \quad h = \max_{K \in \mathcal{T}_h} h_K.\]

Piecewise linear functions:

\[V_h = \{ v \in C(\bar{\Omega}) : v \text{ linear in } K \text{ for each } K, \ v = 0 \text{ on } \partial \Omega \}\]

\[V_h \subset V = H^1_0\]

\(\{P_i\}_{i=1}^{M_h}\) the set of interior nodes

\(v \in V_h\) is uniquely determined by its values at the \(P_j\)

pyramid functions \(\{\Phi_i\}_{i=1}^{M_h} \subset V_h\), defined by

\[\Phi_i(P_j) = \begin{cases} 
1, & \text{if } i = j, \\
0, & \text{if } i \neq j,
\end{cases}\]

forms a basis for \(V_h\), \(v(x) = \sum_{i=1}^{M_h} v_i \Phi_i(x)\), where \(v_i = v(P_i)\).
Finite element method

The finite element equation: find $u_h \in V_h$ such that

$$a(u_h, \chi) = (f, \chi), \quad \forall \chi \in V_h$$

Use basis:

$$u_h(x) = \sum_{i=1}^{M_h} U_i \Phi_i(x)$$

$$\sum_{j=1}^{M_h} U_j a(\Phi_j, \Phi_i) = (f, \Phi_i), \quad i = 1, \ldots, M_h,$$

Matrix form: $AU = b$, where $U = (U_i)$, $A = (a_{ij})$ is the stiffness matrix with elements $a_{ij} = a(\Phi_j, \Phi_i)$, and $b = (b_i)$ the load vector with elements $b_i = (f, \Phi_i)$.

$A$ is symmetric and positive definite, large and sparse.
Approximation theory

Interpolation operator $I_h : C(\Omega) \to V_h$ defined by

$$(I_h v)(x) = \sum_{i=1}^{M_h} v_i \Phi_i(x), \quad v_i = v(P_i)$$

Thus

$$(I_h v)(P_i) = v(P_i), \quad i = 1, \ldots, M_h$$

Interpolation error estimates:

$$\|I_h v - v\| \leq Ch^2 \|v\|_2, \quad \forall v \in H^2 \cap V$$

$$\|I_h v - v\|_1 \leq Ch \|v\|_2, \quad \forall v \in H^2 \cap V$$
Error estimate in $H^1$ norm

Error $u_h - u$.

Energy norm:

$$\| v \|_a = a(v, v)^{1/2} = \left( \int_{\Omega} a|\nabla v|^2 \, dx \right)^{1/2}.$$ 

Standard norm:

$$\| v \|_1 = \left( \| v \|^2 + \| \nabla v \|^2 \right)^{1/2}.$$

Theorem

$$\| u_h - u \|_a = \min_{\chi \in V_h} \| \chi - u \|_a,$$

$$\| u_h - u \|_1 \leq C h \| u \|_2.$$
Proof

\[ u \in V; \quad a(u, v) = (f, v), \quad \forall v \in V \]

\[ u_h \in V_h; \quad a(u_h, \chi) = (f, \chi), \quad \forall \chi \in V_h \]

\( V_h \subset V, \) take \( v = \chi \in V_h \) and subtract

\[ a(u_h - u, \chi) = 0, \quad \forall \chi \in V_h, \]

\( u_h \) is the orthogonal projection of \( u \) onto \( V_h \) with respect to the inner product \( a(\cdot, \cdot) \)

\[ \|u_h - u\|_a = \min_{\chi \in V_h} \|\chi - u\|_a, \]

Equivalence of norms and interpolation error estimate:

\[ \|u_h - u\|_1 \leq C\|u_h - u\|_a \leq C\|I_h u - u\|_a \leq C\|I_h u - u\|_1 \leq Ch\|u\|_2. \]
Error estimate in $L_2$ norm

