KLEINIAN GROUPS AND THIN SETS AT THE BOUNDARY

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ABSTRACT

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Let Γ be a discrete group of Möbius transformations acting on and preserving the unit ball in $\mathbb{R}^d$ (i.e. Fuchsian groups in the planar case). We will put a hyperbolic ball around each orbit point of the origin and refer to their union as the *Archipelago of* Γ.

The main topic of this thesis is the question: "How big is the Archipelago of Γ?" We will study different ways to answer various meanings of that question. One of the answers that will be given says that the Archipelago of Γ is *minimally thin* almost everywhere on the unit sphere if and only if Γ is of convergence type.

We will also study and use other concepts from potential theory such as *rarefiedness* and *boundary layers* in order to answer the question above and to give connections between the theory of discrete groups and small sets in potential theory. We will also consider the discrete orbit as it is. In order to do that, we define and study the discrete form of some potential theoretic concepts such as minimal thinness and boundary layer.

Keywords: Discrete group, Fuchsian group, Kleinian group, Poincaré series, horocycle, minimal thinness, reduced function, harmonic measure, boundary layer, NTA domain, discrete potential, Martin kernel, conditioned random walk, asymptotic capacity.

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To Kajsa

[1]Art Director

Jag kommer för sällan fram till vattnet. Men nu är jag här,
bland stora stenar med fridfulla ryggar.
Stenar som långsamt vandrat baklänges upp ur vågorna.

Ur Långsam musik

Jag står på berget och ser över fjärden,
Båtarna vilar på sommarens yta.
"Vi är sömngångare. Månar på drift."
Så säger de vita seglen.

Ur Från berget

Den som färdas hela dagen i öppen båt
över de glittrande fjärdarna
ska somna till sist inne i en blå lampa
medan öarna kryper som stora nattfjärilar över glaset.

Ur Andrum i juli

Tomas Tranströmer

Contents

CHAPTER I

Introduction

A Möbius mapping of the unit disk onto itself is a conformal mapping of the form

$$\gamma(z) = \frac{az + \bar{c}}{cz + \bar{a}}, \quad \text{where } a, c \in \mathbb{C} \text{ and } |a|^2 - |c|^2 = 1.$$

Let Γ be a Fuchsian group. That is, a group of Möbius mappings in the unit disk sparse enough such that if we consider the orbit of the origin, $\{\gamma(0) : \gamma \in \Gamma\}$, the points in the set do not cluster inside the disk. We can even put non-intersecting small disks on the orbit points. The union of these disks will look like an Archipelago in the "unit disk lake".

Suppose that we have a Brownian particle in the lake. What is the probability that the particle will avoid all the islands before it reaches the "unit circle shore"? What will the probability be if the particle is conditioned to exit at a certain point?

These probabilities are closely related to the size or the density of the Archipelago seen with potential theoretic eyes from the shore.

By studying these and related questions we will try to capture *one* connection between the theory of Kleinian groups[1] and potential theory. To get a such a connection is the main goal of this work.

1. Notation

The following list contains a sample of some notations that we will use.

- $\mathbb{R}^\mathbf{d}$: The Euclidean d-dimensional space.
- U: The unit disk.
- $\mathbb{T}$: The unit circle.
- B: The unit ball, for d ≥ 2.
- ∂B: The unit sphere.
- $B(x, r)$: The ball centered at x and with radius r.
- E: The Kleinian (or Fuchsian) Archipelago.
- r_Γ: The hyperbolic radius of the islands in E.
- Γ: A Kleinian (or Fuchsian) group, see Remark A.4 on page 89.
- γ: An element in Γ, see Appendix A.
- $\gamma(x)$: The image point of x under the mapping γ.
- γx: A slimmer notation of the above.
- $\sigma(\cdot)$: The normalized Lebesgue measure on the unit sphere, i.e. $\sigma(\partial B) = 1$.

[1]Kleinian groups are the generalization of Fuchsian groups to higher dimensions, see Remark A.4 on page 89.

Λ: The limit set.
Λ_c: The non-tangential limit set. Also known as the conical limit set.
Λ_h: The non-osculating limit set.
H: The horospherical limit set.
h_s: The Poincaré series, see Appendix A.
$\{Q_k\}$: A Whitney decomposition.
q_k: The distance to the boundary from the Whitney cube Q_k.
$\rho_k(\tau)$: The distance to the boundary point τ from the Whitney cube Q_k.
$H(\cdot)$: The Hausdorff–measure.
$\omega(\cdot,\cdot,\cdot)$: The harmonic measure.
K_ξ: The Martin kernel at ξ.
W: A Wiener type series for minimal thinness, see Appendix B.
W_0: A variant of the above, see Definition III.3 on page 23.
W^r: A Wiener type series for rarefiedness, see page 25.
$\mathcal{L}(\alpha)$: The α-limit set, see Definition IV.4.
$\mathcal{L}_s(\alpha)$: The strong α-limit set, see Definition IV.11.
$\mathfrak{N}$: The set on the boundary where the Kleinian Archipelago is *not* minimally thin.
X_n: A Markov process.
Z_n: A conditioned Markov process.

2. Plan of the thesis

In order to make this thesis more self contained, we have two appendices, where we give some background and basic facts of the theory of Kleinian groups and potential theory that we use in other parts of this thesis.

Chapter II, which is a revised version of the first part of [31], deals with limit sets of Kleinian groups. We are especially interested in two subsets of the limit set, the classical non-tangential limit set and the "non-osculating limit set" which we define in an analogue way to the non-tangential.

In Chapter III (a revised version of the second part of [31]), we study the connection between the theory of Kleinian groups and potential theory in the sense of the question: "Is the Kleinian Archipelago E thin at the boundary?" and we give both local and global answers to that question.

In Chapter IV, we generalize some results in Chapter II and also indicate an alternative way to prove Proposition II.14 by techniques and results in [33]. Furthermore, some alternative geometric descriptions are given in Section 2. We will also show that the set where the Kleinian Archipelago is not minimally thin is close to the non-tangential limit set. That is, the sets have the same Hausdorff dimension, and they coincide when Γ is of geometrical finite type, see Corollaries IV.13 and IV.14.

Another way to measure the size of the Archipelago of Γ is to consider the concept *boundary layer* which was introduced by Alexander Volberg in [42]. We can give an intuitive picture of that procedure in the following way. Let us consider the planar case, i.e. the Fuchsian case, and let us remove the center island of E. We say that the complement, $U \setminus E$, is a boundary layer if the probability for a Brownian particle started at the origin to hit the arc I at the boundary ∂U (i.e. the unit circle), without hitting the set E, is comparable to the length of I.

In Chapter V, which is a joint work with Hiroaki Aikawa in [6], we study the concept of boundary layers. The goal is twofold. First, we generalize the definition of boundary layers to NTA-domains and to higher dimensions. Secondly, we compare the concept with minimal thinness[2]. The comparison is done in two ways. One is to give a necessary condition and a sufficient condition for for a boundary layer with the help of the Wiener type series used in [20] (see Appendix B) to decide minimal thinness. The other way is to give a weakened definition of boundary layers that exactly correspond to minimal thinness, or more explicit, $U \setminus E$ is a *weak boundary layer* if and only if E is minimally thin everywhere on the boundary ∂U.

In Chapter VI, we use results from the previous chapter to discuss the following question: when is the complement of the Archipelago of Γ a boundary layer? Theorem VI.7 on page 66 will give a complete answer to that question.

Chapter VII is devoted to the discrete world. In that chapter we do not consider the Archipelago but rather the orbit itself. Since we are now dealing with a discrete set of points we are forced into the discrete potential theory. That is, instead of putting "flesh" on the discrete orbit set to go from the discrete world of Kleinian groups to the continuous world of classical potential theory, we keep on to the discreteness. We give a definition for discrete minimal thinness, see Definition VII.6, and study the concept in different ways. We also define discrete boundary layers. As a feed-back to the continuous classical situation, we obtain in Corollary VII.12, a Brownian motion description of ordinary minimal thinness.

In order to study the orbit of a Kleinian group, we view the orbit as a subset of vertex points in an underlying infinite net, where the underlying net is generated by a super-group of the group we study. This is done in Section 5 in Chapter VII. Finally, we study the Schottky group case.

[2]The fact that those two concepts are related are pointed out in [42], [20] and [21].

CHAPTER II

The non-osculating limit set of a Kleinian group

We transform the study of *non-tangential limit sets* to *"non-osculating limit sets"* of discontinuously acting subgroups of the Möbius mappings that preserve the unit ball in $\mathbb{R}^d$. We state a sufficient condition for the non-osculating limit set to have Lebesgue measure zero, using the *Poincaré series* of the subgroup.

1. Introduction

In Appendix A on page 89 we introduce the limit set Λ and its most famous subset, the conical limit set Λ_c. Another term for conical limit points is *non-tangential limit points* which is what Garnett uses in his study [25] of the Lebesgue measure of these limit sets (see also [1]). He constructs the non-tangential limit set using the *non-tangential cone*, i.e. the set $\{x \in B; |x| > 1/2; |x-z| < M(1-|x|)\}$, for the tip of the cone at $z \in \partial B$.

While studying *minimal thinness* the author needed a horocyclic condition and got curious about the possibility to use horocycles instead of cones to describe another, but similar limit set in the same fashion as Garnett does. Another motivation for doing this is the fact that the Poisson kernel in the upper half-plane with a pole on the boundary has horocycles as level-contours.

The aim of this first part is to state something analogous to the following fundamental result. The definitions are given in Appendix A and in Remark II.5 on page 14.

Theorem A. *If Γ is of convergence type then $\sigma(\Lambda_c) = 0$ (where Λ_c is the non-tangential limit set on the sphere and σ the normalized Lebesgue measure on the surface*[1]*.)*

The proof can be found in [25, Theorem 4, p. 29], or in [1, Lemma 3, p. 93].

We will now follow Garnett's presentation in [25] but we will study *horocycles* instead of cones and give an analogue to Theorem A in Proposition II.14 below.

2. Non-Osculating Limit Sets

Let B be the d–dimensional unit sphere, the *horocycles* are defined as:

Definition II.1. *A* horocycle *is the truncated sphere in B which is tangent to ∂B at $z \in \partial B$ with radius $= \frac{M}{M+1}$, or in other words:*

$$\widetilde{H}(z, M) = \{x \in B; |x| > 1/2; |x-z|^2 < M(1-|x|^2)\}.$$

Definition II.2. *A* horocap *is the part of ∂B reached by paths totally inside horocycles from a point x in B, where every path lies in a horocycle containing x. If $M > 0$, the horocap $C_h(x, M)$ is given by*

$$C_h(x, M) = \{z \in \partial B; |x-z|^2 < M(1-|x|^2)\}.$$

Remark II.3. The horocycle and the horocap are related by:

$$x \in \widetilde{H}(z, M) \Leftrightarrow z \in C_h(x, M).$$

[1] i.e. $\sigma(\partial B) = 1$

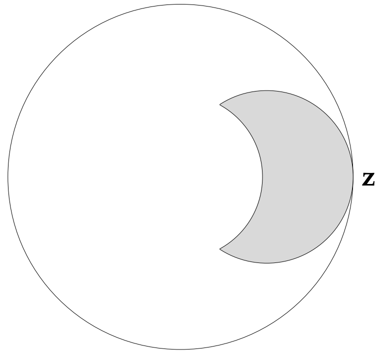

FIGURE II.1. Horocycle, $\widetilde{H}(z, M)$, with $M = 1$ and $z = 1$.

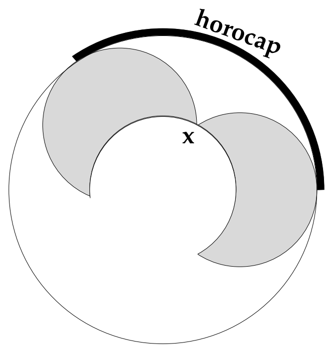

FIGURE II.2. Horocap, $C_h(x, M)$, with $M = 1$.

By taking the union of the horocaps as $|\gamma_n x_0|$ tends to 1 (which is equivalent to $n \to \infty$) we get a pre–version of the desired general limit set in the following definition.

DEFINITION II.4. *Let us by* $\Lambda_h(x_0\,, M)$ *denote the* non-osculating limit set of x_0 and M *defined as*

$$\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} C_h(\gamma_n x_0, M), \quad \gamma_n \in \Gamma.$$

We will later on see that the special choices of x_0 *and* M *are of little importance.*

REMARK II.5. The non-tangential limit set $\Lambda_c(x_0, M)$ used[2] by Garnett is defined in the following way. Let

$$C_c(x, M) := \{z \in \partial B : |x - z| < M(1 - |x|)\}.$$

Then

$$\Lambda_c(x_0, M) := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} C_c(\gamma_n x_0, M).$$

For details, we refer to [25, p. 26].

We will now study the analogue to Theorem A by using Garnett's technique on our horocaps. The analogue is stated below in Proposition II.14.

[2]In [25] the notation Λ is used for the non-tangentially limit set, which we have reserved to denote the whole limit set.

2.1. The size of the horocap. Before we give an estimate of the size of the horocaps, we state the following two lemmas (c.f. [25, Lemma 5.1]).

LEMMA II.6. *There exists a constant $C_1(R)$, only depending on R, such that if $d(x,y) < R$ then $|x-y| < C_1(R)(1-|x|)$.*

PROOF. Without loss of generality, it is enough to study the case where $y = 0$. We then have

$$R > d(x,0) = \log \frac{1+|x|}{1-|x|},$$

see for example [9, p. 38]. Hence,

$$e^R(1-|x|) > 1+|x| > |x|.$$

We can thus choose $C_1(R)$ to be e^R. □

LEMMA II.7. *There exists a constant $C_2(R)$ such that if $d(x,y) < R$ then $|x-y|^2 < C_2(R)(1-|x|^2)$.*

PROOF. The proof follows immediately from Lemma II.6 and the fact that $|x-y| < 2$.

$$|x-y|^2 < 2|x-y| < 2C_1(R)(1-|x|) \leq 2C_1(R)(1-|x|)(1+|x|) = C_2(R)(1-|x|^2).$$

That is, we can choose $C_2(R)$ to be $2e^R$. □

The following lemma compares the Lebesgue measure of the horocap with the distance from the boundary.

LEMMA II.8. *There exist constants $C_3(M)$ and $C_4(M)$ such that*

$$C_3(M)(1-|x|^2)^{\frac{d-1}{2}} \leq \sigma(C_h(x,M)) \leq C_4(M)(1-|x|^2)^{\frac{d-1}{2}}.$$

PROOF. The area of the horocap at x is proportional to the distance of x to the endpoint of the horocap z raised to the dimension of the surface of ∂B, i.e.

$$\sigma(C_h(x,M)) \approx |z-x|^{\mathrm{d}-1},$$

where z is such that:

$$|z-x|^2 = M(1-|x|^2),$$

$$\Downarrow$$

$$\sigma(C_h(x,M)) \approx (1-|x|^2)^{\frac{\mathrm{d}-1}{2}}$$

□

The radius of the horocycles is of no importance if we consider the Lebesgue measure of the resulting limit set, as we will see in the next lemma. This is of great importance when we later will define the general non-osculating limit set.

