TMV035 Analysis and Linear Algebra B, 2005

LECTURE 4.1

In this lecture we present analytic solution of differential equations. This covers AMBS Ch 35, 38, 39.

Analytic solution of differential equations

0.1. **Introduction.** In AMBS Ch 40 we construct a unique solution of the general system of ordinary differential equations:

$$
\begin{aligned}
&u'(x) = f(x, u(x)), \quad x \in [a, b], \\
&u(a) = u_a.
\end{aligned}
$$

This means that all (reasonable) differential equations can be solved. We also have an algorithm for the computation of the solutions. Some of these functions are so important that we give them names. For example,

$$
\begin{aligned}
&u'(x) = u(x), \quad x \geq 0, \\
&u(0) = 1,
\end{aligned}
$$

the solution is called $u(x) = \exp(x)$; and

$$
\begin{aligned}
&u''(x) + u(x) = 0, \quad x \geq 0, \\
&u(0) = u_0,\ u'(0) = u_1,
\end{aligned}
$$

the solution is $u(x) = u_0 \cos(x) + u_1 \sin(x)$.

Sometimes we can express the solution of other initial-value problems as a combination of our elementary functions: polynomials, log, exp, cos, sin. We can only do this in some simple cases, in general we have to solve differential equations approximately on the computer.

We will obtain analytical solution formulas in the following four cases:

(1) linear equation of first order (with constant or variable coefficient);
(2) linear equation of second order with constant coefficients;
(3) systems of linear equations with constant coefficients (in ALA-C);
(4) separable nonlinear equation.

Note that cases 1, 2, 4 are scalar equations (only one equation) while 3 concerns systems of equations. It is important to recognize these types of equations and to know how to solve them.

1. Linear equations

1.1. **Linear equation of first order.** A linear equation of first order is of the form:

$$u' + a(t)u = f(t).$$

Here $u = u(t)$ is an unknown function of an independent variable t (we write t instead of x because the independent variable is often time). The equation is called *homogeneous* if $f(t) \equiv 0$ and *nonhomogeneous* otherwise. The differential operator $Lu = u' + a(t)u$ has *constant coefficient* if $a(t) = a$ is constant and it has *variable coefficient* otherwise. The equation is said to be a *linear equation*, because the operator L is a *linear operator*:

$$L(\alpha u + \beta v) = \alpha Lu + \beta Lv, \quad (\alpha, \beta \in \mathbf{R},\ u = u(t),\ v = v(t))$$

i.e., it preserves linear combinations of functions. Check this!

Date: November 13, 2005, Stig Larsson, Mathematical Sciences, Chalmers University of Technology.

1.2. **Constant coefficient, homogeneous equation.**

$$\begin{aligned} &u'(t) + au = 0, \quad t \geq 0, \\ &u(0) = u_0. \end{aligned}$$

We already know that the unique solution is $u(t) = u_0 \exp(-at)$.

1.3. **Variable coefficient, homogeneous equation.**

$$\begin{aligned} &u'(t) + a(t)u = 0, \quad t \geq 0, \\ &u(0) = u_0. \end{aligned}$$

Guess: $u(t) = u_0 \exp(b(t))$ where b is unknown. The initial condition gives:

$$u_0 = u(0) = u_0 \exp(b(0)) \implies b(0) = 0.$$

The differential equation gives:

$$-a(t)u(t) = u'(t) = u_0 \exp(b(t))b'(t) = u(t)b'(t)$$

so that $b' = -a$. Therefore b is given by

$$\begin{aligned} &b'(t) = -a(t), \quad t \geq 0, \\ &b(0) = 0. \end{aligned}$$

Fundamental theorem:

$$b(t) = b(0) - \int_0^t a(s)\,ds = -A(t), \quad A(t) = \int_0^t a(s)\,ds.$$

Thus:

$$u(t) = u_0 \exp(-A(t)), \quad A(t) = \int_0^t a(s)\,ds.$$

1.4. **Constant coefficient, inhomogeneous equation.**

$$\begin{aligned} &u'(t) + au = f(t), \quad t \geq 0, \\ &u(0) = u_0. \end{aligned}$$

