**Case A:** The parameter of interest is a proportion or probability \( p \in (0, 1) \).

Data is the frequency \( f \) of successes in \( n \) independent trials with success probability \( p \), i.e.,
\[
f|n, p \sim \text{Bin}(n, p)
\]
Or, the number \( x \) of independent trials until the first success, i.e.
\[
x|p \sim \text{Geo}(p)
\]

As prior, we shall take the beta(\( \alpha, \beta \))-density for suitable \( \alpha, \beta > 0 \),
\[
\pi(p) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha, \beta)}
\]
where
\[
B(\alpha, \beta) = \int_0^1 p^{\alpha-1}(1-p)^{\beta-1} \, dp
\]
is the Beta function.

Refer to Wikipedia for plots of various beta-densities. Note now that beta(1, 1) = U(0, 1).

The mean and variance of \( p \) are
\[
Ep = \mu_p = \frac{\alpha}{\alpha + \beta}
\]
\[
\text{var}(p) = \sigma_p^2 = \frac{\alpha \beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}
\]
The mode (or most likely value) is
\[
\text{md}(p) = \hat{p} = \frac{\alpha - 1}{\alpha + \beta - 2} \quad \text{if} \quad \alpha, \beta > 1
\]

**Rules for updating a probability**

If the prior for \( p \) is beta(\( \alpha, \beta \)), then the posterior is either
- beta(\( \alpha + f, \beta + n - f \)), or
- beta(\( \alpha + 1, \beta + x - 1 \)).

depending on whether data is a frequency \( f \) or the number of trials \( x \) needed to obtain one success.

1st rule for updating \( p \)

| If \( p \sim \text{beta}(\alpha, \beta) \) \( f|n, p \sim \text{Bin}(n, p) \) then \( p|n, f \sim \text{beta}(\alpha + f, \beta + n - f) \) |

2nd rule for updating \( p \)

| If \( p \sim \text{beta}(\alpha, \beta) \) \( x|p \sim \text{Geo}(p) \) then \( p|x \sim \text{beta}(\alpha + 1, \beta + x - 1) \) |
In this course, proofs are typically omitted.

When the prior and posterior are from the same distributional family (in this case the beta family), the prior is said to be conjugate to the model.

Thus, the beta prior is conjugate to the binomial and geometric models.

In this course we will only consider the case when data is binomial.

If \( p \sim \text{beta}(0.5, 0.5) \), then

\[
p|n, f \sim \text{beta}(0.5 + f, 0.5 + n - f)
\]

This is the result of a reference analysis and beta(0.5, 0.5) is the reference prior for \( p \).

If nothing is known about \( p \), my advice is to use the reference prior and perform a reference analysis.

However, some authors recommend the uniform density \( \text{beta}(1, 1) \).

The usual frequentist estimate of \( p \) is

\[
\hat{p} = \frac{f}{n}
\]

The Bayesian analyst may prefer the posterior mean

\[
E[p|n, f] = \mu_{p|n,f} = \frac{\alpha + f}{\alpha + \beta + n}
\]

or the posterior mode of \( p \).

We often write just \( \mu \) or \( \hat{p} \) for means and modes, and let it be clear from the context whether it is a prior or posterior mean.

Suppose a new trial, which is independent of the \( n \) earlier trials, will be done. The posterior predictive probability that the trial will be a success \( S \) is

\[
P(S|n, f) = \frac{\alpha + f}{\alpha + \beta + n}
\]

Thus,

\[
P(S|n, f) = E[p|n, f]
\]

Before considering the demonstrations, let us just note the general fact that if \( p_q(\alpha, \beta) \) equals the \( q \)-quantile w.r.t the beta(\( \alpha, \beta \)), then

\[
p_q(\alpha, \beta) + p_{1-q}(\beta, \alpha) = 1
\]

or

\[
p_{1-q}(\beta, \alpha) = 1 - p_q(\alpha, \beta)
\]

This is a consequence of the symmetry in the definition of the beta-densities.

**Demonstration 18** During the last 5 years, there has been 15 accidents on national road 40 between Gothenburg and Borås. The driver was under influence of alcohol (i.e. drunk) in 8 of the 15 accidents. Perform a reference analysis of this data. What is the posterior predictive probability that the driver in the next accident will be drunk?
**Solution:** We postpone the analysis of the accident intensity to part PRA2B. Here we analyze the proportion \( p \) of accidents in which the driver was drunk.