Theorem

$$\|u_h - u\| \leq Ch^2 \|u\|_2.$$
Proof

Duality argument based on the auxiliary problem (where $e = u_h - u$)

\[ A\phi = e \quad \text{in } \Omega, \]
\[ \phi = 0 \quad \text{on } \partial \Omega \]

Weak formulation: find $\phi \in H_0^1$ such that

\[ a(w, \phi) = (w, e), \quad \forall w \in H_0^1. \]

Regularity estimate: $\|\phi\|_2 \leq C\|A\phi\| = C\|e\|$

Take $w = e \in H_0^1$

\[
\|e\|^2 = a(e, \phi) = a(e, \phi - I_h\phi) \leq C\|e\|_1 \|\phi - I_h\phi\|_1 \\
\leq Ch\|e\|_1 \|\phi\|_2 \leq Ch\|e\|_1 \|e\|.
\]

\[
\|e\| \leq Ch\|e\|_1 \leq Ch^2\|u\|_2
\]
Ritz projection

Recall $V_h \subset V$ and

$$a(u_h - u, \chi) = 0, \quad \forall \chi \in V_h$$

$u_h$ is the orthogonal projection of $u$ onto $V_h$ with respect to the inner product $a(\cdot, \cdot)$

We denote it by $R_h : V \to V_h$. It satisfies

$$a(R_h u - u, \chi) = 0, \quad \forall \chi \in V_h$$

With this notation the error estimates are:

$$\|R_h v - v\| \leq Ch^2 \|v\|_2, \quad \|R_h v - v\|_1 \leq Ch \|v\|_2, \quad \forall v \in H^2 \cap H^1_0$$

Also:

$$\|R_h v - v\| \leq Ch \|v\|_1, \quad \forall v \in H^1_0$$
Lecture 3: Finite elements for semilinear parabolic PDE
Abstract framework

Find $u(t) \in V$ such that

$$u' + Au = f(u), \quad t > 0; \quad u(0) = u_0.$$  

Linear homogenous equation:

$$u' + Au = 0, \quad t > 0; \quad u(0) = u_0; \quad u(t) = E(t)u_0$$

By Duhamel's principle:

$$u(t) = E(t)u_0 + \int_0^t E(t-s)f(u(s)) \, ds, \quad t \geq 0.$$
Local existence and regularity

**THEOREM** For any $R_0 > 0$ there is $\tau = \tau(R_0)$ such that there is a unique solution $u \in C([0, \tau], V)$ for any initial value $u_0 \in V$ with $\|u_0\|_1 \leq R_0$. Moreover, there is $c$ such that $\|u\|_{L^\infty([0,\tau], V)} \leq cR_0$.

**THEOREM** Let $R \geq 0$ and $\tau > 0$ be given and let $u \in C([0, \tau], V)$ be a solution. If $\|u(t)\|_1 \leq R$ for $t \in [0, \tau]$, then

\[
\|u(t)\|_2 \leq C(R, \tau)t^{-1/2}, \quad t \in (0, \tau],
\]

\[
\|u_t(t)\|_s \leq C(R, \tau)t^{-1-(s-1)/2}, \quad t \in (0, \tau], \ s = 0, 1, 2.
\]

Global solution on $[0, T]$ if we can prove a priori bound

\[
\|u(t)\|_1 \leq R, \quad t \in [0, T]
\]
Finite element method

Linear elliptic equation:

\[-\nabla \cdot (a \nabla u) = f \quad x \in \Omega\]
\[u = 0 \quad x \in \partial \Omega\]

Weak formulation: find \( u \in V = H^1_0 \) such that

\[a(u, v) = (f, v), \quad \forall v \in V = H^1_0\]

where

\[a(v, w) = \int_{\Omega} a \nabla v \cdot \nabla w \, dx \quad \text{and} \quad (f, v) = \int_{\Omega} f v \, dx\]