Multiply by the integrating factor e^{at}:

$$\mathrm{e}^{at}u'(t) + \mathrm{e}^{at}au(t) = \mathrm{e}^{at}f(t).$$

The left-hand side is an exact derivative:

$$\frac{d}{dt}(\mathrm{e}^{at}u(t)) = \mathrm{e}^{at}u'(t) + \mathrm{e}^{at}au(t) = \mathrm{e}^{at}f(t),$$

so we can integrate:

$$\begin{aligned} \int_0^T \frac{d}{dt}(\mathrm{e}^{at}u(t))\,dt &= \int_0^T \mathrm{e}^{at}f(t)\,dt \\ \mathrm{e}^{aT}u(T) - \mathrm{e}^{a0}u(0) &= \int_0^T \mathrm{e}^{at}f(t)\,dt \\ \mathrm{e}^{aT}u(T) &= u_0 + \int_0^T \mathrm{e}^{at}f(t)\,dt \\ u(T) &= \mathrm{e}^{-aT}u_0 + \int_0^T \mathrm{e}^{-a(T-t)}f(t)\,dt \end{aligned}$$

Finally replace $t \to s$, $T \to t$:

$$u(t) = \mathrm{e}^{-at}u_0 + \int_0^t \mathrm{e}^{-a(t-s)}f(s)\,ds \tag{1}$$

1.5. **Variable coefficient, inhomogeneous equation.**

$$\begin{aligned} &u'(t) + a(t)u = f(t), \quad t \geq 0, \\ &u(0) = u_0. \end{aligned}$$

Multiply by the integrating factor $\mathrm{e}^{A(t)}$ with the primitive function $A(t) = \int_0^t a(s)\,ds$:

$$\mathrm{e}^{A(t)}u'(t) + \mathrm{e}^{A(t)}a(t)u(t) = \mathrm{e}^{A(t)}f(t).$$

The left-hand side is an exact derivative because $D\mathrm{e}^{A(t)} = \mathrm{e}^{A(t)}A'(t) = \mathrm{e}^{A(t)}a(t)$:

$$\frac{d}{dt}(\mathrm{e}^{A(t)}u(t)) = \mathrm{e}^{A(t)}u'(t) + \mathrm{e}^{A(t)}a(t)u(t) = \mathrm{e}^{A(t)}f(t),$$

so we can integrate:

$$\begin{aligned} \int_0^T \frac{d}{dt}(\mathrm{e}^{A(t)}u(t))\,dt &= \int_0^T \mathrm{e}^{A(t)}f(t)\,dt \\ \mathrm{e}^{A(T)}u(T) - \mathrm{e}^{A(0)}u(0) &= \int_0^T \mathrm{e}^{A(t)}f(t)\,dt \\ \mathrm{e}^{A(T)}u(T) &= u_0 + \int_0^T \mathrm{e}^{A(t)}f(t)\,dt \\ u(T) &= \mathrm{e}^{-A(T)}u_0 + \int_0^T \mathrm{e}^{-A(T)+A(t)}f(t)\,dt \end{aligned}$$

Finally: replace $t \to s$, $T \to t$

$$u(t) = \mathrm{e}^{-A(t)}u_0 + \int_0^t \mathrm{e}^{-A(t)+A(s)}f(s)\,ds$$

Summary. The linear equation of first order is solved by the method of integrating factor: multiply by the *integrating factor* $e^{A(t)}$ with $A(t) = \int_0^t a(s)\,ds$, and integrate.

1.6. **Linear differential equation—second order—constant coefficients.**

$$u'' + a_1u' + a_0u = f(t). \tag{2}$$

The equation is called *homogeneous* if $f(t) \equiv 0$ and *nonhomogeneous* otherwise. We assume that the differential operator $Lu = u'' + a_1u' + a_0u$ has *constant coefficients* a_1 and a_0. Check that the operator L is linear!

Variable coefficients: Linear differential equations of second order with variable coefficients $u'' + a_1(t)u' + a_0(t)u = f(t)$, cannot be solved analytically, except in some special cases. One such case can be found in AMBS Ch 35.6. We do not discuss this here.

Homogeneous equation. See AMBS Ch 35.3–35.4. The homogeneous equation (2) may be written

$$D^2u + a_1Du + a_0u = 0, \tag{3}$$

or

$$P(D)u = 0,$$

where

$$P(r) = r^2 + a_1r + a_0$$

is the *characteristic polynomial* of the equation. The *characteristic equation* $P(r) = 0$ has two roots r_1 and r_2. Hence $P(r) = (r - r_1)(r - r_2)$. All solutions of equation (2) are obtained as linear combinations

$$\begin{aligned} u(t) &= c_1e^{r_1t} + c_2e^{r_2t}, && \text{if } r_1 \neq r_2, \\ u(t) &= (c_1 + c_2t)e^{r_1t}, && \text{if } r_1 = r_2, \end{aligned} \tag{4}$$

where c_2, c_2 are arbitrary coefficients. The coefficients may be determined from an initial condition of the form

$$u(0) = u_0, \ u'(0) = u_1.$$

The formula (4) is called the *general solution* of homogeneous linear equation (3).