LEMMA II.9. *If $M' > M$ then*

$$\sigma(\Lambda_h(x_0,M') \setminus \Lambda_h(x_0,M)) = 0.$$

PROOF. Let $x_n = \gamma_n x_0$ and

$$E_k = \bigcap_n \{\partial B \setminus C_h(x_n, M); 1 - |x_n|^2 < \frac{1}{k}\}.$$

Thus,

$$\partial B \setminus \Lambda_h(x_0, M) = \bigcup_{k=1}^{\infty} E_k.$$

Fix k and suppose $z_0 \in \partial B$ is a point of density in E_k, i.e.,

$$\frac{\sigma(E_k \cap \{|z - z_0| < \delta\})}{\sigma(\{|z - z_0| < \delta\})} \to 1 \text{ when } \delta \to 0.$$

Our aim is to show that $z_0 \notin \Lambda_h(x_0, M')$. Assume the contrary to hold, i.e. ,

$$\text{if } z_0 \in \Lambda_h(x_0, M') \Rightarrow \forall \eta > 0, \text{ then } \exists x_n \text{ such that:}$$

$$\begin{cases} (i) & 1 - |x_n|^2 < \frac{1}{k}. \\ (ii) & \delta^2 = (M' + M + 2\sqrt{MM'})(1 - |x_n|^2) < \eta^2 \quad (\delta > 0) \ . \\ (iii) & |x_n - z_0|^2 < M'(1 - |x_n|^2). \end{cases}$$

From (i) we have,

$$E_k \cap C_h(x_n, M) = \emptyset,$$

which leads to

$$C_h(x_n, M) = \{z \in \partial B; |x_n - z|^2 < M(1 - |x_n|^2)\} \setminus E_k.$$

Let us pick a z such that $|x_n - z|^2 < M(1 - |x_n|^2)$, then by (iii)

$$|z - z_0|^2 \leq |z - x_n + x_n - z|^2 \leq |x_n - z|^2 + 2|x_n - z||x_n - z_0| + |x_n - z_0|^2 <$$
$$< M(1 - |x_n|^2) + 2\sqrt{M(1 - |x_n|^2)}\sqrt{M'(1 - |x_n|^2)} +$$
$$+ M'(1 - |x_n|^2) = (M + M' + 2\sqrt{MM'})(1 - |x_n|^2),$$

which is, by (ii), equal to δ^2.

Now, since $\delta > 0$,

$$|x_n - z|^2 < M(1 - |x_n|^2) \Rightarrow |z - z_0| < \delta,$$

hence,

$$C_h(x_n, M) = \{z \in \partial B; |x_n - z|^2 < M(1 - |x_n|^2)\} \setminus E_k \subsetneq$$

$$\subsetneq \{z \in \partial B; |z - z_0| < \delta)\} \setminus E_k. \tag{1}$$

From Lemma II.8, we obtain the following estimate,

$$\sigma(C_h(x_n, M)) \geq C_3(M)(1 - |x_n|^2)^{\frac{d-1}{2}} = (ii) =$$
$$= C_3(M)\left(\frac{\delta^2}{M' + M + 2\sqrt{MM'}}\right)^{\frac{d-1}{2}}$$
$$= \frac{C_3(M)}{(M' + M + 2\sqrt{MM'})^{\frac{d-1}{2}}}(\delta^2)^{\frac{d-1}{2}} = K'(M, M')\delta^{(d-1)} \geq$$

$$\geq K(M, M')\sigma(\{|z - z_0| < \delta\}). \tag{2}$$

Equation (1) gives us ,

$$\sigma(C_h(x_n, M)) < \sigma(\{z \in \partial B; |z - z_0| < \delta\} \setminus E_k) =$$

$$= \sigma(\{z \in \partial B; |z - z_0| < \delta\}) - \sigma(E_k \cap \{z \in \partial B; |z - z_0| < \delta\}).$$

This, together with equation (2), implies:

$$K(M, M')\sigma(\{z \in \partial B; |z - z_0| < \delta\}) <$$
$$< \sigma(\{z \in \partial B; |z - z_0| < \delta\}) - \sigma(E_k \cap \{z \in \partial B; |z - z_0| < \delta\}),$$

i.e.,

$$\frac{\sigma(E_k \cap \{z \in \partial B; |z - z_0| < \delta\})}{\sigma(\{z \in \partial B; |z - z_0| < \delta\})} < 1 - K(M, M')$$

Letting η tend to 0 will force δ to approach 0, in other words,

$$\limsup_{\delta \to 0} \frac{\sigma(E_k \cap \{z \in \partial B; |z - z_0| < \delta\})}{\sigma(\{z \in \partial B; |z - z_0| < \delta\})} < 1.$$

But this means that z_0 is not a point of density in E_k, which is a contradiction. We finally conclude,

$$z_0 \notin \Lambda_h(x_0, M'),$$

which ends the proof of Lemma II.9. □

2.2. Covering horocaps by horocaps. We are aiming at a definition of the non-osculating limit set that is independent of the choice of "starting point" x_0. Lemma II.11 below will provide us with that possibility; but first we need an elementary observation.

LEMMA II.10. *Let x, y and ξ be points in $\mathbb{R}^d$. We can then adjust the triangle inequality.*

$$|x - y|^2 \leq 2(|x - \xi|^2 + |\xi - y|^2).$$

PROOF.

$$|x - y|^2 = |x - \xi + \xi - y|^2 \leq (|x - \xi| + |\xi - y|)^2 =$$
$$= |x - \xi|^2 + |\xi - y|^2 + 2|x - \xi||\xi - y| \leq$$
$$\leq |x - \xi|^2 + |\xi - y|^2 + |x - \xi|^2 + |\xi - y|^2 = 2|x - \xi|^2 + 2|\xi - y|^2.$$

□

LEMMA II.11. *If $d(x_0, x_0') < R$ and $z \in C_h(\gamma_n x_0, M)$,*

then there exists an $M' = M'(R, M)$ such that $z \in C_h(\gamma_n x_0', M')$.

PROOF. If $d(x_0, x_0') < R$, then $d(\gamma_n x_0, \gamma_n x_0') < R$, so we can make use of Lemma II.6:

$$||\gamma_n x_0'| - |\gamma_n x_0|| \leq |\gamma_n x_0' - \gamma_n x_0| \leq C_1(R)(1 - |\gamma_n x_0'|),$$

which leads to,

$$1 - |\gamma_n x_0| \leq (1 - |\gamma_n x_0'|) + C_1(1 - |\gamma_n x_0'|) = (1 + C_1)(1 - |\gamma_n x_0'|),$$

and to,

$$(1 + |\gamma_n x_0|)(1 - |\gamma_n x_0|) < (1 + 1)(1 + C_1)(1 - |\gamma_n x_0'|) \leq$$
$$\leq 2(1 + C_1)(1 - |\gamma_n x_0'|)(1 + |\gamma_n x_0'|).$$

Thus, we obtain,

$$1 - |\gamma_n x_0|^2 < 2(1 + C_1)(1 - |\gamma_n x_0'|^2). \tag{3}$$

Let $z \in C_h(\gamma_n x_0, M)$, we then first obtain the following estimate using Lemma II.10:

$$|z - \gamma_n x_0'|^2 \leq 2(|z - \gamma_n x_0|^2 + |\gamma_n x_0 - \gamma_n x_0'|^2).$$

Lemma II.7 and the fact that z is in $C_h(\gamma_n x_0, M)$ give us now

$$|z - \gamma_n x_0'|^2 < 2(M(1 - |\gamma_n x_0|^2) + C_2(1 - |\gamma_n x_0'|^2)).$$

By using equation (3) we finally end up with

$$|z - \gamma_n x_0'|^2 < 2(M(2(1 + C_1(R))(1 - |\gamma_n x_0'|^2)) + C_2(R)(1 - |\gamma_n x_0'|^2)) =$$

$$= \underbrace{2(2M(1 + C_1(R)) + C_2(R))}_{M'}(1 - |\gamma_n x_0'|^2),$$

i.e.

$$|z - \gamma_n x_0'|^2 < M'(1 - |\gamma_n x_0'|^2),$$

which means that z is in $C_h(\gamma_n x_0', M')$ concluding the proof of Lemma II.11. □

2.3. Definition of the non-osculating limit set. Lemma II.11 implies that $C_h(\gamma_n x_0, M) \subset C_h(\gamma_n x_0', M')$ and hence $\Lambda_h(x_0, M) \subset \Lambda_h(x_0', M')$. Therefore we can now define the following.

Definition II.12. *The* ***non-osculating limit set*** Λ_h *is defined as*

$$\bigcup_{M>0} \Lambda_h(x_0, M)$$

Note that Λ_h is independent of x_0.

Remark II.13. Lemma II.9 gives us that

$$\sigma(\Lambda_h) = \sigma(\Lambda_h(x_0, M)), \text{ for any } x_0 \in B \text{ and any } M > 0. \tag{4}$$

We are now finally in a position to state what we are aiming at, an analogue of Theorem A.

2.4. The case of zero measure.

Proposition II.14. *Let δ be the critical exponent for the discontinuously acting subgroup Γ of $\mathcal{M}$. Then we have the following.*

$$\delta < \tfrac{d-1}{2} \Rightarrow \sigma(\Lambda_h) = 0,$$

or even something stronger,

$$h_{\frac{d-1}{2}} < \infty \Rightarrow \sigma(\Lambda_h) = 0,$$

(see page 90 for the definition of $h_{\frac{d-1}{2}}$).

Remark II.15. On page 36 below, we give alternative ways to obtain the above proposition.

PROOF. Let δ be the critical exponent (see definition A.7 on page 90) and $p = \frac{d-1}{2}$,

$\delta < p \Rightarrow h_p(x, y)$ convergent $\Leftrightarrow h_p(0, x_0)$ convergent, see for example [33, p. 20] .

This will lead us to

$$h_p(0, x_0) = \sum_{\gamma_n \in \Gamma} e^{-pd(0,\gamma_n x_0)} = \sum_{\gamma_n \in \Gamma} e^{-p \log\left(\frac{1+|\gamma_n x_0|}{1-|\gamma_n x_0|}\right)} =$$

$$= \sum_{\gamma_n \in \Gamma} \left(\frac{1+|\gamma_n x_0|}{1-|\gamma_n x_0|}\right)^{-p} \geq \frac{1}{2^p} \sum_{\gamma_n \in \Gamma} (1 - |\gamma_n x_0|)^p \geq \frac{1}{4^p} \sum_{\gamma_n \in \Gamma} (1 - |\gamma_n x_0|^2)^p.$$

Hence,

$$\delta < \tfrac{d-1}{2} \Rightarrow h_{\frac{d-1}{2}} < \infty \Rightarrow \sum_{\gamma_n \in \Gamma} (1 - |\gamma_n x_0|^2)^{\frac{d-1}{2}} \text{ converges.} \tag{5}$$

We can now, due to Remark II.13 , estimate the Lebesgue measure of our non-osculating set.

$$\sigma(\Lambda_h) = \sigma(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} C_h(\gamma_n\ x_0, M)) \leq \lim_{k\to\infty} \sum_{n=k}^{\infty} \sigma(C_h(\gamma_n\ x_0, M)),$$

By Lemma II.8 this can be estimated from above by

$$\lim_{k\to\infty} \sum_{n=k}^{\infty} C_4(M)(1 - |\gamma_n x_0|^2)^{\frac{d-1}{2}} =$$

$$= C_4(M) \quad \underbrace{\lim_{k\to\infty} \sum_{n=k}^{\infty} (1 - |\gamma_n x_0|^2)^{\frac{d-1}{2}}}_{\text{the tail of the series in eq. (5)}} \quad = 0.$$

That is,

$$\delta < \tfrac{d-1}{2} \Rightarrow h_{\frac{d-1}{2}} < \infty \Rightarrow \sigma(\Lambda_h) = 0,$$

which is the end of the proof of Proposition II.14. $\square$

2.5. The geometrically finite case. In this section we will give a necessary and sufficient condition for the non-osculating limit set to be of full (or empty) measure by restricting ourselves to a special, but natural, class of subgroups of $\mathcal{M}$.

DEFINITION II.16. *A group Γ is* ***geometrically finite*** *if some convex fundamental polyhedron has finitely many faces. (c.f.* [8, p. 6]*.)*

In the planar case, Γ is geometrically finite if and only if Γ is finitely generated. We will, in this finitely generated situation, give a precise answer to the questions above and obtain a better result than Proposition II.14.

PROPOSITION II.17. *If Γ is a geometrically finite discontinuously acting group of convergence type then $\sigma(\Lambda_h) = 0$.*

PROOF. The following two facts hold.

(a): The non-osculating limit set Λ_h is the set of non-tangential limit points and parabolic fixed points.

(b): The parabolic fixed points in a discontinuous group are countable.

(a) follows from the construction of the horocaps and from the fact that the whole limit set Λ contains only non-tangential limit points and parabolic fixed points when Γ is geometrically finite, see for example Theorem 9.29 in [34, p. 281].

(b) follows simply from the fact that the elements in Γ are denumerable (see for example [25, p. 27]), and the fact that a parabolic element has only one fixed point.

(a) and **(b)** give us now that the set difference $\Lambda_h \setminus \Lambda_c$ is only a set of countably many points. This tells us immediately that $\sigma(\Lambda_h \setminus \Lambda_c) = 0$ and by the use of Theorem A noting that Γ is of convergence type we end up with $0 = \sigma(\Lambda_c) = \sigma(\Lambda_h)$ ending the proof. $\square$

Remark II.18. We can actually say more than this. Since the set difference only consists of countably many points, we have

$$d(\Lambda_h) = d(\Lambda_c),$$

where the set function $d(\cdot)$ is the usual Hausdorff dimension.

Furthermore, since we study the case where Γ is a geometrically finite discontinuous group we will have that the total limit set $\Lambda \equiv \Lambda_h$ and the critical exponent will be equal to the Hausdorff dimension of all three limit sets,

$$\delta = d(\Lambda) = d(\Lambda_h) = d(\Lambda_c).$$

(c.f. [34, p. 285].)

CHAPTER III

Thinness of the Archipelago

1. The Archipelago of Γ

In order to investigate connections between Kleinian groups and thin sets we have to "put on some flesh" on the set of orbit points to make the point set visible for our potential theoretic eyes. We do that by the following construction.

Let Γ be a Kleinian group, see Remark A.4 on page 89. By the fact that Γ is discontinuous it is possible to find an $r_\Gamma > 0$ such that the balls B_j do not intersect each other, where $B_j := \{z \in B; d(z, \gamma_i(0)) < r_\Gamma, \gamma_i \in \Gamma \setminus \{I\}\}$. Let $E := \bigcup_j B_j$. That is, E is the "fattened" orbit of Γ and we call it **the Archipelago of** Γ.

Remark III.1. In the planar case, i.e. the Fuchsian case, we can be very precise about the constant r_Γ. A. Yamada showed in [45] that it is necessary and sufficient to take the hyperbolic radius to be less than

$$r_\Gamma < \operatorname{arcsinh}\left(\sqrt{\frac{2\cos(2\pi/7)-1}{8\cos(\pi/7)+7}}\right) = .1314\ldots$$

In [26] and in a recent preprint [27] F. Gehring and G. Martin give similar estimates for d = 3.

To make the pictures clearer and the proofs less technical, we start the analysis with the planar case. In Section 5, we will study the higher dimensional case. By construction, E covers the orbit of the origin by Euclidean balls with radii comparable to the distance from the boundary ∂B. Using the similarity between these balls (or disks) and the Whitney cubes (or squares) in some Whitney decomposition, we can obtain relations between discontinuous groups and thin sets, which was our main goal.

