# Finite element and wavelet approximation of a parabolic stochastic partial differential equation

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# Outline

Parabolic stochastic partial differential equation with additive noise

$$\begin{cases} du - \Delta u\,dt = dW, & x \in \mathcal{D},\ t > 0 \\ u = 0, & x \in \partial\mathcal{D},\ t > 0 \\ u(0) = u_0 \end{cases}$$

- Abstract framework
  - Mild solution
  - Finite element approximation
  - Error estimates
- Approximation of the noise
  - Survey of the literature
  - Wavelet multiresolution

# Co-workers

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# Abstract framework

$$\begin{cases} du + Au\,dt = dW, & t > 0 \\ u(0) = u_0 \end{cases}$$

- $H = L_2(\mathcal{D})$, $\|\cdot\|$, $(\cdot,\cdot)$, $\mathcal{D} \subset \mathbf{R}^d$, bounded domain
- $A = -\Delta$, $D(A) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$
- $u(t)$, $H$-valued random process on probability space $(\Omega, \mathcal{F}, \mathbf{P})$
- $W(t)$, $H$-valued Wiener process
- $E(t) = e^{-tA}$, analytic semigroup generated by $-A$

Mild solution:

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\,dW(s), \quad t \ge 0$$

# Wiener process

- $W(t) = \sum_{l=1}^{\infty} \gamma_l^{1/2} \beta_l(t) e_l, \qquad W(t,x,\omega) = \sum_{l=1}^{\infty} \gamma_l^{1/2} \beta_l(t,\omega) e_l(x)$
- $Qe_l = \gamma_l e_l, \quad \gamma_l > 0, \quad \{e_l\}$ ON basis
- covariance operator $Q : H \to H$, self-adjoint, positive definite, bounded, linear operator, cov$(W(t)) = tQ$
- $\beta_l(t)$, independent identically distributed, real-valued, Brownian motions

Two extreme cases:

- $\mathrm{Tr}(Q) < \infty$: $W(t)$ is an $H$-valued Wiener process

  $$\mathbf{E}\Big\| \sum_{l=1}^{\infty} \gamma_l^{1/2} \beta_l(t) e_l \Big\|^2 = \sum_{l=1}^{\infty} \gamma_l \mathbf{E} \beta_l(t)^2 = t \sum_{l=1}^{\infty} \gamma_l = t\,\mathrm{Tr}(Q) < \infty$$
- $Q = I$: $W(t)$ is not $H$-valued, since $\mathrm{Tr}(I) = \infty$, “white noise”

# Stochastic integral

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\,dW(s), \quad t \geq 0$$

- We can define the stochastic integral $\int_0^t B(s)\,dW(s)$ if $B(s)Q^{1/2}$ is a Hilbert-Schmidt operator on $H$
- Isometry property
  $$\mathbf{E}\Big\| \int_0^t B(s)\,dW(s)\Big\|^2 = \mathbf{E}\int_0^t \|B(s)Q^{1/2}\|_{HS}^2\,ds$$

Hilbert-Schmidt operator $B$, $\|B\|_{HS} < \infty$

$\|B\|_{HS}^2 = \sum_{l=1}^\infty \|B\varphi_l\|^2$, $\{\varphi_l\}$ arbitrary orthonormal basis in $H$

Da Prato and Zabczyk, *Stochastic Equations in Infinite Dimensions*

# Regularity

$$|v|_\beta = \|A^{\beta/2} v\|, \quad \dot H^\beta = D(A^{\beta/2}), \quad \beta \in \mathbf{R}$$

$$\|v\|^2_{L_2(\Omega, \dot H^\beta)} = \mathbf{E}(|v|^2_\beta) = \int_\Omega \int_{\mathcal{D}} |A^{\beta/2} v|^2 \, dx \, d\mathbf{P}(\omega), \quad \beta \in \mathbf{R}$$

**Theorem 1.** *If* $\|A^{(\beta-1)/2} Q^{1/2}\|_{HS} < \infty$ *for some* $\beta \geq 0$, *then*

$$\|u(t)\|_{L_2(\Omega,\dot H^\beta)} \leq C\Big(\|u_0\|_{L_2(\Omega,\dot H^\beta)} + \|A^{(\beta-1)/2} Q^{1/2}\|_{HS}\Big)$$