Proof. We write the equation as

$$P(D)u = (D - r_1)(D - r_2)u = 0$$

and solve two first order equations $(D - r_1)v = 0$ and $(D - r_2)u = v$. First we get

$$(D - r_1)v = 0 \implies v(t) = A\mathrm{e}^{r_1 t}, \quad A = v(0).$$

Then the other equation

$$(D - r_2)u(t) = v(t) = A\mathrm{e}^{r_1 t}$$

is solved by multiplying by integrating factor $\mathrm{e}^{-r_2 t}$, see (1),

$$\begin{aligned} u(t) &= B\mathrm{e}^{r_2 t} + \mathrm{e}^{r_2 t}\int_0^t \mathrm{e}^{-r_2 s} v(s)\, ds \quad \left\{ \ B = u(0) \ \right\} \\ &= B\mathrm{e}^{r_2 t} + A\mathrm{e}^{r_2 t}\int_0^t \mathrm{e}^{(r_1 - r_2)s)}\, ds \\ &= B\mathrm{e}^{r_2 t} + \begin{cases} A\dfrac{\mathrm{e}^{r_1 t} - \mathrm{e}^{r_2 t}}{r_1 - r_2}, & r_1 \neq r_2 \\ At\mathrm{e}^{r_2 t}, & r_1 = r_2 \end{cases} \\ &= \begin{cases} c_1\mathrm{e}^{r_1 t} + c_1\mathrm{e}^{r_1 t}, & r_1 \neq r_2, \\ (c_1 + c_2 t)\mathrm{e}^{r_1 t}, & r_1 = r_2. \end{cases} \end{aligned}$$

□

Example 1. We solve

$$u'' + u' - 12u = 0; \quad u(0) = u_0, \ u'(0) = u_1.$$

The equation is written $(D^2 - D - 12)u = 0$ and the characteristic equation is $r^2 + r - 12 = 0$ with roots $r_1 = 3$, $r_2 = -4$. The general solution is

$$u(t) = c_1 e^{3t} + c_2 e^{-4t}$$

with the derivative

$$u'(t) = 3c_1 e^{3t} - 4c_2 e^{-4t}.$$

The initial condition gives

$$\begin{aligned} u_0 &= u(0) = c_1 + c_2 \\ u_1 &= u'(0) = 3c_1 - 4c_2 \end{aligned}$$

which implies $c_1 = (4u_0 + u_1)/7$, $c_2 = (3u_0 - u_1)/7$. The solution is

$$u(t) = \frac{4u_0 + u_1}{7} e^{3t} + \frac{3u_0 - u_1}{7} e^{-4t}.$$

Complex roots. If the characteristic polynomial $P(r)$ has real coefficients, then its roots are either real numbers or a pair of conjugate complex numbers, see AMBS Ch 22.11. In the latter case we have $r_1 = \alpha + i\omega$ and $r_2 = \alpha - i\omega$ and the solution (4) becomes (see AMBS Ch 33.2 for the definition of $\exp(z)$ with a complex variable z)

$$\begin{aligned} u(t) &= c_1 e^{(\alpha+i\omega)t} + c_2 e^{(\alpha-i\omega)t} \\ &= e^{\alpha t}\Big(c_1 e^{i\omega t} + c_2 e^{-i\omega t}\Big) \\ &= e^{\alpha t}\Big(c_1\big(\cos(\omega t) + i\sin(\omega t)\big) + c_2\big(\cos(\omega t) - i\sin(\omega t)\big)\Big) \\ &= e^{\alpha t}\Big(d_1\cos(\omega t) + d_2\sin(\omega t)\Big), \end{aligned}$$

with $d_1 = c_1 + c_2$, $d_2 = i(c_1 - c_2)$.

Nonhomogeneous equation. See AMBS Ch 35.5. The solution of the nonhomogeneous equation $P(D)u = f(t)$ is given by

$$u(t) = u_h(t) + u_p(t), \tag{5}$$

where u_h is the general solution (4) of the corresponding homogeneous equation, i.e., $P(D)u_h = 0$, and u_p is a *particular solution* of the nonhomogeneous equation, i.e., $P(D)u_p = f(t)$.