The reference posterior is

\[ p \sim \text{beta}(0.5 + f, 0.5 + n - f) = \text{beta}(8.5, 7.5) \]

In the beta-quantiles tables we see that \( p_{0.05}(8.5, 7.5) = 0.329 \) and \( p_{0.05}(7.5, 8.5) = 0.271 \). Hence \( p_{0.05}(8.5, 7.5) = 1 - 0.271 = 0.729 \). Thus the interval

\[ 0.33 \leq p \leq 0.73 \]

has 90\% credibility.

**Note** that it is important to provide an answer that is more informative than just saying the the posterior is beta with some parameters. You must also explain what this means in terms of appropriate posterior quantities like mean, mode, variance and credibility interval(s).

The posterior mean and mode are

\[
\mu = \frac{8.5}{16} = 0.531 \\
\hat{\beta} = \frac{7.5}{14} = 0.536 
\]

The posterior predictive probability that the driver in the next accident is drunk equals the posterior mean, and is

\[ P(S|n = 15, f = 8) = 0.531 \]

**Demonstration 19** It is believed that

\[ p \leq 0.25 \]

holds true with credibility 0.95. Suggest a suitable beta-density for \( p \).

**Solution:** Thus, we seek \( \alpha, \beta \) such that

\[ p_{0.05}(\alpha, \beta) = 0.25 \iff p_{0.05}(\beta, \alpha) = 0.75 \]

Clearly, there is an infinite range of possible choices.

In the beta-quantiles tables we see \( p_{0.05}(7.0, 0.5) = 0.753 \).

Hence \( p_{0.05}(0.5, 7.0) = 1 - 0.753 = 0.247 \approx 0.25 \).

Thus, \( \alpha = 0.5, \beta = 7.0 \) is a possibility.

Let’s look for other instances where \( p_{0.05}(\beta, \alpha) \approx 0.75 \)

One such instance is \( p_{0.05}(27.5, 4.5) = 0.748 \), implying that \( p_{0.05}(4.5, 27.5) = 1 - 0.748 = 0.252 \approx 0.25 \).

So, another possibility is \( \alpha = 4.5, \beta = 27.5 \).

Note that \( p_{0.05}(4.5, 27.5) = 0.055 \).

Note also that \( p_{0.05}(0.5, 7.0) = 0 \).

In cases like this, it is often wise to suggest the choice with smallest sum \( \alpha + \beta \), since it minimizes \( p_{0.05} \).

**Demonstration 20** It is believed that

\[ 0.05 \leq p \leq 0.25 \]

holds true with credibility 0.90. Suggest suitable values for the beta-parameters. Calculate the prior mode of \( p \).

In \( n = 8 \) independent trials, in which the probability of success \( S \) is \( p \), the observed frequency of \( S \) was \( f = 2 \). Calculate the posterior mode and a posterior 90\% interval for \( p \).
**Solution:** We have already seen that \( p_{0.05}(0.5, 7.0) = p_{0.05}(4.5, 27.5) = 0.25 \). Cf. the solution to Demonstration 19.

We furthermore see in the beta-quantiles tables that \( p_{0.05}(0.5, 7.0) \approx 0 < 0.05 \) and \( p_{0.05}(4.5, 27.5) = 0.055 > 0.05 \). We conclude that suitable values must lie between these “extremes.” Probably closer to the latter.

A search in the quantiles tables now yields the suggestion \( \alpha = 4.0, \beta = 25.0 \), since then \( p_{0.05} = 0.050 \), and \( p_{0.05} = 1 - 0.746 = 0.254 \). The prior mode of \( p \) then is

\[
\hat{p} = \frac{4 - 1}{4 + 25 - 2} = \frac{3}{27} = 0.111
\]

Note also that the prior mean is

\[
\mu = \frac{4}{29} = 0.138
\]

Recall that data is \( n = 8, f = 2 \).

