Nedelec elements for computational electromagnetics

Per Jacobsson, June 5, 2007

1 Maxwell's equations

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{1}$$
$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J} \tag{2}$$
$$\nabla \cdot (\varepsilon\mathbf{E}) = \rho \tag{3}$$
$$\nabla \cdot (\mu\mathbf{H}) = 0 \tag{4}$$
$$\nabla \cdot \mathbf{J} = -j\omega\rho \tag{5}$$

Only three of the equations are needed, (1), (2) and (3) or (1), (2) and (5). Combining (1) and (2) gives the wave equation

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{E}) - \omega^2\varepsilon\mathbf{E} = -j\omega\mathbf{J} \tag{6}$$

for $\mathbf{E}$ and similarly for $\mathbf{H}$.

$\mathbf{E}, \mathbf{J} \in H(\text{curl}; \Omega)$ corresponds to finite energy solutions, where

$$H(\text{curl}; \Omega) = \{\mathbf{u} \in [L^2(\Omega)]^3 : \nabla \times \mathbf{u} \in [L^2(\Omega)]^3\} \tag{7}$$

in three dimensions. The norm associated with $H(\text{curl}; \Omega)$ is defined as

$$||\mathbf{v}||_{H(\text{curl};\Omega)} = (||\mathbf{v}||_{[L^2(\Omega)]^3} + ||\nabla \times \mathbf{v}||_{[L^2(\Omega)]^3})^{1/2} \tag{8}$$

Some other vector spaces used:

$$H^1(\Omega) = \{v \in L^2(\Omega) : \ \nabla v \in [L^2(\Omega)]^3\} \tag{9}$$
$$H_0^1(\Omega) = \{v \in H^1(\Omega) : \ v = 0 \text{ on } \partial\Omega\} \tag{10}$$
$$H_0(\text{curl}; \Omega) = \{\mathbf{v} \in H(\text{curl}; \Omega) : \ \boldsymbol{\nu} \times \mathbf{v} = 0\} \tag{11}$$
$$H(\text{div}; \Omega, \varepsilon) = \{\mathbf{v} \in [L^2(\Omega)]^3 : \ \nabla \cdot (\varepsilon\mathbf{v}) \in L^2(\Omega)\} \tag{12}$$
$$H_0(\text{div}; \Omega, \varepsilon) = \{\mathbf{v} \in [L^2(\Omega)]^3 : \ \nabla \cdot (\varepsilon\mathbf{v}) = 0\} \tag{13}$$
$$V = H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega, \varepsilon) \tag{14}$$

where $\boldsymbol{\nu}$ is the outward normal to Ω.

2 Problems when using nodal elements

When using nodal elements spurious solutions can occur and we can also get convergence to solutions of a problem different from our original problem. As an example we look at a simple cavity eigenvalue problem with perfectly conducting walls

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{E}) = \omega^2\varepsilon\mathbf{E} \qquad \text{in } \Omega \tag{15}$$
$$\nabla \cdot (\varepsilon\mathbf{E}) = 0 \qquad \text{in } \Omega \tag{16}$$
$$\boldsymbol{\nu} \times \mathbf{E} = 0 \qquad \text{on } \partial\Omega \tag{17}$$

The weak form of this eigenvalue problem is to find $\omega \in \mathbb{R}$, $\mathbf{E} \in H_0(\text{curl};\Omega) \cap H_0(\text{div};\Omega,\varepsilon)$, $\mathbf{E} \neq 0$ such that

$$(\mu^{-1}\nabla \times \mathbf{E}, \nabla \times \mathbf{v}) = \omega^2(\varepsilon\mathbf{E}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0(\text{curl};\Omega) \cap H_0(\text{div};\Omega,\varepsilon). \tag{18}$$

We see that $\omega^2 = 0$ is not an eigenvalue to (18) since that would imply $\mathbf{E} = 0$. This means that we cannot have a physical mode with non-zero energy and $\omega = 0$, the only physical modes are the ones with $\omega > 0$. For this case, (15) implies (16) and we can write the problem on weak form as
Find $\omega \in \mathbb{R}$, $\mathbf{E} \in H_0(\text{curl};\Omega)$, $\mathbf{E} \neq 0$ s.t.

$$(\mu^{-1}\nabla \times \mathbf{E}, \nabla \times \mathbf{v}) = \omega^2(\varepsilon\mathbf{E}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0(\text{curl};\Omega). \tag{19}$$

This adds an infinite-dimensional space of non-physical eigensolutions that coincides with $\nabla H_0^1(\Omega)$ with zero eigenvalues. It is important to recognize these non-physical solutions so that they can be discarded.