Remark: $W(t) \in L_2(\Omega, \dot H^{-(1-\beta)}) \subset L_2(\Omega, \dot H^{-1})$

Two cases:

- If $\|Q^{1/2}\|^2_{HS} = \mathrm{Tr}(Q) < \infty$, then $\beta = 1$
- If $Q = I, \ d = 1, \ A = -\frac{\partial^2}{\partial x^2}$, then $\|A^{(\beta-1)/2}\|_{HS} < \infty$ for $\beta < 1/2$

  $\|A^{(\beta-1)/2}\|^2_{HS} = \sum_j \lambda_j^{-(1-\beta)} \approx \sum_j j^{-(1-\beta)2/d} < \infty$ iff $d = 1, \beta < 1/2$

# The finite element method

- triangulations $\{\mathcal{T}_h\}_{0<h<1}$, mesh size $h$
- finite element spaces $\{S_h\}_{0<h<1}$
- $S_h \subset H_0^1(\mathcal{D}) = \dot{H}^1$
- $S_h$ continuous piecewise linear functions
- $A_h : S_h \to S_h,$ discrete Laplacian, $(A_h\psi, \chi) = (\nabla\psi, \nabla\chi), \forall \chi \in S_h$
- $P_h : L_2 \to S_h,$ orthogonal projection, $(P_h f, \chi) = (f, \chi), \forall \chi \in S_h$

$$\begin{cases} u_h(t) \in S_h, \quad u_h(0) = P_h u_0 \\ du_h + A_h u_h \, dt = P_h dW \end{cases}$$

More rigorously, with $E_h(t) = e^{-tA_h}$,

$$\begin{cases} u_h(t) \in S_h, \quad u_h(0) = P_h u_0 \\ u_h(t) = E_h(t) P_h u_0 + \displaystyle\int_0^t E_h(t-s) P_h \, dW(s) \end{cases}$$

# Error estimates for the deterministic problem

$$\begin{cases} u_t + Au = 0, & t > 0 \\ u(0) = v \end{cases} \qquad \begin{cases} u_{h,t} + A_h u_h = 0, & t > 0 \\ u_h(0) = P_h v \end{cases}$$

$$u(t) = E(t)v \qquad\qquad u_h(t) = E_h(t)P_h v$$

Denote $F_h(t)v = E_h(t)P_h v - E(t)v,\ t \geq 0$. We have, for $0 \leq \beta \leq 2$,

- $\|F_h(t)v\| \leq Ch^\beta |v|_\beta,\ t \geq 0 \qquad |v|_\beta = \|A^{\beta/2}v\|$
- $\Big(\int_0^t \|F_h(s)v\|^2\,ds\Big)^{1/2} \leq Ch^\beta |v|_{\beta-1},\ t \geq 0$
- $|F_h(t)v|_{-1} \leq Ch^{\beta+1}|v|_\beta,\ t \geq 0$
- $\Big(\int_0^t |F_h(s)v|_{-1}^2\,ds\Big)^{1/2} \leq Ch^{\beta+1}\ell_h|v|_{\beta-1}, \quad \ell_h = 1 + \ln(t/h^2),\ t > 0$

V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*

# Strong convergence in $L_2$ norm

**Theorem 2.** *If* $\|A^{(\beta-1)/2}Q^{1/2}\|_{HS} < \infty$ *for some* $\beta \in [0,2]$*, then*

$$\|u_h(t) - u(t)\|_{L_2(\Omega,H)} \le Ch^\beta \Big( \|u_0\|_{L_2(\Omega,\dot{H}^\beta)} + \|A^{(\beta-1)/2}Q^{1/2}\|_{HS} \Big)$$

Recall: $\|u_h(t) - u(t)\|^2_{L_2(\Omega,H)} = \mathbf{E}(\|u_h(t) - u(t)\|^2)$

Two cases:

- If $\|Q^{1/2}\|^2_{HS} = \mathrm{Tr}(Q) < \infty$, then the convergence rate is $O(h)$.
- If $Q = I,\ d = 1,\ A = -\frac{\partial^2}{\partial x^2}$, then the rate is almost $O(h^{1/2})$.