The finite element equation: find \( u_h \in V_h \) such that

\[a(u_h, \chi) = (f, \chi), \quad \forall \chi \in V_h\]
Error estimates

\[ \|u_h - u\|_1 \leq Ch\|u\|_2 \]

\[ \|u_h - u\| \leq Ch^2\|u\|_2 \]
**Ritz projection**

Recall $V_h \subset V$ and

$$a(u_h - u, \chi) = 0, \quad \forall \chi \in V_h$$

$u_h$ is the orthogonal projection of $u$ onto $V_h$ with respect to the inner product $a(\cdot, \cdot)$

We denote it by $R_h : V \rightarrow V_h$. It satisfies

$$a(R_h u - u, \chi) = 0, \quad \forall \chi \in V_h$$

With this notation the error estimates are:

$$\|R_h v - v\| \leq Ch^2 \|v\|_2, \quad \|R_h v - v\|_1 \leq Ch \|v\|_2, \quad \forall v \in H^2 \cap H^1_0$$

Also:

$$\|R_h v - v\| \leq Ch \|v\|_1, \quad \forall v \in H^1_0$$
Spatially semidiscrete approximation

\[ u' + Au = f(u), \; t > 0; \; u(0) = u_0. \]

The weak formulation: find \( u(t) \in V \) such that

\[ (u', v) + a(u, v) = (f(u), v), \; \forall v \in V, \; t > 0, \]
\[ u(0) = u_0, \tag{27} \]

where \( a(u, v) = (\nabla u, \nabla v) = (\Delta u, v) = (Au, v) \) is the bilinear form associated with \( A \).

Finite element spaces: \( \{V_h\} \subset V \)

Spatially semidiscrete finite element equation: find \( u_h(t) \in V_h \) such that

\[ (u'_h, \chi) + a(u_h, \chi) = (f(u_h), \chi), \; \forall \chi \in V_h, \; t > 0, \]
\[ u_h(0) = u_{h,0}, \tag{28} \]

where \( u_{h,0} \in V_h \) is an approximation of \( u_0 \).
Abstract framework

Linear operator $A_h : V_h \rightarrow V_h$

Orthogonal projection $P_h : H \rightarrow V_h$

\[
(A_h \psi, \chi) = a(\psi, \chi), \quad (P_h g, \chi) = (g, \chi) \quad \forall \psi, \chi \in V_h, \ g \in H,
\]

Finite element equation becomes:

\[
u_h' + A_h u_h = P_h f(u_h), \ t > 0; \quad u_h(0) = u_{h,0}.
\]

$A_h$ is self-adjoint positive definite (uniformly in $h$);

The corresponding semigroup $E_h(t) = \exp(-tA_h) : V_h \rightarrow V_h$

therefore has the smoothing properties (uniformly in $h$):

\[
\|D^l_t E_h(t)v\| = \|A^l_h E_h(t)v\| \leq C_l t^{-l} \|v\|, \quad t > 0, \ v \in V_h, \ l \geq 0.
\]

Moreover, for the operator $A_h$ we have the equivalence of norms

\[
c\|v\|_1 \leq \|A^{1/2}_h v\| = \sqrt{a(v, v)} = \|A^{1/2} v\| \leq C\|v\|_1, \quad v \in V_h.
\]
Abstract formulation

\[ \| A_h^{1/2} v \| \text{ controls } \| v \|_1 \text{ and } \| v \|_1 \text{ controls the Lipschitz constant of } f. \]

We also have

\[ \| P_h f \| \leq \| f \|, \quad f \in H, \]

and

\[ \| A_h^{-1/2} P_h f \| \leq C \| f \|_{-1}, \quad f \in H, \]

which follows from

\[
\| A_h^{-1/2} P_h f \| = \sup_{v_h \in V_h} \frac{|(A_h^{-1/2} P_h f, v_h)|}{\| v_h \|} = \sup_{v_h \in V_h} \frac{|(f, A_h^{-1/2} v_h)|}{\| v_h \|} \\
= \sup_{w_h \in V_h} \frac{|(f, w_h)|}{\| A_h^{1/2} w_h \|} \leq C \sup_{w_h \in V_h} \frac{|(f, w_h)|}{\| w_h \|_1} \leq C \sup_{w \in V} \frac{|(f, w)|}{\| w \|_1} = C \| f \|_{-1}.
\]
Abstract formulation