When using FE we seek solutions in a finite-dimensional subspace V_h of $H_0(\text{curl};\Omega)$:
Find $\omega_h \in \mathbb{R}$, $\mathbf{E}_h \in V_h$, $\mathbf{E}_h \neq 0$ s.t.

$$(\mu^{-1}\nabla \times \mathbf{E}_h, \nabla \times \mathbf{v}) = \omega_h(\varepsilon\mathbf{E}_h, \mathbf{v}) \quad \forall \mathbf{v} \in V_h. \tag{20}$$

For nodal elements the approximation space is $V_h = [P_k]^3 \cap H_0(\text{curl};\Omega)$. When using nodal based FE on an unstructured grid, we obtain many spurious eigenvalues approaching zero and it can be hard to recognize these so that they can be discarded. For certain structured grids, the zero eigenvalues can be well approximated but other spurious solutions appear in the spectrum. Since these spurious eigenvalues can be distributed all over the spectrum, they are hard to distinguish from the physical eigenvalues. For non-convex domains, it has been proved that the discrete eigenvalues computed using nodal elements converge to the eigenvalues of a problem different than the original problem.

One possible explanation for the failure of nodal based FE in the case of non-convex domains in two dimensions is that the nodal elements are constrained in $[H^1(\Omega)]^2$, which is a proper and closed subset of V. This means that there are eigenfunctions in V that cannot be approximated by nodal elements in $[H^1(\Omega)]^2$.

Another important drawback of nodal-based FE when solving Maxwell's equations is the inability to model field singularities at conducting corners and tips. The boundary condition for the electric field at a conducting surface is $\boldsymbol{\nu} \times \mathbf{E} = 0$, but enforcing this boundary condition at each node also enforces normal continuity, which is not desired.

3 Nedelec's edge elements

We start by introducing a subspace $\mathcal{S}_k$ of homogeneous vector polynomials of degree k in d dimensions by

$$\mathcal{S}_k = \{\mathbf{p} \in [\tilde{P}_k]^d : \ \mathbf{x} \cdot \mathbf{p} = 0\}, \tag{21}$$

where $\tilde{P}_k$ is the space of homogeneous polynomials of order k. We can now introduce the space $\mathcal{R}_k$ which is needed for Nedelec's vector elements,

$$\mathcal{R}_k = [P_{k-1}]^d \oplus \mathcal{S}_k. \tag{22}$$

The dimension of the space $\mathcal{R}_k$ in three dimensions is $\dim(\mathcal{R}_k) = \frac{1}{2}(k+3)(k+2)k$, so for $k = 1$ we have $\dim(\mathcal{R}_1) = 6$.

In two dimensions, a polynomial $\mathbf{p} \in \mathcal{R}_1$ has dimension 3 and can be represented as

$$\mathbf{p} = (c_1 + c_2 y, c_3 - c_2 x). \tag{23}$$

For $k = 1$, the local degrees of freedom in two dimensions are given by

$$N_i(\mathbf{p}) = \int_{e_i} \mathbf{p} \cdot \boldsymbol{\tau}_i ds, \tag{24}$$

where $\boldsymbol{\tau}_i$ is the tangential unit vector of edge e_i.

The local shape functions can be expressed in terms of barycentric coordinates. In two dimensions we can define the local shape function $\boldsymbol{\theta}_1$ associated with edge 1 oriented from node 1 to node 2 as

$$\boldsymbol{\theta}_1 = \lambda_1 \nabla \lambda_2 - \lambda_2 \nabla \lambda_1, \tag{25}$$

where the barycentric coordinates for node 1 and 2 can be written as $\lambda_1 = a_1 + b_1 x + c_1 y$ and $\lambda_2 = a_2 + b_2 x + c_2 y$ respectively, where a_i and b_i depend on the nodal coordinates of the triangle. Now

$$\begin{aligned}\boldsymbol{\theta}_1 &= \lambda_1 \nabla \lambda_2 - \lambda_2 \nabla \lambda_1 = (a_1 + b_1 x + c_1 y)(b_2, c_2) - (a_2 + b_2 x + c_2 y)(b_1, c_1) \\ &= (d_1 + d_2 y, d_3 - d_2 x) \in \mathcal{R}_1.\end{aligned} \tag{26}$$

The local shape functions for triangular elements in two dimensions are shown in Fig. 1.