Proof: If u is given by (5), then $Lu = L(u_h + u_p) = Lu_h + Lu_p = 0 + f = f$, so that u solves the nonhomogeneous equation. On the other hand: if u_p is a particular solution and u is any other solution of the nonhomogeneous equation, then $L(u - u_p) = Lu - Lu_p = f - f = 0$, i.e., $u - u_p$ solves the homogeneous equation. Thus $u - u_p = u_h$, which is (5).

A particular solution can sometimes be found by guess-work: we make an Ansatz for u_p of the same form as f.

Example 2. $u'' - u' - 2u = t$. Here $f(t) = t$ is a polynomial of degree 1 and we make the Ansatz $u_p(t) = At + B$, i.e., a polynomial of degree 1. Substitution into the equation gives $-A - 2(At + B) = t$. Identification of coefficients gives $A = -\frac{1}{2}$, $B = \frac{1}{4}$, so that $u_p(t) = \frac{1}{4} - \frac{1}{2}t$. The general solution of the homogeneous equation is $u_h(t) = c_1 e^{-t} + c_2 e^{2t}$. Hence we get

$$u(t) = u_h(t) + u_p(t) = c_1 e^{-t} + c_2 e^{2t} + \frac{1}{4} - \frac{1}{2}t.$$

Re-writing as a system of first order equations. By setting $w_1 = u$, $w_2 = u'$, $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, we can re-write (2) as a system of first order equations

$$w'(t) = Aw(t) + F(t); \quad w(0) = w_0,$$

where

$$w_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \quad F(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}.$$

To see this we compute

$$w' = \begin{bmatrix} u' \\ u'' \end{bmatrix} = \begin{bmatrix} u' \\ -a_0 u - a_1 u' + f(t) \end{bmatrix} = \begin{bmatrix} w_2 \\ -a_0 w_1 - a_1 w_2 + f(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}.$$

It is necessary to do this re-writing before we can use our MATLAB programs to solve (2).

2.3 System of linear differential equations of first order.

Constant coefficients—homogeneous equations. We finally mention

$$\begin{aligned} &u' + Au = 0, \quad t > 0, \\ &u(0) = u_0, \end{aligned} \tag{6}$$

where $u(t), u_0 \in \mathbf{R}^d$, and $A \in \mathbf{R}^{d\times d}$ is a constant matrix of coefficients. This kind of system will studied by means of eigenvalues and eigenvectors in the following course ALA-C.

2. Nonlinear equation

The only kind of nonlinear equation that we solve analytically is the so-called separable equation.

2.1. **Separable equation.**

A nonlinear differential equation

$$u'(x) = f(x, u(x))$$

is called *separable* if $f(x, u) = h(x)k(u)$. It is called *autonomous* if $f(x, u) = k(u)$. Of course, an autonomous equation is separable. Separable equations can sometimes be solved by integration. Let us write it in the form:

$$\begin{aligned} &u'(x) = \frac{h(x)}{g(u(x))}, \ x \in [a, b], \quad \Big\{\text{it is convenient to write } k(u) = \frac{1}{g(u)}\Big\} \\ &u(a) = u_a. \end{aligned}$$

We separate the variables:

$$g(u(x))u'(x) = h(x).$$

We now assume that we can find primitive functions G and H for g and h. Then the left-hand side is an exact derivative:

$$\frac{d}{dx}G(u(x)) = g(u(x))u'(x) = h(x) = \frac{d}{dx}H(x)$$

and we can integrate:

$$\begin{aligned} \Big[G(u(x))\Big]_{x=a}^{y} &= \Big[H(x)\Big]_{x=a}^{y} \\ G(u(y)) - G(u_a) &= H(y) - H(a) \\ G(u(x)) &= G(u_a) + H(x) - H(a). \end{aligned}$$

We then solve for $u(x)$ if we can. This means that we must find an inverse function to G, i.e., $u(x) = G^{-1}(G(u_a) + H(x) - H(a))$.

This calculation is easier to remember if we write it as follows:

$$\begin{aligned} g(u)\frac{du}{dx} &= h(x) \\ g(u)\,du &= h(x)\,dx \\ \int_{u=u_a}^{u(y)} g(u)\,du &= \int_{x=a}^{y} h(x)\,dx \\ \Big[G(u)\Big]_{u=u_a}^{u(y)} &= \Big[H(x)\Big]_{x=a}^{y} \\ G(u(y)) - G(u_a) &= H(y) - H(a) \\ G(u(x)) &= G(u_a) + H(x) - H(a). \end{aligned}$$

This only works in simple cases. What can go wrong? We cannot find primitive functions for g and h, or we cannot solve for $u(x)$ in the last step.