Using these inequalities we prove:

\[ \|D_t^l E_h(t) P_h f\|_\beta \leq C t^{-l-(\beta-\alpha)/2} \|f\|_\alpha, \quad t > 0, \; f \in \mathcal{D}(A^{\alpha/2}), \]
\[ -1 \leq \alpha \leq \beta \leq 1, \; l = 0, 1. \]

Note that the upper limit to \( \beta \) is 1, while it is 2 in the continuous case.
The initial-value problem is equivalent to the integral equation

\[ u_h(t) = E_h(t)u_{h,0} + \int_0^t E_h(t-s)P_h f(u_h(s)) \, ds, \quad t \geq 0. \]

The proof of the previous local existence theorem carries over verbatim to the semidiscrete case. We thus have:

**Theorem** For any \( R_0 > 0 \) there is \( \tau = \tau(R_0) \) such that there is a unique solution \( u_h \in C([0, \tau], V) \) for any initial value \( u_{h,0} \in V_h \) with \( \|u_{h,0}\|_1 \leq R_0 \). Moreover, there is \( c \) such that \( \|u_h\|_{L_\infty([0,\tau], V)} \leq cR_0 \).

We denote by \( S_h(t, \cdot) \) the corresponding (local) solution operator, so that \( u_h(t) = S_h(t, u_{h,0}) \) is the solution.
Local a priori error estimate

Next goal: estimate the difference between the local solutions \( u(t) = S(t, u_0) \) and \( u_h(t) = S_h(t, u_{h,0}) \)

**THEOREM** Let \( R \geq 0 \) and \( \tau > 0 \) be given. Let \( u(t) \) and \( u_h(t) \) be solutions of (??) and (??) respectively, such that \( u(t), u_h(t) \in B_R \) for \( t \in [0, \tau] \). Then

\[
\|u_h(t) - u(t)\|_1 \leq C(R, \tau) t^{-1/2} (\|u_{h,0} - P_h u_0\| + h), \quad t \in (0, \tau],
\]

\[
\|u_h(t) - u(t)\| \leq C(R, \tau) (\|u_{h,0} - P_h u_0\| + h^2 t^{-1/2}), \quad t \in (0, \tau].
\]

Local: because the constant \( C(R, \tau) \) grows with \( \tau \) and \( R \).
A priori: because the error is evaluated in terms of derivatives of \( u \), which are estimated a priori in a previous regularity theorem.
Proof

Recall the Ritz projection operator $R_h : V \to V_h$ defined by

$$a(R_h v, \chi) = a(v, \chi), \quad \forall \chi \in V_h.$$ 

with error bounds

$$\| R_h v - v \| + h\| R_h v - v \|_1 \leq Ch^s\|v\|_s, \quad v \in V \cap H^s(\Omega), \quad s = 1, 2$$

We divide the error into two parts:

$$e(t) \equiv u_h(t) - u(t) = (u_h(t) - R_h u(t)) + (R_h u(t) - u(t)) \equiv \theta(t) + \rho(t).$$

Then, for $j = 0, 1$ and $s = 1, 2$,

$$\| \rho(t) \|_j \leq Ch^{s-j} \| u(t) \|_s \leq C(R, \tau)h^{s-j}t^{-(s-1)/2}, \quad t \in (0, \tau],$$

$$\| \rho_t(t) \| \leq Ch^s \| u_t(t) \|_s \leq C(R, \tau)ht^{1-(s-1)/2}, \quad t \in (0, \tau].$$
Proof