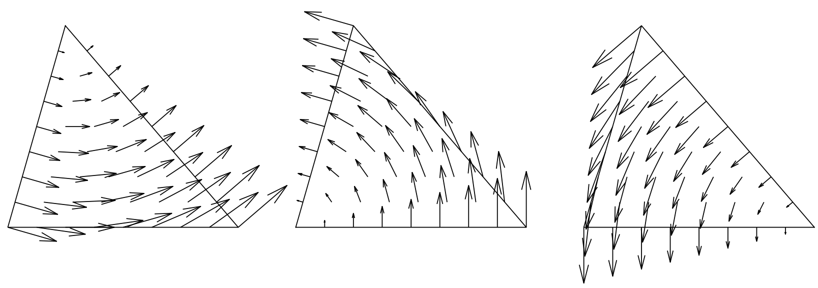

Figure 1: The local shape functions for $k = 1$ in two dimensions on a triangle.

3.1 Affine maps

When transforming to and from the reference element we need to transform in a special way since we are working in $H(\text{curl}; \hat{K})$ with vectors in $\mathcal{R}_k$. For the affine map $F_K \hat{\mathbf{x}} = B_K \hat{\mathbf{x}} + \mathbf{b}_K$ and $\hat{\mathbf{u}} \in \mathcal{R}_k$, the transformation is given by

$$\mathbf{u} \circ F_K = (B_K^T)^{-1} \hat{\mathbf{u}} \tag{27}$$

and the curl transforms as

$$\nabla \times \mathbf{u} = \frac{1}{\det(B_K)} B_K \hat{\nabla} \times \hat{\mathbf{u}}. \tag{28}$$

Lemma 1. *The space $\mathcal{R}_k$ is invariant under the transformation (27)*

Proof. Since for $\hat{\mathbf{u}} \in \mathcal{R}_k$ we can write $\hat{\mathbf{u}} = \hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2$, where $\hat{\mathbf{p}}_1 \in [P_{k-1}]^3$ and $\hat{\mathbf{p}}_2 \in \mathcal{S}_k$. Now,

$$\begin{aligned}\mathbf{u}(\mathbf{x}) &= [(B_K^T)^{-1} \hat{\mathbf{p}}_1 + (B_K^T)^{-1} \hat{\mathbf{p}}_2](F_K^{-1}(\mathbf{x})) \\ &= [(B_K^T)^{-1} \hat{\mathbf{p}}_1](F_K^{-1}(\mathbf{x})) + (B_K^T)^{-1} \hat{\mathbf{p}}_2 (B_K^{-1} \mathbf{x} - B_K^{-1} \mathbf{b}_K).\end{aligned} \tag{29}$$

Since $\hat{\mathbf{p}}_2 \in [\tilde{P}_k]^3$, we have that $\hat{\mathbf{p}}_2(B_K^{-1}\mathbf{x} - B_K^{-1}\mathbf{b}_K) = \hat{\mathbf{p}}_2(B_K^{-1}\mathbf{x}) + \hat{\mathbf{p}}_3(\mathbf{x})$, where $\hat{\mathbf{p}}_3 \in [P_{k-1}]^3$. This means

$$\mathbf{u}(\mathbf{x}) = [(B_K^T)^{-1}\hat{\mathbf{p}}_1(F_K^{-1}(\mathbf{x})) + \hat{\mathbf{p}}_3(\mathbf{x})] + (B_K^T)^{-1}\hat{\mathbf{p}}_2(B_K^{-1}\mathbf{x}). \tag{30}$$

We have that $(B_K^T)^{-1}\hat{\mathbf{p}}_1 \circ F_K^{-1} + \hat{\mathbf{p}}_3 \in [P_{k-1}]^3$ and also that $(B_K^T)^{-1}\hat{\mathbf{p}}_2(B_K^{-1}\mathbf{x}) \cdot \mathbf{x} = \hat{\mathbf{p}}_2(B_K^{-1}\mathbf{x}) \cdot (B_K^{-1}\mathbf{x}) = 0$, since $\hat{\mathbf{p}}_2 \in \mathcal{S}_k$. Thus, $\mathbf{u} \in \mathcal{R}_k$. □