In Section 4 on page 27, we will study three basic Fuchsian Archipelagoes and give the pictures of their orbits, see Figures III.4, III.5 and III.7.

2. Global properties

Proposition III.2. *Let E be the Fuchsian Archipelago of Γ as above. The following are equivalent:*

- *Γ is of convergence type.*
- *E is thin with respect to capacity.*
- *E is thin with respect to measure.*

PROOF. Let us denote $t_i = 1 - |\gamma_i(0)|$. By definition Γ is of convergence type if and only if $\sum_i t_i < \infty$. On the other hand, by Definition B.5 on page 95, E is thin with respect to capacity if and only if

$$\sum_i q_k \left(\log \frac{4q_k}{\operatorname{cap}(Q_k \cap E)} \right)^{-1} < \infty,$$

where $\{Q_k\}$ is a Whitney decomposition and q_k is the distance from the square Q_k to the boundary. By Lemma III.5 below we obtain for the logarithmic capacity the following comparisons,

$$\log \frac{4q_k}{\operatorname{cap}(Q_k \cap E)} \approx c \quad \text{and}$$

$$q_k \approx t_i \ \text{ for } Q_k \cap B_i \neq \emptyset.$$

Hence we conclude that E is thin with respect to capacity if and only if

$$\sum_{E \cap Q_k \neq \emptyset} q_k < \infty \text{ which is equivalent to } \sum_i t_i < \infty, \quad \text{since}$$

$$c_4' b_3 \sum_{i:\gamma_i \in \Gamma} t_i \leq \sum_{E \cap Q_k \neq \emptyset} q_k \leq c_4 b_4 \sum_{i:\gamma_i \in \Gamma} t_i,$$

with the constants taken from the proof of Lemma III.5. We have obtained the first equivalence:

$$\Gamma \text{ is of convergence type } \Leftrightarrow E \text{ is thin with respect to capacity.}$$

By Definition B.6 on page 95, E is thin with respect to measure if $H(E \cap D_t) \to 0$ when $t \to 0$. Suppose now that Γ is of convergence type and consider the upper half–plane case.

$$H(E \cap D_t) \leq \sum_{\text{the tail}} 2t_i,$$

where we by the notation $\sum_{\text{the tail}}$ mean that we sum over all indices k such that the Whitney cube Q_k intersects the set $E \cap D_t$. The sum on the right tends to zero with t.

Thus,

$$\Gamma \text{ is of convergence type } \Rightarrow E \text{ is thin with respect to measure.}$$

On the contrary, let us now assume that E is thin with respect to measure. The *essential projection* E^* of E is defined as

$$E^* = \{X \in \mathbb{R} : \forall t > 0 \ \exists y \text{ such that } 0 < y < t \text{ and } (X, y) \in E\}.$$

We now choose an M, $1 < M < M_{r_\Gamma}$, where M_{r_Γ} is a constant only depending on our hyperbolic radius constant r_Γ which forces us to choose an M close enough to 1. Let us then construct a non-tangential limit set $\Lambda_c(0, M)$ with respect to Γ and the parameter M (cf. Remark II.5 at page 14). The construction in the unit ball is carried out in [25] but can immediately be applied to the upper half–plane as well. Since for every B_i in $E \cap D_t$ the cap $C(\gamma_i(0), M) := \{X \in \partial B : |X - \gamma_i(0)| < M(1 - |\gamma_i(0)|)\}$ lies in the projection of the disk B_i, by the choice of M. We have that

$$\Lambda_c(0, M) \subset E^*,$$

see Figure III.3.

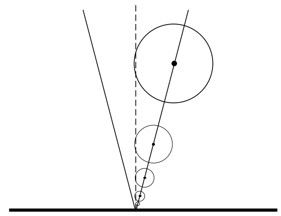

FIGURE III.3. By taking the opening–angle small enough (i.e. M small enough), the vertex of the cone will be contained in the essential projection of E.

The non-tangential limit set is defined as

$$\Lambda_c = \Lambda_c(G) = \bigcup_{M>1} \Lambda_c(0, M).$$

But the limit set is just "slightly dependent" of the parameter M. In fact, the following holds, where $\sigma(\cdot)$ denotes the normalized Lebesgue measure on ∂B,

$$\sigma(\Lambda_c) = \sigma(\Lambda_c(0, M)) \text{ for all } M > 1,$$

see [25, p. 29]. We have then

$$\sigma(E^*) \geq \sigma(\Lambda_c(0, M)) = \sigma(\Lambda_c).$$

Since we assumed that E was thin with respect to measure, we can use Lemma 6.5 in [4] and deduce that $\sigma(E^*) = 0$. Hence $\sigma(\Lambda_c) = 0$ and we can apply [25, Theorem 5, p. 29] which states the following: *If Γ is of divergence type then* $\sigma(\Lambda_c) = 1$. We therefore conclude that Γ can not be of divergence type giving us that the series $\sum_k t_i$ must converge. That is

$$G \text{ is of convergence type } \Leftarrow E \text{ is thin with respect to measure.}$$

We have proved the proposition. □

Let us use a variant of the Wiener type condition in Definition B.2 on page 94.

DEFINITION III.3. *Suppose that $\{Q_k\}$ is a Whitney decomposition of B, then we define the following.*

$$W_0(\tau) := \sum{}' (q_k/\rho_k(\tau))^2,$$

where $q_k = \operatorname{dist}(\partial B, Q_k)$ and $\rho_k(\tau) = \operatorname{dist}(Q_k, \tau)$. The notation $\sum'$ means that we sum over all indices k such that $Q_k \cap E \neq \emptyset$. (c.f [20, p. 88].)

REMARK III.4. We say that two positive functions u and v are comparable, i.e. $u \approx v$ if there is a constant $C \geq 1$ such that $C^{-1}u \leq v \leq Cu$ holds.

LEMMA III.5. *Let E be a Kleinian Archipelago, and let $t_i = 1 - |\gamma_i(0)|$, $R_i(\tau) = \operatorname{dist}(B_i, \tau)$. and q_k, $\rho_k(\tau)$ be as above. Then*

- $t_i \approx q_k$, *if* $Q_k \cap B_i \neq \emptyset$.
- $R_i(\tau) \approx \rho_k(\tau)$ *if* $Q_k \cap B_i \neq \emptyset$.
- $\operatorname{cap}(E \cap Q_k) \approx q_k$.
- $W(\tau) \approx W_0(\tau)$.

PROOF. Suppose that $\{Q_k\}$ is a Whitney decomposition such that

$$c_1 q_k < \text{diameter}(Q_k) < c_2 q_k \text{ where } c_1 > 0,\ c_2 < 1 \tag{6}$$

The balls B_i are controlled too, by the choice of the hyperbolic radius[1], in a similar way,

$$b_1 t_i < \text{radius}(B_i) < b_2 t_i \text{ where } b_1 > 0 \text{ and } b_2 < 1. \tag{7}$$

We can now get the first two estimates in the following way. Let Q_{i_k} be the Whitney cubes (or squares) that intersects the ball (or disk) B_i. Then

$$q_{i_k} \geq t_i - \text{radius}(B_i) - \text{diam}(Q_{i_k}) \geq t_i - b_2 t_i - c_2 q_{i_k}.$$

Thus we have by putting $b_3 := \frac{1-b_2}{1+c_2}$ the estimate $q_{i_k} \geq b_3 t_i$. In a similar way we have by $b_4 := 1 + b_2$ that $q_{i_k} \leq b_4 t_i$. Thus $t_i \approx q_{i_k}$. The argument holds without any change when we compare ρ_k and R_i. Hence

$$b_3 R_i(\tau) \leq \rho_{i_k}(\tau) \leq b_4 R_i(\tau).$$

The first two statements in the lemma are shown.

Let us now turn to the last two estimates. We have also a size relation between intersecting balls and cubes. Let $Q_k \cap B_i \neq \emptyset$ hold. Then we have the following estimate.

$$\frac{\text{diam}(Q_k)}{\text{diam}(B_i)} \geq \frac{c_1 q_k}{2 b_2 t_i} \geq \frac{c_1 b_3 t_i}{2 b_2 t_i}.$$

So, by letting $c_3 := \frac{c_1 b_3}{2 b_2}$ we have that

$$\min_k(\text{diam}(Q_k) : Q_k \cap B_i \neq \emptyset) \geq c_3 \text{diam}(B_i). \tag{8}$$

The number of Whitney–cubes that intersect a hyperbolic ball is then estimated above by $c_4 := (\frac{1}{c_3} + 1)^2$. Analogously, we can get an upper estimate of the number of balls B_{k_i} that intersects a Whitney cube Q_k by $c_4' := (\frac{1}{c_3'} + 1)^2$, where $c_3' := \frac{2b_1}{c_2 b_4}$.

The logarithmic capacity of a square of side-length a is bounded above by $0.6a$, see [30, p. 172]. Therefore we have

$$\text{cap}(E_k) = \text{cap}(E \cap Q_k) \leq \text{cap}(Q_k) \leq 0.6 \frac{1}{\sqrt{2}} \text{diam}(Q_k) \leq \frac{0.6}{\sqrt{2}} c_2 q_k < q_k,$$

where we use the notation $E_k := E \cap Q_k$. Hence

$$\left(\log\left(\frac{4q_k}{\text{cap}(E_k)}\right)\right)^{-1} \leq \left(\log(4)\right)^{-1} \leq 1. \tag{9}$$

It follows that

$$W(\tau) \leq W_0(\tau). \tag{10}$$

We will now obtain an opposite inequality.

$$W(\tau) = \sum_{E \cap Q_k \neq \emptyset} (q_k/\rho_k(\tau))^2 \left(\log\left(\frac{4q_k}{\text{cap}(E_k)}\right)\right)^{-1} \geq$$

$$\geq \sum_{i:\gamma_i \in \Gamma} (q_{k_i}/\rho_{i_k}(\tau))^2 \left(\log\left(\frac{4q_{i_k}}{\text{cap}(E_{i_k})}\right)\right)^{-1},$$

[1] Our choice is r_Γ.

where the E_{i_k} is chosen as the largest[2] E_k that intersects the ball B_i (i.e. $\operatorname{cap}(E_{i_k}) = \max_k \operatorname{cap}(E_k \cap B_i)$). Let us estimate the logarithmic capacity of E_{i_k}.

$$\operatorname{cap}(E_{i_k}) \geq \frac{1}{c_4} \operatorname{cap}(B_i) \geq \frac{1}{c_4} b_1 t_i \geq \frac{1}{c_4} b_1 \frac{1}{b_4} q_{i_k},$$

where we used that the logarithmic capacity of a ball of radius a is a, see [30, p. 172]. Hence

$$\frac{4q_i}{\operatorname{cap}(E_{k_i})} \leq \frac{4c_4 b_4}{b_1}.$$

Let $c_5 := \left(\log(\frac{4c_4 b_4}{b_1})\right)^{-1}$, then

$$W(\tau) \geq \sum_{i:\gamma_i \in \Gamma} (q_{k_i}/\rho_{k_i}(\tau))^2 c_5 \geq \frac{1}{c_4} c_5 \sum_{E \cap Q_k \neq \emptyset} (q_k/\rho_k(\tau))^2.$$

We conclude

$$W(\tau) \leq W_0(\tau) \leq c_W W(\tau).$$

□

We are now ready to state a result concerning the relation between the non-osculating limit set and rarefiedness. Let $\complement\Lambda_h$ denote the set $\partial B \setminus \Lambda_h$.

Proposition III.6. *If $h_{\frac{1}{2}} < \infty$ and $\tau \in \complement\Lambda_h$ then E is rarefied at τ.*

Proof. To simplify the notation let us denote as above $t_i := 1 - |\gamma_i(0)|$ and $R_i = R_i(\tau) := |1 - \overline{\gamma_i(0)}\tau| = |\gamma_i(0) - \tau|$. Let us also recall the notion of the non-osculating limit set, Λ_h. For the definition see II.4 and II.12.

Let $\tau \in \complement\Lambda_h$, then $\tau \notin \cup_{M>0}\Lambda_h(0, M)$ or in other words $\tau \in \complement\Lambda_h(0, M)$, for all $M > 0$. Let us now fix $M > 0$ then

$$\tau \in \complement\Lambda_h(0, M) = \complement \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} C_h(\gamma_i(0), M) = \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} \complement C_h(\gamma_i(0), M).$$

This is a "lim inf"–construction telling us that there exists a natural number $I = I(M)$ such that if $i > I$ then $\tau \notin C_h(\gamma_i(0), M)$.

Aikawa gives in [4, Theorem 3.2] (see Theorem L on page 94) a necessary and sufficient condition for a set to be rarefied at τ (this is also implicitly given in [19]). E is rarefied at τ if

$$W^r(\tau) := \sum_k q_k \frac{(\log(4q_k/\operatorname{cap}(E_k)))^{-1}}{\rho_k(\tau)} < \infty, \tag{11}$$

where we have the same notation as in Lemma III.5 above. Let us now define an auxiliary series in the same spirit as W_0 in Definition III.3 above.

$$W_0^r(\tau) := {\sum_k}' (q_k/\rho_k(\tau)). \tag{12}$$

(Recall that we only sum over those k for which $Q_k \cap E \neq \emptyset$.) Since we have in Lemma III.5 showed that $t_i \approx q_k$ and $R_i(\tau) \approx \rho_k(\tau)$ if $Q_k \cap E \neq \emptyset$, and that the number of intersections are controlled (by c_4, c_4'). Thus we have that

$${\sum_k}' (q_k/\rho_k(\tau)) \approx \sum_i (t_i/R_i(\tau)).$$

[2]in a capacity sense

Let us divide the series into two parts.

$$\sum_i (t_i/R_i(\tau)) = \sum_{i\le I}(t_i/R_i) + \sum_{i>I}(t_i/R_i).$$

Denote the finite summation $c_0 = \sum_{i\le I}(t_i/R_i)$. We see that $c_0 \le I$. For the other series we have $i > I$ which implies $\tau \notin C_h(\gamma_i(0), M)$, which in turn implies

$$|\gamma_i(0) - \tau| \ge \sqrt{M(1-|\gamma_i(0)|^2)} \text{ and } R_i \ge \sqrt{Mt_i},$$

by the construction of the horocap, see Definition II.2. Therefore,

$$\sum_i (t_i/R_i(\tau)) \le c_0 + \sum_{i>I} \frac{t_i}{\sqrt{Mt_i}} = c_0 + \frac{1}{\sqrt{M}} \sum_{i>I} t_i^{1/2}.$$

Since $h_{\frac{1}{2}} < \infty$ the series $\sum_{i>I} t_i^{1/2}$ converges. Hence we have that $W_0^r(\tau) < \infty$ and it follows that $W^r(\tau) < \infty$ by Equation (9) in the proof of Lemma III.5. Hence E is rarefied at τ. $\square$

REMARK III.7. Let us argue as in the proof above, where we used arguments from the proof of Lemma III.5 to obtain the following chain of comparisons.

$$W^r(\tau) \approx W_0^r(\tau) = {\sum_k}'(q_k/\rho_k(\tau)) \approx \sum_i (t_i/R_i(\tau)). \tag{13}$$

If $h_{\frac{1}{2}} < \infty$ we can use Proposition II.14 which tell us that the non-osculating limit set has empty measure, i.e. $\sigma(\Lambda_h) = 0$. Hence we have the following corollary.