It remains to estimate $\theta(t) \in V_h$. It satisfies, for $\chi \in V_h$,

$$(\theta_t, \chi) + a(\theta, \chi) = (u_{ht}, \chi) + a(u_h, \chi) - (R_h u_t, \chi) - a(R_h u, \chi)$$

$$= (u_{ht}, \chi) + a(u_h, \chi) - (R_h u_t, \chi) - a(u, \chi)$$

$$= (f(u_h) - f(u), \chi) - (R_h u_t - u_t, \chi)$$

$$= (f(u_h) - f(u), \chi) - (\rho_t, \chi)$$

equivalently

$$\theta_t + A_h \theta = P_h (f(u_h) - f(u) - \rho_t)$$

(30)

Hence, by Duhamel’s principle,

$$\theta(t) = E_h(t)\theta(0) + \int_0^t E_h(t - \sigma) P_h \left( f(u_h(\sigma)) - f(u(\sigma)) - D_\sigma \rho(\sigma) \right) d\sigma$$
Proof

Integration by parts yields

\[-\int_0^{t/2} E_h(t - \sigma) P_h D_\sigma \rho(\sigma) \, d\sigma = E_h(t) P_h \rho(0) - E_h(t/2) P_h \rho(t/2)\]

\[+ \int_0^{t/2} (D_\sigma E_h(t - \sigma)) P_h \rho(\sigma) \, d\sigma.\]

Hence

\[
\theta(t) = E_h(t) P_h e(0) - E_h(t/2) P_h \rho(t/2) + \int_0^{t/2} (D_\sigma E_h(t - \sigma)) P_h \rho(\sigma) \, d\sigma
\]

\[-\int_{t/2}^t E_h(t - \sigma) P_h D_\sigma \rho(\sigma) \, d\sigma + \int_0^t E_h(t - \sigma) P_h \left(f(u_h(\sigma)) - f(u(\sigma))\right) \, d\sigma.\]
Proof

Using the smoothing property of $E_h(t)P_h$, the error estimates for $\rho$ with $j = 0$, $s = 1$, and the Lipschitz condition $\|f(u) - f(v)\| \leq C(R)\|u - v\|_1$, we obtain

$$\|\theta(t)\|_1 \leq C t^{-1/2} (\|P_h e(0)\| + \|\rho(t/2)\|) + C \int_0^{t/2} (t - \sigma)^{-3/2} \|\rho(\sigma)\| \, d\sigma$$

$$+ C \int_{t/2}^{t} (t - \sigma)^{-1/2} \|D_\sigma \rho(\sigma)\| \, d\sigma$$

$$+ C \int_0^{t} (t - \sigma)^{-1/2} \|f(u_h(\sigma)) - f(u(\sigma))\| \, d\sigma$$

$$\leq C(R, \tau) t^{-1/2} (\|P_h e(0)\| + h) + C(R, \tau) h \left( \int_0^{t/2} (t - \sigma)^{-3/2} \, d\sigma + \int_{t/2}^{t} (t - \sigma)^{-1/2} \sigma^{-1} \, d\sigma \right)$$

$$+ C(R) \int_0^{t} (t - \sigma)^{-1/2} \|e(\sigma)\|_1 \, d\sigma$$

$$\leq C(R, \tau) t^{-1/2} (\|P_h e(0)\| + h) + C(R) \int_0^{t} (t - \sigma)^{-1/2} \|e(\sigma)\|_1 \, d\sigma,$$

for $t \in (0, \tau)$. 

– p.49/65
Proof

Since $e = \theta + \rho$ this yields

$$\|e(t)\|_1 \leq C(R, \tau) t^{-1/2} (\|P_h e(0)\| + h) + C(R) \int_0^t (t - \sigma)^{-1/2} \|e(\sigma)\|_1 \, d\sigma, \quad t \in (0, \tau],$$

and we use the generalized Gronwall lemma. This proves the $H^1$-estimate, because $P_h e(0) = u_{h,0} - P_h u_0$.