Lemma 2. *If τ_h is a regular mesh and $s \geq 0$, then we have for $\mathbf{v}$ transformed by (27) to give $\hat{\mathbf{v}}$*

$$|\hat{\mathbf{v}}|_{[H^s(\hat{K})]^3} \leq C h_K^{s-\frac{1}{2}} |\mathbf{v}|_{[H^s(K)]^3} \tag{31}$$

and

$$|\hat{\nabla} \times \hat{\mathbf{v}}|_{[H^s(\hat{K})]^3} \leq C h_K^{s+\frac{1}{2}} |\nabla \times \mathbf{v}|_{[H^s(K)]^3}. \tag{32}$$

3.2 Degrees of freedom

For the reference element $\hat{K}$, which is a tetrahedron defined by the vertices $\hat{\mathbf{a}}_1, \ldots, \hat{\mathbf{a}}_4$ given by $\hat{\mathbf{a}}_1 = (0,0,0)^T$, $\hat{\mathbf{a}}_2 = (1,0,0)^T$, $\hat{\mathbf{a}}_3 = (0,1,0)^T$ and $\hat{\mathbf{a}}_4 = (0,0,1)^T$, we define the curl-conforming finite element $(\hat{K}, \mathcal{P}_{\hat{K}}, \mathcal{N}_{\hat{K}})$ by

- $\hat{K}$ is the reference element
- $\mathcal{P}_{\hat{K}} = \mathcal{R}_k$
- Three types of degrees of freedom associated with edges $\hat{e}$ of $\hat{K}$, faces $\hat{f}$ of $\hat{K}$ and $\hat{K}$ itself. The unit vector along edge $\hat{e}$ is denoted $\hat{\boldsymbol{\tau}}$.

$$M_{\hat{e}}(\hat{\mathbf{u}}) = \left\{ \int_{\hat{e}} \hat{\mathbf{u}} \cdot \hat{\boldsymbol{\tau}} \hat{q} d\hat{s}, \quad \forall \hat{q} \in P_{k-1}(\hat{e}) \right\}, \tag{33}$$

$$M_{\hat{f}}(\hat{\mathbf{u}}) = \left\{ \frac{1}{\text{area}(\hat{f})} \int_{\hat{f}} \hat{\mathbf{u}} \cdot \hat{\mathbf{q}} d\hat{A}, \quad \forall \hat{\mathbf{q}} \in [P_{k-2}(\hat{f})]^3 \text{ and } \hat{\mathbf{q}} \cdot \hat{\boldsymbol{\nu}} = 0 \right\}, \tag{34}$$

$$M_{\hat{K}}(\hat{\mathbf{u}}) = \left\{ \int_{\hat{K}} \hat{\mathbf{u}} \cdot \hat{\mathbf{q}} d\hat{V}, \quad \forall \hat{\mathbf{q}} \in [P_{k-3}(\hat{K})]^3 \right\}. \tag{35}$$

The total set of degrees of freedom is then $\mathcal{N}_{\hat{K}} = M_{\hat{e}}(\hat{\mathbf{u}}) \cup M_{\hat{f}}(\hat{\mathbf{u}}) \cup M_{\hat{K}}(\hat{\mathbf{u}})$.

By using the transformation (27) we can define the finite element $(K, \mathcal{P}_K, \mathcal{N}_K)$ for a general tetrahedron K as

- K is a tetrahedron
- $\mathcal{P}_K = \mathcal{R}_k$
- Three types of degrees of freedom associated with edges e of K, faces f of K and K itself. The unit vector along edge e is denoted $\boldsymbol{\tau}$.

$$M_e(\mathbf{u}) = \left\{ \int_e \mathbf{u} \cdot \boldsymbol{\tau} q ds, \quad \forall q \in P_{k-1}(e) \right\}, \tag{36}$$

$$M_f(\mathbf{u}) = \left\{ \frac{1}{\text{area}(f)} \int_f \mathbf{u} \cdot \mathbf{q} dA, \quad \forall \mathbf{q} = B_K \hat{\mathbf{q}}, \quad \hat{\mathbf{q}} \in [P_{k-2}(\hat{f})]^3 \text{ and } \hat{\mathbf{q}} \cdot \hat{\boldsymbol{\nu}} = 0 \right\}, \tag{37}$$

$$M_K(\mathbf{u}) = \left\{ \int_K \mathbf{u} \cdot \mathbf{q} dV, \quad \forall \mathbf{q} \text{ mapped by } q \circ F_K = \frac{1}{\det(B_K)} B_K \hat{\mathbf{q}}, \quad \hat{\mathbf{q}} \in [P_{k-3}(K)]^3 \right\}. \tag{38}$$