COROLLARY III.8. *If $h_{\frac{1}{2}} < \infty$ then E is rarefied a.e..*

REMARK III.9. For the case Γ is finitely generated, see Corollary III.18 below.

3. Local properties

Except Proposition III.6, all the above propositions are global. Let us now turn to questions of local behavior. What can we say about a given point on the boundary? To answer this question, we will again consider the limit sets Λ_c and Λ_h of the discontinuous group Γ.

PROPOSITION III.10. *Let Γ be of convergence type. If $\tau \in \complement\Lambda_h$, then E is minimally thin at τ.*

REMARK III.11. This result will be considerably sharpened in Theorem IV.12 on page 39.

PROOF. Since τ is in $\complement\Lambda_h$, we can argue as in the proof of Proposition III.6 and obtain the following estimate,

$$W_0(\tau) \approx \sum_i (t_i/R_i)^2 = \sum_{i\le I}(t_i/R_i)^2 + \sum_{i>I}(t_i/R_i)^2.$$

Denote the finite summation $c_1 = \sum_{i\le I}(t_i/R_i)^2$. For the other series we have $i > I$ which implies $R_i \ge \sqrt{Mt_i}$. Therefore,

$$\sum_i (t_i/R_i)^2 \le c_1 + \sum_{i>I} \frac{t_i^2}{Mt_i} = c_1 + \frac{1}{M}\sum_{i>I} t_i.$$

Since Γ is of convergence type, $\sum_{i>I} t_i < \infty$ and we have that $W_0(\tau) < \infty$. Lemma III.5 gives us then the result. $\square$

PROPOSITION III.12. *If $\tau \in \Lambda_c$ then E is **not** minimally thin at τ.*

REMARK III.13. This holds independently of the value of δ.

PROOF. Since τ is in $\Lambda_c = \cup_{M>0}\Lambda_c(0,M)$, there exists $M > 0$ such that $\tau \in \Lambda_c(0,M) = \cap_{j=1}^{\infty} \cup_{i>j}^{\infty} C(\gamma_i(0),M)$. This is a "lim sup"–construction and we conclude $\tau \in C(\gamma_i(0),M)$ for infinitely many i, say, for all i in the index set $I(M)$.

We will now estimate the series.

$$W_0(\tau) \approx \sum_i \left(\frac{t_i}{R_i(\tau)}\right)^2 \geq \sum_{i\in I(M)} \left(\frac{1-|\gamma_i(0)|}{|1-\overline{\gamma_i(0)}\tau|}\right)^2.$$

Since $\tau \in C(\gamma_i(0),M)$, we have

$$|1-\overline{\gamma_i(0)}\tau| = |\gamma_i(0)-\tau| < M(1-|\gamma_i(0)|).$$

Hence,

$$W_0(\tau) \gtrapprox \sum_{i\in I(M)} \frac{1}{M^2} = \frac{1}{M^2}\sum_{i\in I(M)} = \infty.$$

This implies that also $W(\tau)=\infty$ by Lemma III.5 and we conclude that E is not minimally thin at τ. We also note that we only use the fact that the cardinality of the index set $I(M)$ is infinite. We do not use any convergence properties. That is, the result is independent of δ. $\square$

REMARK III.14. Since we from Lemma III.5 know that t_i and q_k are comparable and the number of intersections are bounded (by the estimates c_4 and c_4'), we will from now on not make any difference in notation of q_k and t_i (or in ρ_k and R_i)[3].

In the following section, we take a look at some concrete examples of orbits of a discontinuous group to get a picture of the situation.

4. Three basic examples of orbits

Let us plot three basic but fundamental cases. First an orbit of a group generated by a single parabolic element in Figure III.4. In Figure III.5 we do the same thing for a hyperbolic mapping. We then combine them in Figure III.7.

For simplicity, we will give the generators in the upper-half-plane-model and then use the map $z \rightarrowtail \frac{z-i}{z+i}$ to transform the picture to the disk-model.

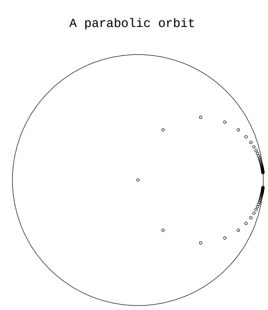

FIGURE III.4. Here is an example of the orbit of a discontinuous group. The group is generated by the parabolic mapping $z \rightarrowtail z+1$ in the upper half–plane which is then conformally mapped onto the unit disc. The critical exponent δ is $\frac{1}{2}$ and the Fuchsian Archipelago is minimally thin but not rarefied at 1.

[3]If we want to be more careful, we can always argue as in the above proofs.

4.1. Parabolic. In the parabolic example in Figure III.4, we have that $\delta = \frac{1}{2}$ and the Fuchsian Archipelago is not rarefied at 1. Let us give a proof of these statements by stating and proving the following standard lemma.

LEMMA III.15. *For a Fuchsian group with a single parabolic generator we have that*

$$e^{-d(0,\gamma_n 0)} \approx \frac{1}{n^2}.$$

PROOF. We expand the left hand side.

$$e^{-d(0,\gamma_n 0)} = \frac{1-|\gamma_n 0|}{1+|\gamma_n 0|} \approx 1 - |\gamma_n 0|.$$

If we now use the fact that the orbit points approach the fixed point parabolically, i.e. "$y = x^2$–like", and then use the so called normalization map depicted in in Figure VI.10, we will by the similarity of triangles get that

$$\frac{1-|\gamma_n 0|}{\sqrt{1-|\gamma_n 0|}} \approx \frac{1}{n}.$$

Hence,

$$e^{-d(0,\gamma_n 0)} \approx \frac{1}{n^2}.$$

□

We use the lemma above to compute the Poincaré series.

$$h_s = \sum_{\gamma_n \in \Gamma} e^{-sd(0,\gamma_n 0)} \approx \sum_{\gamma_n \in \Gamma} \frac{1}{n^{2s}} = \begin{cases} = \infty & \text{if } s \le \frac{1}{2}, \\ < \infty & \text{if } s > \frac{1}{2}. \end{cases}$$

From this and Lemma A.9 we see that $\delta = \frac{1}{2}$. To see that the Fuchsian Archipelago E is minimally thin at 1, we use the Poincaré series in the following way. By a simple geometry argument we see that

$$W_0(1) \approx \sum{}' \frac{t_n^2}{\sqrt{t_n}^2} \approx h_1 < \infty, \quad \text{where } t_n = 1 - |\gamma_n 0|.$$

What about rarefiedness at 1? Well, we know that E is rarefied at 1 if W_0^r converges, see equation (13) on page 26. Let us do the same geometric argument as above.

$$W_0^r(1) \approx \sum{}' \frac{t_n}{\sqrt{t_n}} \approx h_{\frac{1}{2}} = \infty.$$

However, the condition $W_0^r(1)$ converges is not a necessary condition for rarefiedness. We have to examine the situation more carefully. Let us look at the sufficient *and* necessary condition given in Theorem L on page 94.

We see there that there is a possibility to divide E into two parts E' and E'', where, to obtain rarefiedness, $W^r(1, E') < \infty$. It is not difficult to see that if we choose a $B(X_i, r_i)$ to cover each hyperbolic ball B_i in the Fuchsian Archipelago, we will obtain the same W_0^r condition for E''. But maybe we can choose the balls $B(X_i, r_i)$ in a more efficient way to cover more than one island for each ball? Yes, we can, but not more efficient then we gain a factor $\frac{c}{2}$ for the last series in Theorem L, where c is the step length when we have conjugated our parabolic mapping to have the fixed point at ∞ ($c = 1$ in Figure III.4) in the upper half-plane. To obtain this we just study the upper half-plane where the orbit points lie horizontal. We cover them one by one and compare that contribution to the series with a bigger covering.

We conclude that for our studied case, $W_0^r(1)$ converges is both a necessary and sufficient condition for E to be rarefied at 1.

Hence we see that the Fuchsian Archipelago, E, is not rarefied at 1 if Γ is a Fuchsian group with a single parabolic generator.

4.2. Hyperbolic. Next, we turn to a Fuchsian group generated by a single hyperbolic map. Let us go to the upper-half-plane-model and choose $g : z \rightarrowtail 2z$ as the generator. Let us compute the critical exponent δ for this group (see

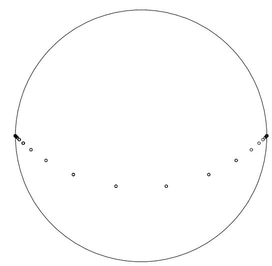

FIGURE III.5. This is another example. The group is generated by the hyperbolic mapping $z \rightarrowtail 2z$ in the upper-half-plane. (In order to make the picture more visible, we choose as the base of the orbit not i but $i + 1$, in the upper-half-plane.) $\delta = 0$ but E is neither minimally thin nor rarefied at 1, or at -1.

Definition A.7). Since we have chosen to map the unit disk to the upper-half-plane by letting 0 go to i, we are interested in the orbit $\{\gamma_n i\}$, see Figure III.5. We have then that the whole orbit lies on the imaginary axis. Therefore, consecutive hyperbolic distances between the orbit points can easily be computed.

$$d(\gamma_n i, g(\gamma_n i)) = \left|\log \frac{\operatorname{Im}(\gamma_n i)}{\operatorname{Im}(g(\gamma_n i))}\right| = \log 2.$$

Hence the orbital counting function $n(r)$, i.e. the number of orbit points $\gamma_n i$ such that $d(i, \gamma_n i) < r$, is

$$n(r) = \left\langle \frac{2r}{\log 2} \right\rangle,$$

where $\left\langle \frac{\alpha}{\beta} \right\rangle$ stands for the integer part of $\frac{\alpha}{\beta}$. That is, $n(r) \approx r$. Thus, by the definition of δ, we get immediately that $\delta = 0$.

4.3. Combination. Let us now study a Fuchsian group generated by the two generators above. We can not do that right away, we have to separate the fixed points first. We accomplish that by conjugating the parabolic mapping. Let the group be generated by the following two maps, $z \rightarrowtail h(f(h^{-1}(z)))$ and $z \rightarrowtail 2z$ in the upper half–plane, where $f(z) = z + 1$ and $h(z) = \frac{2z+1}{z+1}$.

We have to avoid a situation where a parabolic element and a hyperbolic have a common fixed point, that would generate a group which would not be discontinuous, see [9, Theorem 5.1.2].

Since $h(f(h^{-1}(z))) = \frac{z-4}{z-3}$, we have by the conjugation moved the fixed point from ∞ to 2. See Figure III.6 for the result in the unit disk. In Figure III.7, that means that we have have the hyperbolic fixed points in 1 and -1, and the parabolic ditto is in $e^{-\arctan(4/3)i}$.

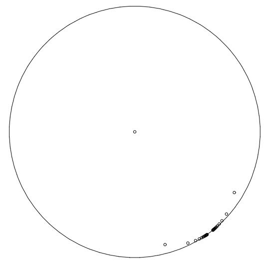

FIGURE III.6. The parabolic fixed point is now in a new position.

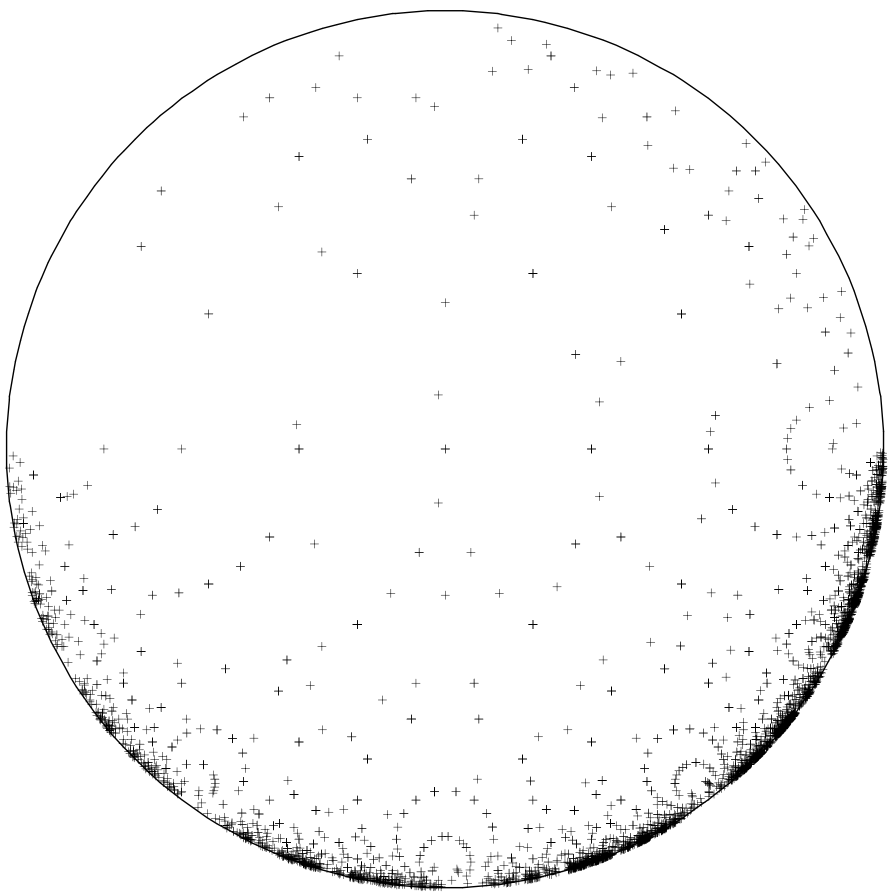

FIGURE III.7. Here we combine a parabolic and a hyperbolic generator. We add the hyperbolic generator from Figure III.5 with the parabolic generator in Figure III.6.

5. Higher dimensions

Some of the above results are still valid in higher dimensions, $\mathrm{d} \geq 3$. Proposition III.2 holds without any changes.[4]

[4]Remember that Γ is of **convergence type** iff $\sum(1-|\gamma_k|)^{\mathrm{d}-1}$ converges.

PROPOSITION III.16. *The following are equivalent:*

- Γ *is of convergence type.*
- E *is thin with respect to capacity.*
- E *is thin with respect to measure.*

PROOF. Γ is of convergence type if and only if $\sum_k t_k^{\mathrm{d}-1} < \infty$. By definition, E is thin with respect to capacity if and only if $\sum_i q_i \operatorname{cap}(E \cap Q_i) < \infty$, where $\operatorname{cap}(\cdot)$ is the Newtonian capacity. We can now adjust Lemma III.5 by natural changes with respect to dimension, i.e. $\operatorname{cap}(Q_i) \approx q_i^{\mathrm{d}-2}$ and $\operatorname{cap}(B_k) \approx t_k^{\mathrm{d}-2}$ (see page 165 in [30]) and let the constants c_4 and c_4' instead be$(\frac{1}{c_3}+1)^{\mathrm{d}}$ and $(\frac{1}{c_3'}+1)^{\mathrm{d}}$ respectively.

This will lead to

$$\sum_i q_i \operatorname{cap}(E \cap Q_i) < \infty \text{ if and only if } \sum_k t_k^{\mathrm{d}-1} < \infty.$$

Hence Γ is of convergence type if and only if E is thin with respect to capacity.