To prove the $L_2$ estimate we use the Lipschitz condition $\|f(u) - f(v)\|_{-1} \leq C(R) \|u - v\|$ instead.
Generalized Gronwall lemma

**Lemma** Let the function $\varphi(t, \tau) \geq 0$ be continuous for $0 \leq \tau < t \leq T$. If

$$
\varphi(t, \tau) \leq A (t - \tau)^{-1+\alpha} + B \int_{\tau}^{t} (t - s)^{-1+\beta} \varphi(s, \tau) \, ds, \quad 0 \leq \tau < t \leq T,
$$

for some constants $A, B \geq 0$, $\alpha, \beta > 0$, then there is a constant $C = C(B, T, \alpha, \beta)$ such that

$$
\varphi(t, \tau) \leq CA (t - \tau)^{-1+\alpha}, \quad 0 \leq \tau < t \leq T.
$$

The constant in Gronwall’s lemma grows exponentially with the length $T$ of the time interval. Hence, results derived by means of this lemma are often useful only for short time intervals.
Proof

Iterating the given inequality $N - 1$ times, using the identity

$$
\int_{\tau}^{t} (t - s)^{-1+\alpha}(s - \tau)^{-1+\beta} \, ds = C(\alpha, \beta) (t - \tau)^{-1+\alpha+\beta}, \quad \alpha, \beta > 0,
$$

(Abel's integral) and estimating $(t - \tau)^\beta$ by $T^\beta$, we obtain

$$
\varphi(t, \tau) \leq C_1 A (t - \tau)^{-1+\alpha} + C_2 \int_{\tau}^{t} (t - s)^{-1+N\beta} \varphi(s, \tau) \, ds, \quad 0 \leq \tau < t \leq T,
$$

where $C_1 = C_1(B, T, \alpha, \beta, N)$, $C_2 = C_2(B, \beta, N)$. We now choose the smallest $N$ such that $-1 + N\beta \geq 0$, and estimate $(t - s)^{-1+N\beta}$ by $T^{-1+N\beta}$. If $-1 + \alpha \geq 0$ we obtain the desired conclusion by the standard version of Gronwall's lemma. Otherwise we set

$$
\psi(t, \tau) = (t - \tau)^{1-\alpha} \varphi(t, \tau)
$$

to obtain

$$
\psi(t, \tau) \leq C_1 A + C_3 \int_{\tau}^{t} (s - \tau)^{-1+\alpha} \psi(s, \tau) \, ds, \quad 0 \leq \tau < t \leq T,
$$

and the standard Gronwall lemma yields $\psi(t, \tau) \leq CA$ for $0 \leq \tau < t \leq T$, which is the desired result.
Error estimate reformulated

If $S_h(t, v_h), S(t, v) \in B_R$ for $t \in [0, 2\tau]$ then, for $l = 0, 1,$

$$\|S_h(t, v_h) - S(t, v)\|_l \leq C(R, \tau) t^{-1/2} (\|v_h - P_h v\| + h^{2-l}), \quad t \in (0, 2\tau],$$

and

$$\|S_h(t, v_h) - S(t, v)\|_l \leq C(R, \tau) (\|v_h - P_h v\| + h^{2-l}), \quad t \in [\tau, 2\tau].$$
Time discretization

Completely discrete scheme based on the backward Euler method.

Difference quotient \( \partial_t U_j = (U_j - U_{j-1})/k \)

\( k \) is a time step

\( U_j \) is the approximation of \( u_j = u(t_j) \) and \( t_j = jk \).

The discrete solution \( U_j \in V_h \) is defined by:

\[
\partial_t U_j + A_h U_j = P_h f(U_j), \quad t_j > 0; \quad U_0 = u_{h,0}.
\]

Duhamel's principle yields

\[
U_j = E_{kh}^j u_{h,0} + k \sum_{l=1}^{j} E_{kh}^{j-l-1} P_h f(U_l), \quad t_j \geq 0,
\]

where \( E_{kh} = (I + kA_h)^{-1} \).
Since $A_h$ is self-adjoint positive definite we have (uniformly in $h$ and $k$)

$$\| \partial_t^l E_{kh}^j v \| = \| A_h E_{kh}^j v \| \leq C_l t_j^{-l} \| v \|, \quad t_j \geq t_l, \ v \in V_h, \ l \geq 0,$$