Example.

$$\begin{aligned} u'(x) &= u(x) \\ u(0) &= u_0 \end{aligned}$$

The equation is separable:

$$\begin{aligned} \int_{u_0}^{u(y)} \frac{du}{u} &= \int_0^y dx \\ \log\Big(\Big|\frac{u(y)}{u_0}\Big|\Big) &= y \end{aligned}$$

We can solve for $u(y)$ by taking the exponential:

$$\begin{aligned}
&\left|\frac{u(y)}{u_0}\right| = \exp(y) \\
&|u(y)| = |u_0| \exp(y) \\
&u(y) = \pm u_0 \exp(y) = u_0 \exp(y) \quad \Big\{\text{must be } + \text{ by taking } y = 0\Big\}
\end{aligned}$$

Example.

$$\begin{aligned}
&u'(t) = ku^3(t), \quad t > 0, \\
&u(0) = u_0
\end{aligned}$$

The equation is separable:

$$\begin{aligned}
&\frac{1}{u^3}\frac{du}{dt} = k \\
&\int_{u_0}^{u(T)} \frac{du}{u^3} = \int_0^T k\,dt \\
&\left[\frac{u^{-2}}{-2}\right]_{u_0}^{u(T)} = kT \\
&\frac{1}{u(T)^2} = \frac{1}{u_0^2} - 2kT
\end{aligned}$$

We can easily solve for $u(T)$:

$$u(T) = \pm\sqrt{\frac{u_0^2}{1 - 2kTu_0^2}} = \pm\frac{u_0}{\sqrt{1-2kTu_0^2}} = \frac{u_0}{\sqrt{1-2kTu_0^2}}, \quad T < \frac{1}{2ku_0^2},$$

where we decided about the plus sign by taking $T = 0$. Therefore:

$$u(t) = \frac{u_0}{\sqrt{1-2kTu_0^2}}, \quad \text{for } 0 \le t < \frac{1}{2ku_0^2}.$$

Example. The logistic equation. We know that $u' = ku$ $(k > 0)$ leads to exponential growth: $u(t) = u_0 e^{kt}$. In real-world problems the exponential growth is usually broken when u becomes large. One simple way of modelling this is to replace the rate ku by $ku(1 - u/M)$ for some large number M. This means that the rate is $\approx ku$ when u is small but the rate is ≈ 0 when u approaches M. We thus consider

$$\begin{aligned}
&u' = ku(1 - u/M), \quad t > 0, \\
&u(0) = u_0
\end{aligned}$$

The equation is separable:

$$\begin{aligned}
&\frac{1}{u(1-u/M)}\frac{du}{dt} = k \\
&\int_{u_0}^{u(T)} \frac{du}{u(1-u/M)} = k\int_0^T k\,dt
\end{aligned}$$

Partial fractions:

$$\frac{1}{u(1-u/M)} = \frac{M}{u(M-u)} = \frac{1}{u} - \frac{1}{u-M}$$

Now we can integrate:

$$\int_{u_0}^{u(T)} \left(\frac{1}{u} - \frac{1}{u-M}\right) du = kT$$
$$\Big[\log(|u|) - \log(|u-M|)\Big]_{u_0}^{u(T)} = kT$$
$$\log\left(\left|\frac{u(T)}{M-u(T)}\frac{M-u_0}{u_0}\right|\right) = kT$$
$$\left|\frac{u(T)}{M-u(T)}\frac{M-u_0}{u_0}\right| = \mathrm{e}^{kT}$$
$$\left|\frac{u(T)}{M-u(T)}\right| = \left|\frac{u_0}{M-u_0}\right|\mathrm{e}^{kT}$$
$$\frac{u(T)}{M-u(T)} = \pm\frac{u_0}{M-u_0}\mathrm{e}^{kT}$$
$$\frac{u(T)}{M-u(T)} = \frac{u_0}{M-u_0}\mathrm{e}^{kT}$$

where we decided about the plus sign by taking $T = 0$. We solve for $u(t)$:

$$u(t) = \frac{Mu_0}{u_0 + (M-u_0)\mathrm{e}^{-kt}}$$

Notice that $\mathrm{e}^{-kt} \to 0$ as $t \to \infty$ so that

$$u(t) \to \frac{Mu_0}{u_0 + 0} = M \quad \text{as } t \to \infty$$

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