The posterior beta-density for \( p \) has parameters \( \alpha = 4 + 2 = 6, \beta = 25 + 6 = 31 \). Thus the posterior mode is

\[
\hat{p} = \frac{5}{35} = 0.143
\]

Also note that

\[
p_{0.05}(6, 31) \approx 0.075
\]

and

\[
p_{0.05}(6, 31) = 1 - p_{0.05}(31, 6) \approx 1 - 0.73 = 0.27
\]

Unfortunately, we must extrapolate in the beta-quantiles tables to get the last number.

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**Demonstration 21** Let \( p \) be the probability of success \( S \) in a certain trial. Assume that \( p \) has a beta(\( \alpha, \beta \))-density with \( \mu = 0.6 \) and \( \sigma = 0.2 \). What is the predictive probability that the first trial is a success?

**Solution:** The predictive probability of the event that the first trial is a success, equals the mean of \( p \) and is therefore \( P(S) = 0.6 \).

Also, if we were asked to calculate \( \alpha, \beta \), in order to perform some further analysis, we would have concluded that \( \alpha = 3 \) and \( \beta = 2 \) from the two equations

\[
0.6 = \frac{\alpha}{\alpha + \beta} \quad \text{and} \quad 0.2^2 = \frac{\alpha \beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}
\]

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**Remark 3** The Bayesian way of writing that an unknown parameter \( p \) is the probability of an event \( S \) is

\[
P(S|p) = p
\]

If, further, \( p \sim \text{beta}(\alpha, \beta) \), then the predictive probability of \( S \) is

\[
P(S) = \int_0^1 P(S|p) \pi(p) \, dp
\]

\[
= \int_0^1 p \pi(p) \, dp = Ep = \frac{\alpha}{\alpha + \beta}
\]

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**Exercise 12** It is believed that \( p \geq 0.7 \) holds with 90% credibility. Suggest suitable parameters for a prior beta density.

**Answer:** \( \alpha = 4, \beta = 0.5 \) (since \( p_{0.1}(4, 0.5) = 0.698 \))
**Exercise 13** It is believed that $p \geq 0.7$ holds with 95% credibility. Suggest suitable parameters for a prior beta density.

Solution: A look in the beta tables yields $p_{0.95}(0.5, 1.5) = 1 - 0.229 = 0.771$ and $p_{0.95}(0.5, 2) = 1 - 0.342 = 0.658$, so if $\alpha = 0.5$, then $\beta$ is between 1.5 and 2.0. Since none of these beta values seem satisfactory, linear interpolation might be appropriate. We then get

$$\beta = 1.5 + \frac{2 - 1.5}{0.771 - 0.658} \cdot (0.771 - 0.7) = 1.8$$

Thus, a possible choice is $\alpha = 0.5$, $\beta = 1.8$. 

**Exercise 14** It is believed that $0.7 \leq p \leq 0.95$ holds with 90% credibility. Suggest suitable parameters for a prior beta density.

Answer: $\alpha = 18.5$, $\beta = 3.5$ ($p_{0.95}(18.5, 3.5) = 0.7$, and $p_{0.05}(3.5, 18.5) = 0.053 \Rightarrow p_{0.95}(18.5, 3.5) = 0.947$)

**Exercise 15** It is believed that $0.7 \leq p \leq 0.95$ holds with 80% credibility. Suggest suitable parameters for a prior beta density.

Solution: If $\alpha = 8.5$, $\beta = 1.5$, then $p_{0.1} = 0.699$ and $p_{0.9} = 1 - 0.033 = 0.967$. This interval is too large, hence the sum of $\alpha + \beta$ should be larger than 10.

If $\alpha = 12$, $\beta = 2.5$, then $p_{0.1} = 0.695$ and $p_{0.9} = 1 - 0.061 = 0.939$. This interval is somewhat too small, so probably $\beta = 2$.

If $\alpha = 10.5$, $\beta = 2$, then $p_{0.1} = 0.702$ and $p_{0.9} = 1 - 0.047 = 0.953$.

I therefore suggest $\alpha = 10.5$, $\beta = 2$. 

**Exercise 16** It is believed that $0.4 \leq p \leq 0.6$ holds with (a) 80% and (b) 90% credibility. Suggest suitable parameters for a prior beta density.

Solution: If $\alpha = 8.5$, $\beta = 1.5$, then $p_{0.1} = 0.699$ and $p_{0.9} = 1 - 0.033 = 0.967$. This interval is too large, hence the sum of $\alpha + \beta$ should be larger than 10.