Now, let us assume that Γ is of convergence type and for convenience, let us be situated in the upper half–plane. The Hausdorff–type measure, $H(\cdot)$, is used to check if E is thin with respect to measure.

$$H(E \cap D_t) \leq \sum_{\text{the tail}} (2t_k)^{\mathrm{d}-1},$$

where D_t is the hyper–strip $\{x \in D;\ 0 < x_N < t,\ x = (x_1, \ldots, x_d)\}$. Hence E is thin with respect to measure.

On the contrary, assume that E is thin with respect to measure. Then there exists a constant $M > 1$ such that the non-tangential limit set $\Lambda_c(0, M)$, has the following property.

$$\Lambda_c(0, M) \subset E^*, \text{ the essential projection of } E.$$

Lemma 6.5 in [4] tells us now that the Lebesgue measure of E^* is 0. In a remark [25, p. 29] we have $\sigma(\Lambda_c(0, M)) = \sigma(\Lambda_c)$. Hence we have

$$0 = \sigma(E^*) \geq \sigma(\Lambda_c(0, M)) = \sigma(\Lambda_c).$$

Thus, $\sigma(\Lambda_c) = 0$ and by Theorem J on page 91 we conclude that Γ is of convergence type. $\square$

We also have validity of a generalization of Corollary III.8

PROPOSITION III.17. *If* $h_{\frac{d-1}{2}} < \infty$ *then* E *is rarefied a.e.*

PROOF. Let $d \geq 3$. E is rarefied at τ if

$$W^r(\tau) := \sum_k \frac{q_k}{\rho_k(X)^{\mathrm{d}-1}} \operatorname{cap}(E \cap Q_k) < \infty.$$

See Theorem L on page 94 for the complete statement and [4, Theorem 3.2] for the proof.

As above we have for the Newtonian capacity

$$\operatorname{cap}(E \cap Q_k) \approx \operatorname{cap}(B_k) \approx t_k^{\mathrm{d}-2}.$$

As in the proof of Proposition III.6 we know there is an integer K such that $\tau \notin C_h(\gamma_k(0), M)$ for a fixed M and all $k > K$. For those k:s we have the estimate $R_k \geq \sqrt{M t_k}$. Hence the following holds.

$$W^r(\tau) \approx \sum_k \left(\frac{t_k}{R_k(\tau)}\right)^{\mathrm{d}-1} =$$

$$= \sum_{k \leq K} \left(\frac{t_k}{R_k(\tau)}\right)^{\mathrm{d}-1} + \sum_{k > K} \left(\frac{t_k}{R_k(\tau)}\right)^{\mathrm{d}-1} \leq$$

$$\leq K + \sum_{k > K} \left(\frac{t_k}{\sqrt{M t_k}}\right)^{\mathrm{d}-1} = K + \frac{1}{M^{\frac{\mathrm{d}-1}{2}}} \sum_{k > K} t_k^{\frac{\mathrm{d}-1}{2}} < \infty.$$

Hence E is rarefied at τ and thus almost everywhere, since $\sigma(\Lambda_h) = 0$ if $h_{\frac{\mathrm{d}-1}{2}} < \infty$, see Proposition II.14. $\square$

Let us now also consider the case where Γ is geometrically finite (or in the planar case Γ is finitely generated).

Corollary III.18. *Let $d \geq 2$.*
If Γ is geometrically finite and $\delta < d - 1$ then E is rarefied a.e.

Proof. Since Γ is geometrically finite δ (the critical exponent of Γ) equals the Hausdorff–measure of the total limit set, that is $d(\Lambda) = \delta < \mathrm{d} - 1$. Thus $\sigma(\Lambda) = 0$ and almost every point τ on the boundary is in the complement of Λ. Since $\partial B \setminus \Lambda$ is an open set there is an open ball centered at τ and with radius r_τ such that the ball does not intersect E. Therefore we have for all k, $r_\tau \leq R_k(\tau)$ and as in the proof above,

$$W^r(\tau) \approx \sum_k \left(\frac{t_k}{R_k(\tau)}\right)^{\mathrm{d}-1} \leq \sum_k \left(\frac{t_k}{r_\tau}\right)^{\mathrm{d}-1} = \frac{1}{r_\tau} \sum_k (t_k)^{\mathrm{d}-1} < \infty.$$

Hence E is rarefied at τ and thus for almost every point on the boundary. $\square$

Even the two propositions dealing with local behavior have their counterparts.

Proposition III.19. *If $\tau \in \complement\Lambda_h$ and $h_{d-1} < \infty$, then E is minimally thin at τ.*

Remark III.20. Theorem IV.12 will give a much stronger statement.

Proposition III.21. *If $\tau \in \Lambda_c$ then E is* ***not*** *minimally thin at τ.*

The proofs of the above propositions are straightforward generalizations of Propositions III.10 and III.12 once we know what minimal thinness means in higher dimensions. The following lemma provides us with that information.

Lemma III.22. *Let $d \geq 3$, E as above and $\{\gamma_k\}$ the elements in Γ. E is minimally thin at $\tau \in \partial B$ if and only if*

$$\sum_k \left(\frac{1 - |\gamma_k(0)|}{|1 - \overline{\gamma_k(0)}\tau|}\right)^d < \infty.$$

PROOF. In Theorem K on page 94 a necessary and sufficient condition for a set E to be minimally thin is given, i.e. E is minimally thin at τ if and only if

$$\sum_k \frac{q_k^2}{\rho_k(\tau)^{\mathrm{d}}} \operatorname{cap}(E \cap Q_k) < \infty.$$

(As usual q_k is the distance from Q_k to the boundary, and $\rho_k(\tau)$ is the distance from Q_k to τ.) If we transform this condition in the usual manner, see the proof of Lemma III.5, and use the fact that $\operatorname{cap}(E_k) \approx t_k^{\mathrm{d}-2}$, we will end up with the following convergence criterion.

$$\sum_k (t_k/R_k)^d < \infty,$$

which is the desired condition. □

5.1. The main result. Our main result concerning the global size of E follows now easily from the Propositions III.2, III.12, III.16 and III.21 above.

THEOREM III.23. *Let, for $d \geq 2$, E be a Kleinian Archipelago as above. Then Γ is of convergence type if and only if E is minimally thin a.e. on the boundary.*

PROOF. Propositions III.2 and III.16 gives the necessary part, since we know by [4, Theorem 1.2] that either *thin with respect to capacity* or *thin with respect to measure* gives minimal thinness a.e.

The sufficient part is obtained by the following reasoning. Suppose that Γ is not of convergence type, then we know that the conical limit set has full Lebesgue measure. That is, almost every point on the boundary is in the conical limit set. If τ is such a point, we know by Propositions III.12 and III.21 that E is not minimally thin at τ. We conclude that E is not minimally thin at almost every point on the boundary. □

CHAPTER IV

Generalized limit sets and convergence criteria

In the previous chapters some results were established concerning minimal thinness of hyperbolically covered Kleinian orbits and two different limit sets, the non-tangential and the non-osculating limit set. We will here generalize some of these results to a general family of limit sets. We will also sharpen some of the above results with the help of these more general limit sets.

The family and its notion are cited from [33], where we also will collect some results to be able to do the above mentioned generalizations.

1. Notations, definitions and basic relations

Let B be the d-dimensional unit ball and ∂B its boundary. The following definition is cited from [33, p. 5].

DEFINITION IV.1. *Let $a \in B$ and $k, \alpha > 0$. We define*

$$I(a : k, \alpha) = \{x \in \partial B : \left| x - \frac{a}{|a|} \right| < k(1 - |a|)^{\alpha}\}.$$

Let Γ be a Kleinian group that preserves the unit ball, i.e. if d = 2 Γ is a Fuchsian group. We denote the elements in Γ by γ_n.

Let us also cite page 23 in [33] for the following definition.

DEFINITION IV.2.

$$L(0 : k, \alpha) = \bigcap_{m=1}^{\infty} \bigcup_{n>m}^{\infty} I(\gamma_n(0) : k, \alpha).$$

From the definitions of the non-osculating limit set of 0 and M in Definition II.4 and the non-tangential limit set of 0 and M at page 14, we obtain the following.

LEMMA IV.3. *For the non-tangential*

$$L(0 : k, 1) = \Lambda_c(0, \sqrt{1 + k^2}),$$

and for the non-osculating limit set

$$L(0 : k, \frac{1}{2}) = \Lambda_h(0, \frac{k^2}{2}).$$

PROOF. Let us reformulate with the help of the parameter l

$$I(a : k, \alpha) = \{x \in \partial B : \left| x - \frac{a}{|a|} \right| < k(1 - |a|)^{\alpha}\} =$$

$$= \{x \in \partial B : |x - a| < l(1 - |a|)^{\alpha}\}.$$

Let us for simplicity write t for $(1-|a|)$, we will then obtain the following asymptotic[1] relation $t^2+k^2t^{2\alpha}=l^2t^{2\alpha}$, i.e.

$$l=\sqrt{t^{2(1-\alpha)}+k^2}. \tag{14}$$

Thus we immediately obtain the first result for $\alpha=1$ and the definition of the non-tangential limit set of 0 and M.

If $\alpha<1$ we have that $l\to k$ as $t\to 0$. Definition IV.2 tells us that to obtain the limit sets we will produce a lim sup process[2]. In other words: we are in fact only interested in the limit case as $t\to 0$.

Let us now, to consider the latter statement in the lemma, put $\alpha=\frac{1}{2}$.

$$\{x\in\partial B:|x-a|^2<\frac{1}{2}k^2(1-|a|^2)\}=\{x\in\partial B:|x-a|<k\sqrt{\frac{1+|a|}{2}}\sqrt{1-|a|}\}.$$

Since $\alpha<1$ we have that

$$\{x\in\partial B:|x-a|<k\sqrt{\frac{1+|a|}{2}}\sqrt{1-|a|}\}\to\{x\in\partial B:|x-\frac{a}{|a|}|<k(1-|a|)^{\frac{1}{2}}\}$$

as $|a|\to 1$, which is what happends when we construct the limit set. We obtain the last statement in the lemma. $\square$

To get the opening angle independent non-tangential limit set one takes the union over all opening angles. We can do the same thing in our generalized situation.

DEFINITION IV.4. *Let us denote the α-limit set by*

$$\mathcal{L}(\alpha)=\bigcup_{k>0}L(0:k,\alpha)$$

Let us cite Theorem 2.1.1 in [33].

THEOREM B. *Let γ be a discrete group acting in B for which the series*

$$\sum_{\gamma\in\Gamma}(1-|\gamma(0)|)^{(d-1)\alpha}\ \textit{converges.}$$

Then $|\mathcal{L}(\alpha)|=0$, where $|\cdot|$ stands for the Lebesgue measure on the surface ∂B.

Thanks to Lemma IV.3 we have that the non-osculating limit set, see Definition II.12, is in fact $\mathcal{L}(\frac{1}{2})$. We can therefore obtain the result in Proposition II.14 by applying Theorem B above for $\alpha=\frac{1}{2}$. We thus have an alternative proof for that result.

REMARK IV.5. Let me here indicate and give the reference to two other ways to obtain Proposition II.14.

- We can use Lemma II.8 together with the Borel-Cantelli type result in [40, p. 218].

[1] $t\to 0$

[2] i.e. $\cap_{m=1}^{\infty}\cup_{n>m}^{\infty}$

- One can rather easily see that the, by Pommerenke in [35] defined set of *oricyclic points,* is the complement to the non-osculating limit set. Furthermore, Γ is said to be of fully accessible type if the set of orocyclic points is of full Lebesgue measure, see [35]. Varopoulos showed in Proposition 9.1 in [41] that for a Fuchsian group that satisfies $h_{\frac{1}{2}} < \infty$ the group is of fully accessible type.

Nicholls defines, on page 37 in [33], the following limit set.

DEFINITION IV.6. *The* ***horospherical limit set****, H, is the union of points $\xi \in \partial B$ such that for every $a \in B$ there is a sequence $\{\gamma_n\} \subset \Gamma$ such that*

$$\frac{|\xi - \gamma_n(a)|^2}{1 - |\gamma_n(a)|} \to 0 \ as \ n \to \infty.$$

THEOREM C. *Let γ be a discrete group acting in B. Then the horospherical limit set is given by*

$$H = \bigcap_{k>0} L(a : k, 1/2).$$

REMARK IV.7. Note that the horospherical limit set is a subset of the non-osculating limit set. It can be rather easily shown that the set difference, $\Lambda_h \setminus H$, is the so called set of *Garnett points.* Theorem 2.6.6 in [33] tells us that the Lebesgue measure of the set of Garnett points is always zero.

REMARK IV.8. The author was kindly informed by L. Ward that in [32], K. Matsuzaki defines in fact the non-osculating limit set. He calls it *the weak horocyclic limit set* and denotes it by $\Lambda_{\bar{h}}$.

2. Equivalent geometric interpretations

PROPOSITION IV.9. *Let r_τ be the ray from the origin to $\tau \in \partial B$ and let $\alpha < 1$. The following are then equivalent.*

- *τ is in $L(0 : k, \alpha)$.*
- *There is a sequence $\{\gamma_i\}$ of members in Γ such that $\gamma_i 0 \to \tau$ and such that*

$$\cosh(d(\gamma_i 0, r_\tau)) < k\Big(\cosh(d(\gamma_i 0, 0))\Big)^{1-\alpha}.$$

- *There is a sequence $\{\gamma_i\}$ of members in Γ such that $\gamma_i 0 \to \tau$ and*

$$d(\gamma_i 0, r_\tau) < (1-\alpha) d(\gamma_i 0, 0) + \alpha \log 2 + \log k.$$

PROOF. Theorem 1.2.1 in [33] gives us an expression of the hyperbolic distance to a geodesic between ξ and η on the boundary. In our case we put $\xi = \tau$ and $\eta = -\tau$ and obtain the following.

$$\cosh d(\gamma_i 0, r_\tau) = \frac{2|\gamma_i 0 - \tau||\gamma_i 0 + \tau|}{|\tau + \tau|(1 - |\gamma_i 0|^2)},$$

which is approximately $\frac{|\gamma_i 0 - \tau|}{1 - |\gamma_i 0|}$ when $\gamma_i 0$ is close to τ. In other words

$$\cosh d(\gamma_i 0, r_\tau) \approx \frac{R_n}{t_n} \quad \text{for } Q_n \cap \gamma_i 0 \neq \emptyset \text{ and } \gamma_i 0 \text{ close to } \tau.$$

What about the distance to the origin? Since we have that

$$d(\gamma_i 0, 0) = \log \frac{1+|a|}{1-|a|} \quad \text{(see for example [9, p. 38])},$$

we immediately obtain the following.

$$\cosh d(\gamma_i 0, 0) = \frac{1+|\gamma_i 0|^2}{1-|\gamma_i 0|^2} \approx \frac{1}{1-|\gamma_i 0|} \quad \text{as } |\gamma_i 0| \to 1.$$

That is, $\cosh d(\gamma_i 0, 0) \approx \frac{1}{t_n}$ as $|\gamma_i 0| \to 1$.

By the Definitions IV.1 and IV.2 we have that $\tau \in L(0 : k, \alpha)$ if and only if there are infinitely many $\gamma_i \in \Gamma$ such that τ is in $I(\gamma_i 0 : k, \alpha)$. That is,

$$\left| \tau - \frac{\gamma_i 0}{|\gamma_i 0|} \right| < k(1 - |\gamma_i 0|)^\alpha$$

holds for a sequence of elements γ_i in Γ such that $\gamma_i 0$ tends to τ.