Smoothing property

$$\| \partial_t^l E_{kh}^j P_h f \|_\beta \leq C t_j^{-(\beta-\alpha)/2} \| f \|_\alpha, \quad t_j > 0, \ f \in \mathcal{D}(A^\alpha/2),$$

$$-1 \leq \alpha \leq \beta \leq 1, \ l = 0, 1.$$
THEOREM
For any $R_0 > 0$ there is $\tau = \tau(R_0)$ such that there is a unique solution $U_j$, $t_j \in [0, \tau]$, for any initial value $u_{h,0} \in V_h$ with $\|u_{h,0}\|_1 \leq R_0$. Moreover, there is $c$ such that $\max_{t_j \in [0,\tau]} \|U_j\|_1 \leq cR_0$.

THEOREM
Let $R \geq 0$ and $\tau > 0$ be given. Let $u(t)$ and $U_j$ be continuous and discrete solutions, such that $u(t), U_j \in B_R$ for $t, t_j \in [0, \tau]$. Then, for $k \leq k_0(R)$, we have

$$\|U_j - u(t_j)\|_1 \leq C(R, \tau)(\|u_{h,0} - P_h u_0\| t_j^{-1/2} + h t_j^{-1/2} + k t_j^{-1}), \quad t_j \in (0, \tau],$$

$$\|U_j - u(t_j)\| \leq C(R, \tau)(\|u_{h,0} - P_h u_0\| + h^2 t_j^{-1/2} + k t_j^{-1/2}), \quad t_j \in (0, \tau].$$
Lecture 4: Application to dynamical systems theory
Dynamical systems

Nonlinear semigroups:

\[ S(t, \cdot) : V \to V \]

and

\[ S_h(t, \cdot) : V_h \to V_h \]

\[ u(t) = S(t, v) \] is the solution of

\[ u' + Au = f(u), \ t > 0; \ u(0) = v \] (32)

\[ u_h(t) = S_h(t, v_h) \] is the solution of

\[ u'_h + A_h u_h = P_h f(u_h), \ t > 0; \ u_h(0) = v_h \] (33)

Assume that they are defined for all \( t \in [0, \infty) \).
Global attractor

We assume that $S(t, \cdot)$ has a global attractor $\mathcal{A}$, i.e., $\mathcal{A}$ is a compact invariant subset of $V$, which attracts the bounded sets of $V$. Thus, for any bounded set $B \subset V$ and any $\epsilon > 0$ there is $T > 0$ such that

$$S(t, B) \subset \mathcal{N}(\mathcal{A}, \epsilon), \quad t \in [T, \infty),$$

where $\mathcal{N}(\mathcal{A}, \epsilon)$ denotes the $\epsilon$-neighborhood of $\mathcal{A}$ in $V$. Or equivalently,

$$\delta(S(t, B), \mathcal{A}) \to 0 \quad \text{as } t \to \infty,$$

where $\delta(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_1$

denotes the unsymmetric semidistance between two subsets $A, B$ of $V$.

Assume that $S_h(t, \cdot)$ has a global attractor $\mathcal{A}_h$ in $V_h$.

THEOREM $\delta(\mathcal{A}_h, \mathcal{A}) \to 0$ as $h \to 0$.

In other words: for any $\epsilon > 0$ there is $h_0 > 0$ such that $\mathcal{A}_h \subset \mathcal{N}(\mathcal{A}, \epsilon)$ if $h < h_0$.