No result for $Q = I$, $d \ge 2$.

# Strong convergence in $\dot{H}^{-1}$ norm

**Theorem 3.** *If* $\|A^{(\beta-1)/2}Q^{1/2}\|_{HS} < \infty$ *for some* $\beta \in [0,2]$*, then*

$$\|u_h(t) - u(t)\|_{L_2(\Omega,\dot{H}^{-1})} \leq Ch^{\beta+1}\Big(\|u_0\|_{L_2(\Omega,\dot{H}^{\beta})} + \ell_h \|A^{(\beta-1)/2}Q^{1/2}\|_{HS}\Big)$$

Two cases:

- If $\|Q^{1/2}\|_{HS}^2 = \mathrm{Tr}(Q) < \infty$, then the convergence rate is $O(h^2)$.
- If $Q = I,\ d = 1,\ A = -\frac{\partial^2}{\partial x^2}$, then the rate is almost $O(h^{3/2})$.

# Strong convergence: proof

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\,dW(s)$$

$$u_h(t) = E_h(t)P_h u_0 + \int_0^t E_h(t-s)\,P_h dW(s)$$

$$F_h(t) = E_h(t)P_h - E(t)$$

$$u_h(t) - u(t) = F_h(t)u_0 + \int_0^t F_h(t-s)\,dW(s) = e_1(t) + e_2(t)$$

$\|F_h(t)u_0\| \le Ch^\beta |u_0|_\beta$ (deterministic error estimate)

$$\Longrightarrow \quad \|e_1(t)\|_{L_2(\Omega,H)} \le Ch^\beta \|u_0\|_{L_2(\Omega,\dot{H}^\beta)}$$

# Strong convergence: proof

$$\begin{cases} \mathbf{E}\Big\| \displaystyle\int_0^t B(s)\,dW(s)\Big\|^2 = \mathbf{E}\displaystyle\int_0^t \|B(s)Q^{1/2}\|_{HS}^2\,ds \text{ (isometry)} \\ \Big(\displaystyle\int_0^t \|F_h(s)v\|^2\,ds\Big)^{1/2} \le Ch^\beta |v|_{\beta-1}, \text{ with } v = Q^{1/2}\varphi_l \text{ (deterministic)} \end{cases}$$

$\Longrightarrow$

$$\begin{aligned} \|e_2(t)\|_{L_2(\Omega,H)}^2 &= \mathbf{E}\Big\| \int_0^t F_h(t-s)\,dW(s)\Big\|^2 = \int_0^t \|F_h(t-s)Q^{1/2}\|_{HS}^2\,ds \\ &= \sum_{l=1}^\infty \int_0^t \|F_h(t-s)Q^{1/2}\varphi_l\|^2\,ds \le C\sum_{l=1}^\infty h^{2\beta}|Q^{1/2}\varphi_l|_{\beta-1}^2 \\ &= Ch^{2\beta}\sum_{l=1}^\infty \|A^{(\beta-1)/2}Q^{1/2}\varphi_l\|^2 = Ch^{2\beta}\|A^{(\beta-1)/2}Q^{1/2}\|_{HS}^2 \end{aligned}$$

If $\mathrm{Tr}(Q) < \infty$, we may choose $\beta = 1$, otherwise $\beta < 1$.