Since the orbit can only cluster at the boundary, we will have for our Whitney construction that $\rho_n < k q_n^\alpha$ (or $\rho_n / q_n < k(1/q_n)^{1-\alpha}$) for infinitely many n where $Q_n \cap \gamma_i 0 \neq \emptyset$.

Combining the parts above we get the following.

$$\cosh d(\gamma_i 0, r_\tau) < k(\cosh d(\gamma_i 0, 0))^{1-\alpha}$$

for a sequence $\{\gamma_i\}$ in Γ such that $\gamma_i 0$ tends to τ if and only if $\tau \in L(0 : k, \alpha)$. The first equivalence is shown.

For the second one, we see that $\cosh d(\gamma_i 0, 0)$ tends to $\frac{1}{2} e^{d(\gamma_i 0, 0)}$ as i tends to ∞. Furthermore, since $\alpha < 1$

$$\cosh d(\gamma_i 0, r_\tau) \to \frac{e^{d(\gamma_i 0, r_\tau)}}{2} \quad \text{as } i \to \infty.$$

From the first equivalence we have then that $\tau \in L(0 : k, \alpha)$ if and only if, for infinitely many $\gamma_i \in \Gamma$, the following inequality holds.

$$\frac{e^{d(\gamma_i 0, r_\tau)}}{2} < k \left(\frac{e^{d(\gamma_i 0, 0)}}{2} \right)^{1-\alpha}.$$

Hence, $\tau \in L(0 : k, \alpha)$ if and only if

$$d(\gamma_i 0, r_\tau) < (1-\alpha) d(\gamma_i 0, 0) + \alpha \log 2 + \log k$$

holds for a sequence $\{\gamma_i\}$ of members in Γ such that $\gamma_i 0 \to \tau$. □

Remark IV.10. From the above we see that if there exists a sequence of orbit points tending to a boundary point τ in such a way that the (hyperbolic) distance from the points to the ray r_τ is bounded by a constant, then $\tau \in \Lambda_c$. Furthermore, if there exists a sequence of orbit points tending to τ such that for each point in the sequence the distance to the ray r_τ is less than half the distance to the origin, then τ is in the horospherical limit set, H. (Compare this with Definition IV.6.)

3. The non-minimally thin set $\mathfrak{N}$

In this section we will show that, $\mathfrak{N}$, the set on the boundary where the Kleinian Archipelago is not minimally thin, is almost the non-tangential limit set Λ_c.

We can introduce a strong type of the limit set $\mathcal{L}(\alpha)$ by taking the intersection instead of the union in the following manner.

Definition IV.11. *We define the* ***strong*** α***-limit set*** *to be*

$$\mathcal{L}_s(\alpha) = \bigcap_{k>0} L(0 : k, \alpha).$$

Thus we have that $H = \mathcal{L}_s(\frac{1}{2})$ and that

$$\partial B \supset \mathcal{L}(\alpha) \supset \mathcal{L}_s(\alpha) \supset \mathcal{L}(\alpha+\varepsilon) \quad \text{for all } \varepsilon > 0.$$

Propositions III.12 and III.21 above says that if $\tau \in \mathcal{L}(1)$ then the Kleinian Archipelago is not minimally thin at τ. We will in this section study the opposite relation.

We will show the following theorem.

THEOREM IV.12. *If $\alpha < 1$ and $\tau \notin \mathcal{L}_s(\alpha)$ then the Kleinian Archipelago is minimally thin at τ.*

In the proof we will need the following result which we state in the form given in [4, p. 357].

THEOREM D. (e.g. [14] (for the planar case), [2, p. 440], [15, p. 98]) *Let us consider the upper-half-space*

$$\mathbb{H} = \{z = (X, y) \in \mathbb{R}^d : X = (x_1, x_2, \dots, x_{d-1}) \in \mathbb{R}^{d-1} \text{ and } y > 0\}$$

and the subset

$$E_f = \{z = (X, y) \in \mathbb{R}^d : 0 < y < f(|X|)\},$$

where f is a positive non-decreasing function on $(0, \infty)$. Then

$$E_f \text{ is minimally thin at } 0 \text{ if and only if } \int_0 \frac{f(x)}{x^2}\,dx < \infty.$$

PROOF OF THEOREM IV.12 . Let $\tau \notin \mathcal{L}_s(\alpha)$ then there exists a $k > 0$ such that $\tau \notin L(0 : k, \alpha)$, i.e. there are only finitely many orbit points in the truncated α-"cone", $\mathcal{C}_\alpha(k, \tau)$, which we define as $(\mathbb{H} \setminus E_f) \cap B(0, 1)$ where

$$f(x) = (\frac{1}{k}x)^{\frac{1}{\alpha}}.$$

To obtain E, we fatten the point sequence. We will have to take care of the extra intersections — which may be infinitely many, see Figure IV.8 where the point sequence lies outside the "undashed" α-cone, but every hyperbolic ball intersects it.

We will now show that it is possible to get a slightly smaller cone by changing k to $\frac{k}{2}$ in $\mathcal{C}_\alpha$ so that the number of balls B_i in E that intersects $\mathcal{C}_\alpha(\frac{k}{2}, \tau)$ is finite.

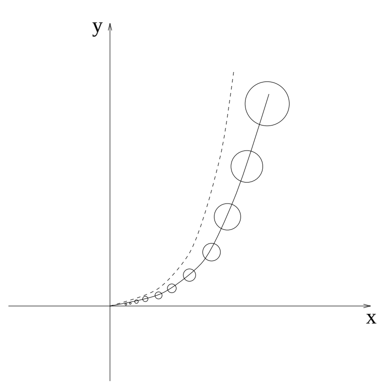

FIGURE IV.8. The undashed curve represents $x = ky^\alpha$ (i.e. $x = f^{-1}(y)$) and the dashed line $x = ky^\alpha - py$, with $p = p(r_\Gamma)$.

We see that no balls B_i in E can reach inside the α-cone $\mathcal{C}_\alpha(k, \tau)$ more than a hyperbolic distance r_Γ. In the $\mathbb{H}$ model, the hyperbolic distance is approximately the Euclidean divided by the distance to the boundary (i.e. y) for small horizontal hyperbolic distances. Due to the fact that the hyperbolic distances we are interested in are bounded by the constant r_Γ we can find a p only depending on r_Γ such that a hyperbolic ball with radius r_Γ with its (hyperbolic) center on the curve $x = ky^\alpha$ does not intersect the upper "dashed curve" $x = ky^\alpha - py$, see Figure IV.8.

We see that

$$\frac{k}{2}y^\alpha < ky^\alpha - py \text{ holds for every } y < \left(\frac{k}{2p}\right)^{\frac{1}{1-\alpha}}.$$

Since Γ is discrete we know that the orbit can not have a cluster-point inside $\mathbb{H}$, hence there are only finitely many $\gamma_i \in \Gamma$ such that

$$\operatorname{Im}(\gamma_i 0) \geq \left(\frac{k}{2p}\right)^{\frac{1}{1-\alpha}}.$$

We conclude that the number of balls B_i in E that intersect $\mathcal{C}_\alpha(\frac{k}{2},\tau)$ is finite.

Now, let us split E into two parts

$$E_1 := E \setminus \mathcal{C}_\alpha(\frac{k}{2},\tau) \text{ and } E_2 := E \cap \mathcal{C}_\alpha(\frac{k}{2},\tau).$$

From Theorem D above with $f(x) = (\frac{2}{k}x)^{\frac{1}{\alpha}}$ and the fact that minimal thinness is a local property, we know that $A = \mathbb{H} \setminus \mathcal{C}_\alpha(\frac{k}{2},\tau)$ is minimally thin at 0. Since $A \supset E_1$ we have that E_1 is minimally thin at the origin.

For the "inner set" E_2, we consider a slightly bigger set

$$\tilde{E}_2 := \bigcup_{B_i \cap \mathcal{C}_\alpha(\frac{k}{2},\tau) \neq \emptyset} B_i.$$

By Lemma III.5 and the fact that B_i intersects $\mathcal{C}_\alpha(\frac{k}{2},\tau)$ at most finitely many times give us the following W-series for $\tilde{E}_2$.

$$W(\tau,\tilde{E}_2) \leq W_0(\tau,\tilde{E}_2) < \infty.$$

In other words, E_2 is minimally thin at τ, c.f. Theorem K on page 94.

If we put this together we will obtain

$$W(\tau,E) \leq W_0(\tau,E) \leq W_0(\tau,E_1) + W_0(\tau,E_2) \leq$$

$$\leq c_W(W(\tau,E_1) + W(\tau,E_2)) < \infty,$$

where c_W is the constant from the proof of Lemma III.5. We have shown that the Kleinian Archipelago E is minimally thin at τ. □

3.1. The Hausdorff dimension of $\mathfrak{N}$. From Propositions III.12 and III.21 we learn that Λ_c is a subset of $\mathfrak{N}$, but Theorem IV.12 tell us that for all $\alpha < 1$ $\mathcal{L}_s(\alpha) \supset \mathfrak{N}$ where we have that $\mathcal{L}_s(1) \subseteq \mathcal{L}(1) = \Lambda_c$. This implies that the sets $\mathfrak{N}$ and Λ_c can not differ very much. In fact, they are of the same dimension.

COROLLARY IV.13. *Let Γ be a non-elementary Kleinian group. The Hausdorff dimension of the non-minimal thin set $\mathfrak{N}$ equals the critical exponent of Γ. Or in other words, Λ_c and $\mathfrak{N}$ have the same Hausdorff dimension.*

PROOF. Let us view the situation in the upper half space $\mathbb{H}$. As usual, let us by δ denote the critical exponent of Γ. That implies,

$$\sum_{\gamma_i \in \Gamma} (1 - |\gamma_i 0|)^{\delta+\varepsilon} < \infty \text{ for all } \varepsilon > 0. \tag{15}$$

Let us now try to estimate the 1-dimensional Hausdorff measure of $\mathcal{L}_s(\alpha)$. In the construction of the limit set $\mathcal{L}_s(\alpha)$, we use the the α-caps, see Figure II.2 where a $\frac{1}{2}$-cap is drawn. The radius of a ball in $\mathbb{R}^d$, centered at $\frac{\gamma_i 0}{|\gamma_i 0|}$, that exactly covers the α-cap at $\gamma_i 0$ is $k(1-|\gamma_i 0|)^\alpha$, see Definitions IV.1, IV.2 and IV.11. Hence we get the following estimate of the one-dimensional Hausdorff measure.

$$H_1(\mathcal{L}_s(\alpha)) \le \sum_{\text{tail}} k(1-|\gamma_i 0|)^\alpha.$$

By equation (15) we have that

$$H_{\frac{\delta+\varepsilon}{\alpha}}(\mathcal{L}_s(\alpha)) \le \sum_{\text{tail}} k^{\frac{\delta+\varepsilon}{\alpha}}(1-|\gamma_i 0|)^{\alpha\frac{\delta+\varepsilon}{\alpha}} = \sum_{\text{tail}} k_1(1-|\gamma_i 0|)^{\delta+\varepsilon} < \infty.$$

Thus we see that the Hausdorff dimension of $\mathcal{L}_s(\alpha)$ is less then or equal to $\frac{\delta+\varepsilon}{\alpha}$ for any $\varepsilon > 0$. Since $\mathcal{L}_s(\alpha)$ is independent of ε, we have that $\dim(\mathcal{L}_s(\alpha)) \le \frac{\delta}{\alpha}$.

Theorem IV.12 gives us that $\mathfrak{N} \subset \mathcal{L}_s(\alpha)$ for every $\alpha < 1$. Thus we have immediately that the Hausdorff dimension of $\mathfrak{N}$ is less then or equal to $\frac{\delta}{\alpha}$, but since $\mathfrak{N}$ is independent of α we obtain $\dim(\mathfrak{N}) \le \delta$. Propositions III.12 and III.21 give us that Λ_c is a subset of $\mathfrak{N}$. Since Γ is non-elementary, Theorem 1.1 in [13] gives us that $\delta = \dim(\Lambda_c)$. Hence, $\dim(\mathfrak{N}) = \delta$. □

3.2. The geometrically finite situation. The question is now: Is in fact $\mathfrak{N} = \Lambda_c$? If we limit ourselves to study groups that are of geometrically finite type, we have the following affirmative answer.

COROLLARY IV.14. *If Γ is a geometrically finite discrete group then $\mathfrak{N} = \Lambda_c$.*

PROOF. First, we note that $\mathfrak{N}$ is a subset of the limit set Λ since if τ is not in Λ then there exists a neighborhood in the unit ball of τ such that the Archipelago of Γ, E, do not intersect that neighborhood.

As in (a) in the proof of Proposition II.17, we have that for a geometrically finite group the limit set Λ is the union of non-tangential limit points (i.e. Λ_c) and parabolic fixed points, see [8].

Let now τ be a parabolic fixed point. We have then that $\tau \notin \mathcal{L}_s(\alpha)$ for all[3] $\alpha > \frac{1}{2}$. By choosing an $\alpha \in (\frac{1}{2}, 1)$, we have from Theorem IV.12 that E is minimally thin at τ. We conclude that $\mathfrak{N} \subset \Lambda_c$ and we are done. □

[3] We have in fact $\tau \notin \mathcal{L}_s(\frac{1}{2})$ $(= \Lambda_h)$ but $\tau \in \mathcal{L}(\frac{1}{2})$ if τ is a parabolic fixed point.

CHAPTER V

Boundary Layers
Joint work with H. Aikawa

REMARK V.1. This chapter is in fact [6][1], which is joint work with professor Hiroaki Aikawa, Shimane University, Japan [2].

The concept of boundary layers, introduced by A. Volberg in [42], is generalized from subsets of the unit disk to subsets of general non-tangentially accessible (NTA) domains. Capacitary conditions of Wiener type series of both necessary and sufficient type for boundary layers are presented and the connection between boundary layers and minimally thin sets is studied.

1. Introduction

In [42] A. Volberg studied domains in the plane with harmonic measures comparable to the Lebesgue measure for boundary arcs and defined the concept *boundary layer*. More precisely, let U be the unit disk $\{|z| < 1\}$. Suppose E is a closed subset of U and $\Omega = U \setminus E$ is a domain containing the origin 0. Volberg [42] said that Ω is a boundary layer if there is a positive constant c such that

$$\omega(0, I) \geq c|I| \text{ for all arcs } I \subset \partial U, \tag{16}$$

where $\omega(0, I)$ is the harmonic measure of I in the domain Ω evaluated at 0 and $|I|$ is the length of I. Loosely speaking, a subset Ω of U is a boundary layer if it is sufficiently "big" and sufficiently "connected", seen from the boundary of U, so that a Brownian particle starting in a given point in the subset should be able to hit any arc of ∂U with probability comparable to the length of the arc. For the historical background and the original motivation for studying boundary layers, see [42].