$\mathcal{A}_h$ is upper semicontinuous at $h = 0$. 
Recall: Error estimate reformulated

THEOREM

If \( S_h(t, v_h), S(t, v) \in B_R \) for \( t \in [0, 2\tau] \) then, for \( l = 0, 1 \),

\[
\|S_h(t, v_h) - S(t, v)\|_l \leq C(R, \tau)t^{-1/2}(\|v_h - P_h v\| + h^{2-l}), \quad t \in (0, 2\tau],
\]

and

\[
\|S_h(t, v_h) - S(t, v)\|_l \leq C(R, \tau)(\|v_h - P_h v\| + h^{2-l}), \quad t \in [\tau, 2\tau].
\]
Exponential stability

Let \( u(t) = S(t, u_0) \).

\( v \) is a perturbed solution starting at \( t_0 \),
if \( v(t) = S(t - t_0, v_0), t \geq t_0 \), with \( v_0 \) near \( u(t_0) \).

\( u \) is exponentially stable, if there are numbers \( \delta, T > 0 \) such that any perturbed solution \( v(t) = S(t - t_0, v_0) \) with \( \|v_0 - u(t_0)\|_1 < \delta \) satisfies

\[
\|v(t) - u(t)\|_1 \leq \frac{1}{2}\|v_0 - u(t_0)\|_1, \quad t \in [t_0 + T, \infty).
\]

Under this assumption we may prove a uniform long-time error estimate.

\[
\|u_h(t) - u(t)\|_1 \leq C\left(1 + t^{-1/2}\right)h, \quad t \in [0, \infty)
\]
Linearization

Let \( \bar{u} \in C([0, T], V) \) be a solution with \( \|\bar{u}(t)\|_1 \leq R, t \in [0, T] \), for some \( T \) and \( R \).

we rewrite the differential equation

\[
    u' + Au + B(t)u = F(t, u),
\]

where

\[
    B(t) = -f'({\bar{u}(t)}) \in \mathcal{L}(V, H),
\]
\[
    F(t, v) = f(v) - f'({\bar{u}(t)})v.
\]

Linearized homogeneous problem:

\[
    v' + Av + B(t)v = 0, \quad t > s; \quad v(s) = \phi \tag{34}
\]

\( v(t) = L(t, s)\phi \) is the solution.
Exponential dichotomy

We assume that the linear evolution operator $L(t, s)$ has an exponential dichotomy in $V$ on the interval $J = [0, T]$.

There are projections $P(t) \in \mathcal{L}(V)$, $t \in J$, and constants $M \geq 1$, $\beta > 0$ such that, for $s, t \in J$, $t \geq s$.

1. $L(t, s)P(s) = P(t)L(t, s)$.

2. The restriction $L(t, s)|_{\mathcal{R}(I - P(s))} : \mathcal{R}(I - P(s)) \to \mathcal{R}(I - P(t))$ is an isomorphism. We define $L(s, t) : \mathcal{R}(I - P(t)) \to \mathcal{R}(I - P(s))$ to be its inverse.

3. $\|L(t, s)P(s)\|_{\mathcal{L}(X)} \leq Me^{-\beta(t-s)}$.

4. $\|L(s, t)(I - P(t))\|_{\mathcal{L}(X)} \leq Me^{-\beta(t-s)}$. 
Shadowing

THEOREM
Let $\bar{u} \in C([0, T], V)$ be a solution of with

$$\max_{t \in [0, T]} \|\bar{u}(t)\|_1 \leq R,$$

for some $T$ and $R$.

Assume that the solution operator $L(t, s)$ of the linearized problem has an exponential dichotomy in $V$ on the interval $[0, T]$. Then there are numbers $\rho_0$ and $C$ such that, for each solution $u_h(t) = S_h(t, u_{h,0})$, $t \in [0, T]$, with

$$\max_{t \in [0, T]} \|u_h(t) - \bar{u}(t)\|_1 \leq \rho_0,$$

there is a solution $u$ such that

$$\|u_h(t) - u(t)\|_1 \leq C(1 + t^{-1/2})h, \quad t \in (0, T].$$

The numerical solution $u_h$ is shadowed by an exact solution $u$ in a neighborhood of $\bar{u}$. 
Open problem

Numerical shadowing: prove that shadowing can be detected “a posteriori” in the numerical computation.