# Time discretization

$$\begin{cases} du + Au\,dt = dW, & t > 0 \\ u(0) = u_0 \end{cases}$$

The implicit Euler method: $k = \Delta t,\ t_n = nk,\ \Delta W^n = W(t_n) - W(t_{n-1})$

$$\begin{cases} U^n \in S_h, \quad U^0 = P_h u_0 \\ U^n - U^{n-1} + kA_h U^n = P_h \Delta W^n, \end{cases}$$

$$U^n = E_{kh} U^{n-1} + E_{kh} P_h \Delta W^n, \quad E_{kh} = (I + kA_h)^{-1}$$

$$U^n = E_{kh}^n P_h u_0 + \sum_{j=1}^{n} E_{kh}^{n-j+1} P_h \Delta W^j$$

$$u(t_n) = E(t_n) u_0 + \int_0^{t_n} E(t_n - s)\, dW(s)$$

# Error estimates for deterministic parabolic problem

Denote $F_n = E_{kh}^n P_h - E(t_n)$

We have the following estimates for $0 \leq \beta \leq 2$:

- $\|F_n v\| \leq C(k^{\beta/2} + h^\beta)|v|_\beta$
- $\left(k \sum_{j=1}^{n} \|F_j v\|^2\right)^{1/2} \leq C(k^{\beta/2} + h^\beta)|v|_{\beta-1}$

# Strong convergence

**Theorem 4.** *If* $\|A^{(\beta-1)/2}Q^{1/2}\|_{HS} < \infty$ *for some* $\beta \in [0,2]$, *then, with* $e^n = U^n - u(t_n)$,

$$\|e^n\|_{L_2(\Omega,H)} \le C(k^{\beta/2} + h^\beta)\Big(\|u_0\|_{L_2(\Omega,\dot H^\beta)} + \|A^{(\beta-1)/2}Q^{1/2}\|_{HS}\Big)$$

J. Printems (2001) (only time-discretization)

Y. Yan, BIT (2004), SIAM J. Numer. Anal (2005)

# Approximating the noise

- No result for $Q = I$, $d \geq 2$.
- Increments $\Delta W^j$ are not directly computable: infinite series with unknown eigenfunctions $e_l$.

# Approximating the noise: white noise in 1-D

$$Q = I, \quad d = 1, \quad \mathcal{D} = (0,1)$$

SPDE: $$\frac{\partial u}{\partial t} + Au = \frac{\partial^2 W}{\partial x \partial t}$$

Piecewise constant approximation:

$$\frac{\partial^2 \widehat{W}}{\partial x \partial t} := \frac{1}{\Delta x \Delta t} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} \frac{\partial^2 W}{\partial x \partial t} \, dx \, dt = \frac{1}{\Delta x \Delta t} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} dW$$

$$\eta_{ij} := \frac{1}{\sqrt{\Delta x \Delta t}} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} dW \in \mathcal{N}(0,1)$$

$$\frac{\partial^2 W}{\partial x \partial t}(x,t) \approx \frac{\partial^2 \widehat{W}}{\partial x \partial t}(x,t) = \frac{1}{\Delta x \Delta t} \sum_{i=0}^{N} \sum_{j=0}^{M} \eta_{ij} \sqrt{\Delta x \Delta t} \, \chi_{[x_i, x_{i+1}]}(x) \, \chi_{[t_j, t_{j+1}]}(t)$$

PDE: $$\frac{\partial \hat{u}}{\partial t} + A\hat{u} = \frac{\partial^2 \widehat{W}}{\partial x \partial t}$$

# Approximating the noise: white noise in 1-D

PDE: $$\frac{\partial \hat{u}}{\partial t} + A\hat{u} = \frac{\partial^2 \widehat{W}}{\partial x \partial t}$$

Finite element or finite difference approximation $\hat{U}$:

$$\left(\mathbf{E}|\hat{U}_{ij} - u(x_i, t_j)|^p\right)^{1/p} \leq C\big(\Delta t^{1/4} + \Delta x^{1/2}\big)$$

Gyöngy (1999)
Allen, Novosel, and Zhang (1998)
Davie and Gaines (2000) (also lower bounds)
Walsh (2005)

Proof technique:

$$u(x,t) = \int_0^1 G(x,y,t) u_0(y)\,dy + \int_0^t \int_0^1 G(x,y,t-s) \frac{\partial^2 W}{\partial x \partial t}(y,s)\,dy\,ds$$

# Approximating the noise: general noise in rectangle

$\|A^{(\beta-1)/2}Q^{1/2}\|_{HS} < \infty, \quad \mathcal{D} = (0,1)^d$

Explicit eigenfunctions:

$Ae_l = \lambda_l e_l$

$Qe_l = \gamma_l e_l$

know: $\lambda_l \approx l^{2/d}, \quad$ assume: $\gamma_l \approx l^{-\alpha}$

$$\|A^{(\beta-1)/2}Q^{1/2}\|_{HS}^2 = \sum_{l=1}^{\infty} \lambda_l^{\beta-1}\gamma_l \approx \sum_{l=1}^{\infty} l^{(\beta-1)d/2-\alpha} < \infty$$

if $\alpha > 1 - (\beta - 1)d/2$

# Approximating the noise: general noise in rectangle

$\|A^{(\beta-1)/2}Q^{1/2}\|_{HS} < \infty, \quad \mathcal{D} = (0,1)^d,$ explicit eigenfunctions

1. Spectral Galerkin approximation in $x$, difference method in $t$

$$W^N(t) = \sum_{j=1}^{N} \gamma_j^{1/2} \beta_j(t) e_j, \quad u_N(x,t) = \sum_{j=1}^{N} \hat{u}_j(t) e_j(x)$$

Shardlow (1999)
Lord and Rougemont (2003)
Müller-Gronbach and Ritter (2004), (2005) (lower bounds)

# Approximating the noise: general noise in rectangle

$\|A^{(\beta-1)/2}Q^{1/2}\|_{HS} < \infty, \quad \mathcal{D} = (0,1)^d,$ explicit eigenfunctions

2. Finite element approximation in $x$, difference method in $t$

$$W^J(t) = \sum_{j=1}^{J} \gamma_j^{1/2} \beta_j(t) e_j$$

$$du_h^J + A_h u_h^J \, dt = P_h \, dW^J$$

$$\Big(\mathbf{E}\|u_h^J(t) - u_h(t)\|^2\Big)^{1/2} \le Ch^\beta \|A^{(\beta-1)/2}Q^{1/2}\|_{HS}$$

if $J \ge N_h = \dim(S_h)$

Du and Zhang (2002) ($d = 1$)
Yan (2004)

# Approx. noise: general noise in general domain

Wavelet multiresolution may provide a solution.

Initial attempt: $d = 1$, Haar basis

$$\phi = \phi_{0,0} = \chi_{[0,1]}, \quad \psi(x) = \phi(2x) - \phi(2x-1), \quad \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$$

$\{\phi_{0,0}, \psi_{j,k}\}_{k=0,\ j=0}^{2^j-1,\ \infty}$ is an ON basis in $L_2((0,1))$.

$$W(t) = \gamma_{0,0}^{1/2}\beta_{0,0}(t)\phi_{0,0}(x) + \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}\gamma_{j,k}^{1/2}\beta_{j,k}(t)\psi_{j,k}(x)$$

$$W^J(t) = \gamma_{0,0}^{1/2}\beta_{0,0}(t)\phi_{0,0}(x) + \sum_{j=0}^{J}\sum_{k=0}^{2^j-1}\gamma_{j,k}^{1/2}\beta_{j,k}(t)\psi_{j,k}(x)$$

# Wavelet multiresolution

$$W^J(t) = \gamma_{0,0}^{1/2}\beta_{0,0}(t)\phi_{0,0}(x) + \sum_{j=0}^{J}\sum_{k=0}^{2^j-1}\gamma_{j,k}^{1/2}\beta_{j,k}(t)\psi_{j,k}(x)$$

Framework of Yan is directly applicable. For example:

$$\sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}\gamma_{j,k} < \infty \Longrightarrow \mathrm{Tr}(Q) < \infty \Longrightarrow$$

$$\left(\mathbf{E}\|u_h^J(t) - u(t)\|^2\right)^{1/2} \le Ch\Big(\|u_0\|_{L_2(\Omega,\dot{H}^1)} + \mathrm{Tr}(Q)^{1/2}\Big)$$

For general domains $\mathcal{D}$ with $d > 1$ we may have to give up orthogonality. For example: biorthogonal wavelets.

Hint: Le Maître, et al.