In [42, Propositions 1.1 and 1.2] Volberg presents capacitary conditions of Wiener type for boundary layers. Volberg's work was then continued by M. Essén in [20, Chapter 5]. The following formulation is taken from Essén [20]. Let $\{Q_k\}$ be a Whitney decomposition of U and let $q_k = \operatorname{dist}(Q_k, \partial U)$ and $\rho_k(\xi) = \operatorname{dist}(Q_k, \xi)$. We put

$$W(\xi) = W(\xi, E) = \sum_k \frac{q_k^2}{\rho_k(\xi)^2} \left(\log \frac{4q_k}{\operatorname{cap}(E \cap Q_k)} \right)^{-1},$$

where cap denotes the logarithmic capacity.

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THEOREM E. *Let* $\frac{1}{2}U \subset \Omega$. *Then there exist positive constants* M_1, M_2 *and* $q_0 < 1$ *with the following properties:*

(i) *If* $\sup_{\xi \in \partial U} W(\xi) \le (1-c)/M_1$ *then* (16) *holds, i.e.* Ω *is a boundary layer.*
(ii) *If* (16) *holds with* $c \ge 1 - q_0$, *then* $\sup_{\xi \in \partial U} W(\xi) \le M_2(1-c)$.

M. Essén gave in [20, Chapter 5] a relationship between boundary layers and minimally thin sets. Namely, [20, Theorem 3] says: *A necessary but not a sufficient condition for* $\Omega = U \setminus E$ *to be a boundary layer is that* E *is a minimally thin set everywhere on* ∂U. At the International Conference in Potential Theory 1994, [21], Essén raised the following question: *"Can we characterize boundary layers in terms of concepts from potential theory?"* Our motivation of this paper is to give an answer to this question. In fact, Theorem E will be generalized and improved in our Theorem V.15.

The paper is organized in the following way: In Section 2 we generalize the notion of boundary layers to general non-tangentially accessible (NTA) domains instead of the unit disk. Since the Martin boundary of an NTA domain is homeomorphic to the Euclidean boundary and every boundary point is minimal ([22]), it is natural to deal with these domains. Section 3 contains the main characterization of boundary layers based on series of reduced functions. We shall use some subtle estimates of the Martin kernels, which can be proved by the boundary Harnack principle. In Section 4 we shall restrict ourselves to smoother domains, namely Liapunov or $C^{1,\alpha}$ domains. For such domains the Martin kernels behave like those for the unit disk. Hence we can give a direct extension of Theorem E. Boundary layers are characterized by Wiener type series based on capacities (analogous series were studied in [5], [20] and [42]). In particular, Theorem V.15 shows that the constant q_0 in Theorem E may be arbitrarily close to 1. Of course, the constant M_2 tends to ∞ as $q_0 \to 1$. We can estimate its growth. In Section 5, we shall discuss a stronger type of boundary layers, which are called good boundary layers. We shall observe that good boundary layers are characterized by the uniform convergence of a certain series involving capacities. In Section 6, we shall discuss a weaker type of boundary layers which turns out to have a precise connection to minimal thinness. See Proposition V.24. In the last section, relationships among various types of boundary layers will be given.

2. Equivalent definitions of boundary layers

In [22] Jerison and Kenig introduced the notion of non-tangentially accessible domains, NTA domains. Hereafter, we let D be a bounded domain in the Euclidean space $\mathbb{R}^{\mathrm{d}}$ with $\mathrm{d} \ge 2$. By $\delta(x)$ we denote the distance $\mathrm{dist}(x, \partial D)$. We say that D is an NTA domain if there exist positive constants M and r_0 such that:

(a) For any $\xi \in \partial D$, $r < r_0$ there exists a point $A_r(\xi) \in D$ such that $M^{-1}r < |A_r(\xi) - \xi| < r$ and $\delta(A_r(\xi)) > M^{-1}r$. (Corkscrew condition.)
(b) The complement of D satisfies the corkscrew condition.
(c) If $\varepsilon > 0$ and x_1 and x_2 belong to to D, $\delta(x_j) > \varepsilon$ and $|x_1 - x_2| < C\varepsilon$, then there exists a Harnack chain from x_1 to x_2 whose length depends on C, but not on ε. (Harnack chain condition.)

In this and the next sections we let D be an NTA domain. As mentioned above, it is known that the Martin boundary of D is homeomorphic to the Euclidean boundary ∂D and every boundary point is minimal ([22]). To be precise, we fix a point $x_0 \in D$. Let $G(x,y)$ be the Green function for D and put $g(x) = G(x, x_0)$. Let $K(x,y) = G(x,y)/g(y)$. Then $K(x,y)$ has a continuous extension to $D \times \overline{D}$. We denote the continuous extension by the same symbol. Sometimes we write K_ξ for $K(\cdot, \xi)$. The kernel K is referred to as the Martin kernel for D. For each $\xi \in \partial D$ the Martin kernel K_ξ is a minimal harmonic function with $K_\xi(x_0) = 1$.

Throughout this paper we let E be a relatively closed subset in D and assume that $\Omega = D \setminus E$ is a domain. We fix $x_0 \in \Omega$. In general, we denote by $\omega(x, I, V)$ the harmonic measure for an open set V of $I \subset \partial V$ evaluated at $x \in V$. For simplicity we let, for $I \subset \partial D$,

$$\omega(x, I) = \omega(x, I, \Omega),$$
$$\widetilde{\omega}(x, I) = \omega(x, I, D).$$

DEFINITION V.2. *Let $c \in (0,1)$. We say that Ω is a c-boundary layer (at x_0) if*

$$\omega(x_0, I) \geq c\widetilde{\omega}(x_0, I) \quad \textit{for every Borel set } I \subset \partial D.$$

We sometimes drop the prefix "c-" if Ω is a c-boundary layer for some $c > 0$.

REMARK V.3. Let D be the unit disk U and $x_0 = 0$. Then $\widetilde{\omega}(0, I) = (2\pi)^{-1}|I|$. Hence our definition generalizes Volberg's boundary layer.

Let $E \subset D$ and let u be a non-negative superharmonic function on D. We put

$$R_u^E(x) = \inf v(x),$$

where the infimum is taken over all non-negative superharmonic functions v such that $v \geq u$ on E. It is known that the lower regularization

$$\widehat{R}_u^E(x) = \liminf_{y \to x} R_u^E(y)$$

is superharmonic in D and $R_u^E = \widehat{R}_u^E$ q.e. on D, i.e. the equality holds outside a polar set. Moreover, $\widehat{R}_u^E = u$ q.e. on E. The function $\widehat{R}_u^E$ is called the (regularized) reduced function of u with respect to E.

PROPOSITION V.4. *The following statements are equivalent:*

(i) *Ω is a c–boundary layer.*
(ii) *$\widehat{R}_{K_\xi}^E(x_0) \leq 1 - c$ for every $\xi \in \partial D$.*
(iii) *$\dfrac{1}{h(x_0)}\widehat{R}_h^E(x_0) \leq 1 - c$ for every positive harmonic function h in D.*

PROOF. For a moment, we fix a Borel set I on the boundary ∂D and write $\omega = \omega(\cdot, I)$ and $\widetilde{\omega} = \widetilde{\omega}(\cdot, I)$. Since

$$\widetilde{\omega} - \omega = \begin{cases} 0 & \text{q.e. on } \partial D, \\ \widetilde{\omega} & \text{q.e. on } E, \end{cases}$$

it follows that

$$\widetilde{\omega} - \omega = \widehat{R}_{\widetilde{\omega}}^E \text{ on } \Omega.$$

Hence Ω is a c-boundary layer if and only if

$$c\widetilde{\omega}(x_0) \leq \widetilde{\omega}(x_0) - \widehat{R}_{\widetilde{\omega}}^E(x_0),$$

or equivalently

$$\widehat{R}_{\widetilde{\omega}}^E(x_0) \leq (1 - c)\widetilde{\omega}(x_0) \quad \text{for every Borel set } I \subset \partial D. \tag{17}$$

In general, a positive harmonic function h is called a kernel function with respect to x_0 at $\xi \in \partial D$ if h vanishes continuously on $\partial D \setminus \{\xi\}$ and $h(x_0) = 1$. It is known that a kernel function at ξ is unique and coincides with K_ξ (cf. [22, Theorem 5.5]). Hence if $r_n \to 0$ and $\widetilde{\omega}_n = \widetilde{\omega}(\cdot, B(\xi, r_n) \cap \partial D)$, then the limit of the ratio $\widetilde{\omega}_n / \widetilde{\omega}_n(x_0)$ exists and is equal to K_ξ. Hence (17) yields

$$\widehat{R}_{K_\xi}^{E}(x_0) \le 1 - c \text{ for every } \xi \in \partial D. \tag{18}$$

Thus (i) $\Longrightarrow$ (ii). The Martin representation theorem (e.g. [18, 1.XII.9]) yields the equivalence (ii) $\Longleftrightarrow$ (iii). Letting $h = \widetilde{\omega}(\cdot, I)$ in (iii), we observe that (17) follows. Thus (iii) $\Longrightarrow$ (i). Proposition V.4 follows. □

3. Series of reduced functions and boundary layers

In this and the next sections we give more concrete characterizations of boundary layers. We shall need many positive constants. So, for simplicity, by the symbol M we denote a positive constant whose value is unimportant and may change from line to line. If necessary, we use $M_1, M_2, \ldots,$ to specify them. We shall say that two positive functions f_1 and f_2 are comparable, written $f_1 \approx f_2$, if and only if there exists a constant $M \ge 1$ such that $M^{-1} f_1 \le f_2 \le M f_1$. The constant M will be called the constant of comparison.

Since our Martin kernel $K(x,y)$ has a reference point x_0, it is necessary to assume that the set E is apart from x_0. In this and the next sections we assume that

$$E \subset D_0 = D \setminus B(x_0, r_1) \text{ with } r_1 > 0. \tag{19}$$

This assumption corresponds to $\frac{1}{2}U \subset \Omega$ in Theorem E. For a boundary point ξ, let us define a Wiener type series of reduced functions.

Definition V.5. *Let $I_j(\xi) = \{x : 2^{-j} \le |x - \xi| < 2^{1-j}\}$ and $E_j(\xi) = E \cap I_j(\xi)$. We define*

$$\Phi(\xi) := \sum_{j=1}^{\infty} \widehat{R}_{K_\xi}^{E_j(\xi)}(x_0).$$

We have the following theorem.

Theorem V.6. *There exists a positive constant M_3 depending only on D, x_0 and r_1 with the following property:*

(i) *If $\sup_{\xi \in \partial D} \Phi(\xi) \le q < 1$, then $\Omega = D \setminus E$ is a $(1-q)$-boundary layer.*
(ii) *If $\Omega = D \setminus E$ is a $(1-q)$-boundary layer, then*

$$\sup_{\xi \in \partial D} \Phi(\xi) \le M_3 \frac{q}{1-q} \log \frac{2}{1-q}.$$

Theorem V.6 (ii) has an immediate corollary.

Corollary V.7. *Let $0 < q_0 < 1$. Then there is a positive constant M_{q_0} depending only on D, r_1 and q_0 such that if Ω is a $(1-q)$-boundary layer with $0 < q \le q_0$, then*

$$\sup_{\xi \in \partial D} \Phi(\xi) \le M_{q_0} q.$$

Moreover, $M_{q_0} \approx (1 - q_0)^{-1} \log[2/(1 - q_0)]$.

PROOF OF THEOREM V.6 (I). We note that the constant M_3 is not involved in this part. This is straightforward from the countable subadditivity of reduced functions. We have

$$\widehat{R}_{K_\xi}^{E}(x_0) \le \sum \widehat{R}_{K_\xi}^{E_j(\xi)}(x_0).$$

Hence by Proposition V.4, we see that if $\sup_{\xi\in\partial D}\Phi(\xi) \le q < 1$, then Ω is a $(1-q)$-boundary layer. $\square$

The second part of Theorem V.6 is not so obvious. We need several lemmas about the estimates of the Martin kernels.

LEMMA V.8. *There are positive constants α and M_4 such that if $\xi \in \partial D$, $x, y \in D_0$ and $2|y-\xi| \le |x-\xi|$, then*

$$\left|\frac{K(x,y)}{K(x,\xi)} - 1\right| \le M_4\left(\frac{|y-\xi|}{|x-\xi|}\right)^{\alpha}.$$

We have in particular,

$$K(x,y) \le \left(1 + M_4\left(\frac{|y-\xi|}{|x-\xi|}\right)^{\alpha}\right)K(x,\xi).$$

PROOF. If $y \in \partial D$, then this is the Hölder continuity of $K(x,y)/K(x,\xi)$ of order α given in [22, Theorem 7.1]. The same proof works, provided $y \in D$ and $2|y-\xi| \le |x-\xi|$. $\square$

LEMMA V.9. *There are positive constants β and M_5 such that if $\xi \in \partial D$, $x, y \in D_0$ and $2|x-\xi| \le |y-\xi|$, then*

$$K(x,y) \le M_5\left(\frac{|x-\xi|}{|y-\xi|}\right)^{\beta}K(x,\xi).$$

PROOF. Let $r = |x-\xi|$ and $R = |y-\xi|$. Since g is a positive harmonic function outside x_0 and vanishes on the boundary, it follows from [22, Lemmas 4.1 and 4.4] that there is $\beta > 0$ such that

$$g \le M(\frac{r}{R})^{\beta} g(A_R(\xi)) \text{ on } B(\xi, r)\cap D.$$

Hence, in particular

$$\frac{g(A_r(\xi))}{g(A_R(\xi))} \le M\left(\frac{r}{R}\right)^{\beta}. \tag{20}$$

Next we show

$$K(y,x) \approx K(y,\xi). \tag{21}$$

Observe that $G(\cdot,y)$ and g are both positive and harmonic on $B(\xi, Mr)\cap D$ and vanish on $B(\xi, Mr)\cap\partial D$. It follows from the boundary Harnack principle [22, Lemma 4.10] that

$$\frac{G(z,y)}{G(A_r(\xi),y)} \approx \frac{g(z)}{g(A_r(\xi))} \text{ for } z \in B(\xi,r)\cap D.$$

This is equivalent to

$$K(y,z) = \frac{G(z,y)}{g(z)} \approx \frac{G(A_r(\xi),y)}{g(A_r(\xi))}.$$

Since the above comparison holds uniformly for $z \in D \cap B(\xi, r)$, we obtain (21) by letting $z \to x$ and $z \to \xi$.