If $\alpha = 12$, $\beta = 2.5$, then $p_{0.1} = 0.695$ and $p_{0.9} = 1 - 0.061 = 0.939$. This interval is somewhat too small, so probably $\beta = 2$.

If $\alpha = 10.5$, $\beta = 2$, then $p_{0.1} = 0.702$ and $p_{0.9} = 1 - 0.047 = 0.953$.

I therefore suggest $\alpha = 10.5$, $\beta = 2$. 

Solution: By symmetry, $\alpha = \beta$. Looking in the diagonal, shows that $\alpha = \beta = 19$ is a good choice in (a) since then $p_{0.1} = 0.397 \Rightarrow p_{0.9} = 1 - 0.397 = 0.603$. In (b) we see that $p_{0.05} = 0.395$ if $\alpha = \beta = 30$, so it seems that we have to extrapolate along the diagonal. A guess then would be that $\alpha = \beta = 32.6$ is a good choice. (According to Matlab, $p_{0.05} = 0.399$ and $p_{0.95} = 0.601$ for this choice, so perhaps $\alpha = \beta = 32$ is a better choice.)

Exercise 17 Assume that you are working with remediation of contaminated land. At a certain site the object of interest is the proportion $p$ of the total area $A$ that is contaminated. A total of $5$ soil samples are taken in independent and randomly chosen locations, the outcome of which is that contamination is found in four of the five samples.

Solution: (a) The reference posterior is beta(1.4.45). A posterior 90% interval is $0.036 \leq p \leq 0.563$ and the posterior mean is $\mu = 1.5/6 = 0.25$.

(b) The prior parameters are $\alpha = 0.6$, $\beta = 2.4$. Hence the posterior parameters are $\alpha = 1.6$, $\beta = 2.4$. A posterior 90% interval is $(0.03, 0.44)$ (calculated with $\alpha = 1.5$, $\beta = 2.5$). The posterior mean is $\mu = 1.6/8 = 0.2$.

(c) The posterior parameters are $\alpha = 3 + 1 = 4$, $\beta = 12 + 4 = 16$. A 90% interval is $0.075 \leq p \leq 0.359$. The posterior mean is $\mu = 4/16 = 0.25$.

Finally, the predictive probability that the next sample is contaminated equals $0.25$, $0.2$ and $0.25$, resp, since this probability equals the posterior mean.

Exercise 18 During the last 5 years a Nuclear Power Plant (NPP) has released detectable amounts of radioactivity 15 times. Typically the amounts are small and of little harm to the environment. But one of the 15 releases was severe, since the amount of discharged radioactivity was large and posed a threat to the local environment. Let $S$ denote the event that a release is severe and let $p$ be its probability. Then the frequency of $S$ in $n = 15$ independent trials is $f = 1$.

Assume that it was believed beforehand, i.e., more than 5 years ago, that $p \leq 0.20$ with 95% credibility. Calculate a posterior 0.95 upper credibility bound for $p$. Also, what are the prior and posterior means of $p$?

Compare with the results from a reference analysis.
Solution: We see in the appropriate quantile table that $p_{0.05}(9,0.5) = 0.803$, so $p_{0.05}(0.5,9) = 1 - 0.803 = 0.197 \approx 0.20$. We therefore let our prior parameters be $\alpha = 0.5$, $\beta = 9$.

Data is $f = 1$ in $n = 15$ independent trials, so the posterior density is $\text{beta}(0.5 + f, 9 + n - f) = \text{beta}(1.5, 23)$. The posterior UCL$_{95} = 0.155$ (since $p_{0.05}(23,1.5) = 0.845$).

The prior mean of $p$ is $\mu_p = 0.5/(9 + 0.5) = 0.053$, and the posterior mean is $\mu_p = 1.5/(23 + 1.5) = 0.061$. Note also that the frequentist estimate of $p$ equals $\hat{p} = 0.067$.

The reference posterior is $\text{beta}(1.5,14.5)$. The posterior mean thus is $\mu_p = 1.5/(1.5 + 14.5) = 0.094$, and the UCL$_{95} = 0.233$ (since $p_{0.05}(14.5,1.5) = 0.767 \Rightarrow p_{0.95}(1.5,14.5) = 1 - 0.767 = 0.233$).