The total set of degrees of freedom for a general tetrahedron is then $\mathcal{N}_K = M_e(\mathbf{u}) \cup M_f(\mathbf{u}) \cup M_K(\mathbf{u})$.

Lemma 3. *If $\det(B_K) \geq 0$, $\boldsymbol{\tau} = \frac{B_K \hat{\boldsymbol{\tau}}}{|B_K \hat{\boldsymbol{\tau}}|}$ and the transformation (27) is used, the degrees of freedom (36)–(38) on K are identical to those on $\hat{K}$.*

3.3 Unisolvence

Theorem 1. *If $\mathbf{u} \in \mathcal{R}_k$ is such that all degrees of freedom (36)–(38)vanish, then $\mathbf{u} = 0$.*

To prove Theorem 1 we need two Lemmas.

Lemma 4. *If $\mathbf{u} \in \mathcal{R}_k$ is such that the degrees of freedom (37) vanish on f and the degrees of freedom (36) vanish on all edges of f, then $\mathbf{u} \times \boldsymbol{\nu} = 0$ on f.*

Lemma 5. *If $\mathbf{u} \in \mathcal{R}_k$ is such that $\nabla \times \mathbf{u} = 0$, then $\mathbf{u} = \nabla p$ for $p \in P_k$.*

Proof of Theorem 1. Since the degrees of freedom on the general tetrahedron and the reference tetrahedron are identical, we prove the theorem on the reference element. Because of Lemma 4 we know that $\hat{\mathbf{u}} \times \hat{\boldsymbol{\nu}} = 0$ on $\partial \hat{K}$. By integration by parts and using that the degrees of freedom (38) are zero, we have

$$\int_{\hat{K}} \hat{\nabla} \times \hat{\mathbf{u}} \cdot \hat{\mathbf{q}} d\hat{V} = \int_{\hat{K}} \hat{\mathbf{u}} \cdot \hat{\nabla} \times \hat{\mathbf{q}} d\hat{V} = 0, \quad \forall \hat{\mathbf{q}} \in [P_{k-2}]^3. \tag{39}$$

Using Stokes theorem on each face $\hat{f}$ of $\hat{K}$ and that the degrees of freedom (37) are zero, we have

$$\int_{\hat{f}} \hat{\nabla}_{\hat{f}} \times \hat{\mathbf{u}}_T \hat{q} d\hat{A} = \int_{\hat{f}} \hat{\mathbf{u}}_T \cdot \vec{\hat{\nabla}}_{\hat{f}} \times \hat{q} d\hat{A} = 0, \quad \forall \hat{q} \in P_{k-1}(\hat{f}), \tag{40}$$

where $\hat{\mathbf{u}}_T = (\hat{\boldsymbol{\nu}} \times \hat{\mathbf{u}}) \times \hat{\boldsymbol{\nu}}$ on $\partial \hat{K}$. The surface vector curl operator $\vec{\hat{\nabla}}_{\partial \hat{f}} \times$ is defined as $\vec{\hat{\nabla}}_{\hat{f}} \times q = -\hat{\boldsymbol{\nu}} \times \hat{\nabla}_{\hat{f}} q$. This means that $\hat{\nabla}_{\hat{f}} \times \hat{\mathbf{u}}_T = (\hat{\nabla} \times \hat{\mathbf{u}}) \cdot \hat{\boldsymbol{\nu}} = 0$ on $\hat{f}$ and hence on $\partial \hat{K}$. Because of this and the fact that $\hat{\nabla} \times \hat{\mathbf{u}} \in [P_{k-1}]^3$, we can write

$$\hat{\nabla} \times \hat{\mathbf{u}} = (\hat{x}_1 \Psi_1, \hat{x}_2, \Psi_2, \hat{x}_3 \Psi_3)^T, \tag{41}$$

where $\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \Psi_3)^T \in [P_{k-2}]^3$. Picking $\hat{\mathbf{q}} = \boldsymbol{\Psi}$ in (39) shows that $\hat{\nabla} \times \hat{\mathbf{u}} = 0$ in $\hat{K}$.

Using Lemma 5 this means that we can write $\hat{\mathbf{u}} = \hat{\nabla} \hat{p}$ for some $\hat{p} \in P_k$. The fact that $\hat{\mathbf{u}} \times \hat{\boldsymbol{\nu}} = 0$ on $\partial \hat{K}$ implies that we can take $\hat{p} = 0$ on $\partial \hat{K}$ and write $\hat{p}$ as

$$\hat{p} = \hat{x}_1 \hat{x}_2 \hat{x}_3 \hat{r}, \quad \text{for some } \hat{r} \in P_{k-3}. \tag{42}$$

The vanishing of the degrees of freedom (38) implies that $\hat{p} = 0$ and thus $\hat{\mathbf{u}} = 0$. ☐

3.4 The interpolant

We can define the global finite element space on a mesh τ_h as

$$V_h = \{\mathbf{u} \in H(\text{curl}; \Omega) : \mathbf{u}|_K \in \mathcal{R}_k \ \forall K \in \tau_h\}. \tag{43}$$

The local interpolant $\mathbf{r}_K \mathbf{u} \in \mathcal{R}_k$ for $K \in \tau_h$ is characterized by the vanishing of the degrees of freedom on $\mathbf{u} - \mathbf{r}_K \mathbf{u}$,

$$M_e(\mathbf{u} - \mathbf{r}_K \mathbf{u}) = M_f(\mathbf{u} - \mathbf{r}_K \mathbf{u}) = M_K(\mathbf{u} - \mathbf{r}_K \mathbf{u}) = 0. \tag{44}$$

The global interpolant is then defined by

$$\mathbf{r}_h \mathbf{u}|_K = \mathbf{r}_K \mathbf{u}, \quad \forall K \in \tau_h. \tag{45}$$

Lemma 6. *If $\nabla \times \mathbf{u} \in [L^p(K)]^3$, $\mathbf{u} \in [L^p(K)]^3$ and $\mathbf{u} \times \boldsymbol{\nu} \in [L^p(\partial K)]^3$ for $p > 2$, then $\mathbf{r}_h \mathbf{u}$ is well-defined and bounded.*

Theorem 2. *Let τ_h be a regular mesh on Ω. If $\mathbf{u} \in [H^s(\Omega)]^3$ and $\nabla \times \mathbf{u} \in [H^s(\Omega)]^3$ for $s \leq k$, then*

$$||\mathbf{u} - \mathbf{r}_h \mathbf{u}||_{[L^2(\Omega)]^3} + ||\nabla \times (\mathbf{u} - \mathbf{r}_h \mathbf{u})||_{[L^2(\Omega)]^3} \leq Ch^s (||\mathbf{u}||_{[H^s(\Omega)]^3} + ||\nabla \times \mathbf{u}||_{[H^s(\Omega)]^3}). \tag{46}$$

The error estimates for the interpolant holds for $(\mathbf{u} - \mathbf{r}_h\mathbf{u})$ and $(\nabla \times (\mathbf{u} - \mathbf{r}_h\mathbf{u})$ separately:

$$||\mathbf{u} - \mathbf{r}_h\mathbf{u}||_{[L^2(\Omega)]^3} \leq Ch^s(||\mathbf{u}||_{[H^s(\Omega)]^3} + ||\nabla \times \mathbf{u}||_{[H^s(\Omega)]^3}) \tag{47}$$

$$||\mathbf{u} - \mathbf{r}_h\mathbf{u}||_{[L^2(\Omega)]^3} \leq Ch^s||\mathbf{u}||_{[H^s(\Omega)]^3} \quad \text{(for } s > 1\text{)} \tag{48}$$

$$||\nabla \times (\mathbf{u} - \mathbf{r}_h\mathbf{u})||_{[L^2(\Omega)]^3} \leq Ch^s||\nabla \times \mathbf{u}||_{[H^s(\Omega)]^3}. \tag{49}$$

Theorem 3. *Let τ_h be a regular mesh on Ω. If $\mathbf{u} \in [H^{\frac{1}{2}+\delta}(K)]^3$, $0 < \delta \leq \frac{1}{2}$ and $\nabla \times \mathbf{u}|_K \in D_k = [P_{k-1}]^3 \oplus \tilde{P}_{k-1}\mathbf{x}$, then*

$$||\mathbf{u} - \mathbf{r}_h\mathbf{u}||_{[L^2(K)]^3} \leq C(h_K^{\frac{1}{2}+\delta}||\mathbf{u}||_{[H^{1/2+\delta}(K)]^3} + h_K||\nabla \times \mathbf{u}||_{[L^2(K)]^3}). \tag{50}$$

The error estimates show that the convergence is $||\mathbf{u} - \mathbf{r}_h\mathbf{u}||_{[L^2(\Omega)]^3} = \mathcal{O}(h^k)$, according to the Bramble-Hilbert lemma since $[P_{k-1}]^3 \subset \mathcal{R}_k$, but $[P_k]^3 \not\subset \mathcal{R}_k$. For triangular elements in two dimensions it has been shown that superconvergence is obtained, so that the order of convergence is similar to nodal based elements, which have $\mathcal{O}(h^{k+1})$ convergence for degree k.

To obtain higher order convergence, a second family of vector elements was introduced by Nedelec:

- K is a tetrahedron
- $\mathcal{P}_K = [P_k]^3$
- Three types of degrees of freedom associated with edges e of K, faces f of K and K itself. The unit vector along edge e is denoted $\boldsymbol{\tau}$.

$$M_e(\mathbf{u}) = \left\{ \int_e \mathbf{u} \cdot \boldsymbol{\tau} q ds, \quad \forall q \in P_{k-1}(e) \right\}, \tag{51}$$

$$M_f(\mathbf{u}) = \left\{ \frac{1}{\text{area}(f)} \int_f \mathbf{u}_T \cdot \mathbf{q} dA, \quad \forall \mathbf{q} \in D_{k-1}(f) \right\}, \tag{52}$$

$$M_K(\mathbf{u}) = \left\{ \int_K \mathbf{u} \cdot \mathbf{q} dV, \quad \forall \mathbf{q} \in D_{k-1}(K) \right\}. \tag{53}$$

The total set of degrees of freedom is then $\mathcal{N}_K = M_e(\mathbf{u}) \cup M_f(\mathbf{u}) \cup M_K(\mathbf{u})$.

The dimension of the second family of elements is $\frac{1}{2}(k+1)(k+2)(k+3)$.

References

[1] C. Amrouce, C. Bernardi, M. Dauge, and V. Girault. Vector potentials in three-dimensional non-smooth domains. *Mathematical Methods in the Applied Sciences*, 21:823–864, 1998.

[2] D. Boffi, P. Fernandes, L. Gastaldi, and I. Perugia. Computational models of electromagnetic resonators: Analysis of edge element approximation. *SIAM J. Numer. Anal.*, 36:1264–1290, 1999.

[3] Daniele Boffi, Marco Farina, and Lucia Gastaldi. On the approximation of Maxwell's eigenproblem in general 2D domains. *Computers and Structures*, 79:1089–1096, 2001.

[4] Daniele Boffi and Lucia Gastaldi. Interpolation estimates for edge finite elements and application to band gap computation. *Applied Numerical Mathematics*, 56:1283–1292, 2006.

[5] Martin Costabel and Monique Dauge. Maxwell and lame eigenvalues on polyhedra. *Mathematical Methods in the Applied Sciences*, 22:243–258, 1999.

[6] R. Hiptmair. Finite elements in computational electromagnetics. *Acta Numerica*, 11:237–339, 2002.

[7] Jianming Jin. *The Finite Element Method in Electromagnetics*. John Wiley & Sons, 2 edition, 2002.

[8] Daniel R. Lynch and Keith D. Paulsen. Origin of vector parasites in numerical Maxwell solutions. *IEEE Transactions on Microwave Theory and Techniques*, 39, 1991.

[9] Peter Monk. *Finite Element Methods for Maxwell's Equations*. Oxford Science Publications, 2003.

[10] J. C. Nedelec. Mixed finite elements in R^3. *Numerische Mathematik*, 35(3):315–341, September 1980.

[11] J.C. Nedelec. A new family of mixed finite elements in R^3. *Numerische Mathematik*, 50:57–81, 1986.