By the maximum principle we have

$$\sup_{D \cap \partial B(\xi,R)} K(\cdot, \xi) \le \sup_{D \cap \partial B(\xi,r)} K(\cdot, \xi).$$

Hence the boundary Harnack principle yields

$$K(A_R(\xi), \xi) \le AK(A_r(\xi), \xi). \tag{22}$$

Once more, we use the boundary Harnack principle to get

$$\frac{K(x,\xi)}{K(A_r(\xi),\xi)} \approx \frac{g(x)}{g(A_r(\xi))}, \quad \frac{K(y,\xi)}{K(A_R(\xi),\xi)} \approx \frac{g(y)}{g(A_R(\xi))},$$

or equivalently,

$$\frac{K(x,\xi)}{g(x)} \approx \frac{K(A_r(\xi),\xi)}{g(A_r(\xi))}, \quad \frac{K(y,\xi)}{g(y)} \approx \frac{K(A_R(\xi),\xi)}{g(A_R(\xi))}. \tag{23}$$

Now (20), (21), (22) and (23) imply

$$\begin{aligned} K(x,y) &= \frac{K(y,x)}{g(y)} g(x) \approx \frac{K(y,\xi)}{g(y)} g(x) \approx \frac{K(A_R(\xi),\xi)}{g(A_R(\xi))} g(x) \\ &\le M \frac{K(A_r(\xi),\xi)}{g(A_r(\xi))} \frac{g(A_r(\xi))}{g(A_R(\xi))} g(x) \le M \frac{K(x,\xi)}{g(x)} (\frac{r}{R})^\beta g(x) \\ &= M \left(\frac{|x-\xi|}{|y-\xi|} \right)^\beta K(x,\xi), \end{aligned}$$

which finishes the proof of the lemma. □

For a positive integer k and $\xi \in \partial D$ we let

$$I_{j,k}(\xi) = \{x \in D : 2^{-j-k} \le |x - \xi| < 2^{k+1-j}\}.$$

LEMMA V.10. *Let α, β, M_4 and M_5 be as in Lemmas* V.8 *and* V.9. *Fore $\varepsilon > 0$ we define*

$$k_0(\varepsilon) = \max \left\{ \frac{1}{\alpha \log 2} \log \frac{M_4}{\varepsilon}, \frac{M_5}{\beta \log 2} \right\}.$$

If k is an integer such that $k \ge k_0(\varepsilon)$, then

$$K(x,y) \le (1+\varepsilon) K(x,\xi) \quad \textit{for } x \in I_j(\xi) \textit{ and } y \in D_0 \setminus I_{j,k}(\xi).$$

PROOF. Let $x \in I_j(\xi)$ and $y \in D \setminus I_{j,k}(\xi)$. Then one of (a) or (b) below holds,

(a) $|y - \xi| < 2^{-j-k}$,
(b) $|y - \xi| \ge 2^{k+1-j}$.

Case (a). Since $|y - \xi|/|x - \xi| < 2^{-k}$, it follows from Lemma V.8 that

$$\begin{aligned} K(x,y) &\le \left(1 + M_4 \left(\frac{|y-\xi|}{|x-\xi|} \right)^\alpha \right) K(x,\xi) \\ &\le \left(1 + M_4 2^{-k\alpha} \right) K(x,\xi) \le (1+\varepsilon) K(x,\xi). \end{aligned}$$

Case (b). Since $|x - \xi|/|y - \xi| < 2^{-k}$, it follows from Lemma V.9 that

$$K(x,y) \le M_5 2^{-k\beta} K(x,\xi) \le K(x,\xi).$$

Thus in both cases we obtain the required inequality. The proof is complete. □

PROOF OF THEOREM V.6 (II). Let $k_0(\varepsilon)$ be as in Lemma V.10. For $\varepsilon = (1-q)/2$ we can choose and fix a positive integer k such that

$$k_0(\varepsilon) \le k \le M \log \frac{2}{1-q}.$$

Take an arbitrary boundary point $\xi \in \partial D$. For simplicity we will use the notation $I_j^*(\xi) = I_{j,k}(\xi)$. Lemma V.10 gives us that

$$K(x,y) \le (1 + \frac{1-q}{2})K(x,\xi) = \frac{3-q}{2}K(x,\xi) \tag{24}$$

for $x \in I_j(\xi)$ and $y \in D_0 \setminus I_j^*(\xi)$. Let us now use the distribution μ defined by

$$\widehat{R}^E_{K_\xi} = K\mu.$$

By (19) μ is concentrated on D_0. Since $K(x_0, y) = 1$ and since Ω is a $(1-q)$–boundary layer, it follows that

$$\|\mu\| = K\mu(x_0) = \widehat{R}^E_{K_\xi}(x_0) \le 1 - (1-q) = q, \tag{25}$$

We have from (24)

$$\int_{D\setminus I_j^*(\xi)} K(x,y)d\mu(y) \le \frac{q(3-q)}{2}K(x,\xi).$$

On the other hand, since $K\mu \ge K_\xi$ q.e. on E, it follows that for q.e. $x \in E_j(\xi)$

$$\int_{I_j^*(\xi)} K(x,y)d\mu(y) \ge \left(1 - \frac{q(3-q)}{2}\right)K(x,\xi) \ge \frac{1-q}{2}K(x,\xi).$$

The last inequality comes simply from the fact that $0 < q < 1$. Hence, by putting $\mu_j = \mu|_{I_j^*(\xi)}$, we obtain

$$K\mu_j \ge \frac{1-q}{2}\widehat{R}^{E_j(\xi)}_{K_\xi} \quad \text{on } D.$$

Evaluating both sides at x_0, we see that

$$\|\mu_j\| = K\mu_j(x_0) \ge \frac{1-q}{2}\widehat{R}^{E_j(\xi)}_{K_\xi}(x_0).$$

The "annuli" $\{I_j^*(\xi)\}$ overlap each $I_j^*(\xi)$ at most $2k+1$ times. By (25)

$$\frac{1-q}{2}\sum \widehat{R}^{E_j(\xi)}_{K_\xi}(x_0) \le \sum \|\mu_j\| \le (2k+1)q.$$

Therefore

$$\Phi(\xi) \le \frac{2q}{1-q}(2k+1) \le M\frac{q}{1-q}\log\frac{2}{1-q}.$$

Theorem V.6 (ii) is proved. □

REMARK V.11. We have actually proved a pointwise estimate: for each fixed $\xi \in \partial D$

$$\widehat{R}^E_{K_\xi}(x_0) \le q < 1 \implies \Phi(\xi) \le M_3\frac{q}{1-q}\log\frac{2}{1-q}.$$

We say that E is minimally thin at $\xi \in \partial D$ if $\widehat{R}^E_{K_\xi}(x) \ne K_\xi(x)$ for some $x \in D$. The minimal thinness can be characterized by $\Phi(\xi)$.

PROPOSITION V.12. *Let $\xi \in \partial D$. Then the following statements are equivalent:*

(i) *E is minimally thin at ξ.*
(ii) $\widehat{R}^E_{K_\xi}(x_0) < 1$.
(iii) $\Phi(\xi) < \infty$.
(iv) $\sum_{j=1}^\infty \widehat{R}^{E_j(\xi)}_{K_\xi}$ *is a Green potential.*

As an immediate corollary to Theorem V.6 and this proposition, we have the following, which is a generalization of part of Theorem 3 (a) in [20].

COROLLARY V.13. *If $\Omega = D \setminus E$ is a boundary layer, then E is minimally thin at every $\xi \in \partial D$.*

PROOF OF PROPOSITION V.12. (i) $\implies$ (ii): We know that $\widehat{R}^E_{K_\xi} = K_\xi$ q.e. on E and hence (i) implies that there is $x_1 \in \Omega = D \setminus E$ such that $\widehat{R}^E_{K_\xi}(x_1) \neq K_\xi(x_1)$. Since Ω is a domain, it follows from the minimum principle that $\widehat{R}^E_{K_\xi}(x_0) < K_\xi(x_0) = 1$.

(ii) $\implies$ (iii): By Remark V.11 we have $\Phi(\xi) < \infty$.

(iii) $\implies$ (iv): It is easy to see that each $\widehat{R}^{E_j(\xi)}_{K_\xi}$ is a Green potential. By assumption the summation is convergent at x_0 and hence $\sum_{j=1}^\infty \widehat{R}^{E_j(\xi)}_{K_\xi}$ is a Green potential.

(iv) $\implies$ (i): Since $\sum_{j=1}^\infty \widehat{R}^{E_j(\xi)}_{K_\xi}$ is a Green potential, which majorizes K_ξ over $\bigcup_{j=1}^\infty E_j(\xi)$, it follows that $\widehat{R}^E_{K_\xi}$ is a Green potential, and in particular $\widehat{R}^E_{K_\xi} \neq K_\xi$. Thus E is minimally thin at ξ. ☐

4. Wiener type criterion for boundary layers

In this section we study boundary layers in Liapunov or $C^{1,\alpha}$ domains instead of NTA domains. In view of Widman [43] we have the following estimates

$$g(x) \approx \delta(x),\ K(x,\xi) \approx g(x)|x-\xi|^{-d} \text{ for } x \in D_0,\ \xi \in \partial D. \tag{26}$$

From these estimates and the quasiadditivity of the Green energy we will obtain a Wiener type criterion for boundary layers in terms of capacity. The following series was introduced in [42] and considered in [20], [3] and [5] also.

DEFINITION V.14. *Let $\{Q_k\}$ be the Whitney decomposition of D. For the cube Q_k, let $q_k = \operatorname{dist}(Q_k, \partial D)$ and $\rho_k(\xi) = \operatorname{dist}(Q_k, \xi)$. By* cap *we denote the logarithmic capacity when $d = 2$, and the Newtonian capacity when $d \geq 3$. We put*

$$W(\xi) = W(\xi, E) = \begin{cases} \displaystyle\sum_k \frac{q_k^2}{\rho_k(\xi)^2}\left(\log\frac{4q_k}{\operatorname{cap}(E\cap Q_k)}\right)^{-1} & \text{if } d = 2, \\ \displaystyle\sum_k \frac{q_k^2}{\rho_k(\xi)^d}\operatorname{cap}(E\cap Q_k) & \text{if } d \geq 3. \end{cases}$$

THEOREM V.15. *There exist positive constants M_6 and M_7 depending only on D, x_0 and r_1 with the following properties:*

(i) *If $\sup_{\xi\in\partial D} W(\xi) \leq M_6 q$, then Ω is a $(1-q)$–boundary layer.*
(ii) *If Ω is a $(1-q)$–boundary layer, then*

$$\sup_{\xi\in\partial D} W(\xi) \leq M_7 \frac{q}{1-q}\log\frac{2}{1-q}.$$

COROLLARY V.16. *Let $0 < q_0 < 1$. Then there is a positive constant M_{q_0} depending only on D, r_1 and q_0 such that if $\Omega = D \setminus E$ is a $(1-q)$–boundary layer with $0 < q \le q_0$, then*

$$\sup_{\xi \in \partial D} W(\xi) \le M_{q_0} q.$$

Moreover, $M_{q_0} \approx (1-q_0)^{-1} \log[2/(1-q_0)]$.

REMARK V.17. In view of Remark V.11, we have pointwise results in Theorem V.15 and Corollary V.16: for each fixed $\xi \in \partial D$.

(i) $W(\xi) \le M_6 q \implies \widehat{R}^E_{K_\xi}(x_0) \le q$.

(ii) $\widehat{R}^E_{K_\xi}(x_0) \le q < 1 \implies W(\xi) \le M_7 \dfrac{q}{1-q} \log \dfrac{2}{1-q}$.

(iii) $\widehat{R}^E_{K_\xi}(x_0) \le q$ with $0 < q \le q_0 < 1 \implies W(\xi) \le M_{q_0} q$.

For the proof of the above theorem we use the quasiadditivity of the Green energy. For a subset E of D we observe that $\widehat{R}^E_g$ is a Green potential, $G(\cdot, \lambda_E)$. The energy

$$\gamma(E) = \iint G(x,y) d\lambda_E(x) d\lambda_E(y)$$

is called the Green energy of E (relative to g). Observe that

$$\gamma(E) = \int \widehat{R}^E_g d\lambda_E = \int g d\lambda_E = G\lambda_E(x_0) = \widehat{R}^E_g(x_0), \tag{27}$$

where the second equality follows from $\widehat{R}^E_g = g$ q.e. on the support of λ_E. In view of (26), the quasiadditivity of the Green energy [5, Corollary 2] reads as follows.

THEOREM F. *Let $E \subset D_0$. Then*

$$\gamma(E) \approx \begin{cases} \displaystyle\sum_k q_k^2 \left(\log \frac{4q_k}{\operatorname{cap}(E \cap Q_k)} \right)^{-1} & \text{if } d = 2, \\ \displaystyle\sum_k q_k^2 \operatorname{cap}(E \cap Q_k) & \text{if } d \ge 3. \end{cases}$$

PROOF OF THEOREM V.15. Let us for a moment consider the case d ≥ 3. We have from (26)

$$K(x,\xi) \approx g(x)|x-\xi|^{-\mathrm{d}} \approx 2^{j\mathrm{d}} g(x) \quad \text{for } x \in I_j(\xi).$$

Hence we have from (27) and Theorem F

$$\widehat{R}^{E_j(\xi)}_{K_\xi}(x_0) \approx 2^{j\mathrm{d}} \widehat{R}^{E_j(\xi)}_g(x_0) = 2^{j\mathrm{d}} \gamma(E_j(\xi)) \approx 2^{j\mathrm{d}} \sum_k q_k^2 \operatorname{cap}(E_j(\xi) \cap Q_k).$$

Since $\rho_k(\xi) \approx 2^{-j}$ for $E_j(\xi) \cap Q_k \ne \emptyset$, it follows that

$$\Phi(\xi) \approx \sum_j 2^{j\mathrm{d}} \sum_k q_k^2 \operatorname{cap}(E_j(\xi) \cap Q_k) \approx \sum_k \frac{q_k^2}{\rho_k(\xi)^{\mathrm{d}}} \operatorname{cap}(E \cap Q_k) = W(\xi).$$

The same type of arguments hold for the case d $= 2$ and we conclude $\Phi(\xi) \approx W(\xi)$. Hence Theorem V.6 readily yields the theorem. $\square$

In view of $\Phi(\xi) \approx W(\xi)$ and Proposition V.12 we have the following well-known result ([3], [5] and [20]).

COROLLARY V.18. *Let $\xi \in \partial D$. E is minimally thin at ξ if and only if $W(\xi) < \infty$.*

5. Good boundary layers

In this section we shall work with Liapunov or $C^{1,\alpha}$ domains again. So far we have considered boundary layers. There is also a *strong* type called good boundary layer defined by Volberg in [42, p.155] for the case when D is the unit disk. The definition has a natural generalization. Let $D_n := \{x \in D : \delta(x) > 1/n\}$ and define Ω_n to be $\Omega \cup D_n$ and E_n to be $E \setminus D_n$. (We note that $\Omega_n = D \setminus E_n$.)

DEFINITION V.19. *Ω is a good boundary layer if Ω_n is a $(1-\varepsilon_n)$-boundary layer with* $\lim \varepsilon_n = 0$.

The following proposition is a straightforward generalization of Theorem 1.4 in [42].

PROPOSITION V.20. *Ω is a good boundary layer if and only if $W(\xi)$ converges uniformly on the boundary ∂D.*

PROOF. For simplicity we prove the theorem only for $\mathrm{d} \geq 3$. The case when $\mathrm{d} = 2$ is similar. Since D is bounded, we may assume that Whitney cubes Q_k are enumerated as $Q_1, Q_2, \ldots$ so that Q_k approaches to the boundary if and only if $k \to \infty$. We will prove the proposition in two steps.

Suppose that Ω is a good boundary layer. Take an arbitrary $\varepsilon > 0$. We find $q = q(\varepsilon) > 0$ so small that

$$M_7 \frac{q}{1-q} \log \frac{2}{1-q} < \varepsilon,$$

where M_7 is the constant in Theorem V.15. Since Ω is a good boundary layer, by choosing n large enough we see that Ω_n is a $(1-q)$-boundary layer. We have from Theorem V.15 (ii)

$$\sup_{\xi \in \partial D} W(\xi, E_n) \leq M_7 \frac{q}{1-q} \log \frac{2}{1-q} < \varepsilon,$$

which means that

$$\sup_{\xi \in \partial D} \sum_{k > k_n} \frac{q_k^2}{\rho_k(\xi)^{\mathrm{d}}} \operatorname{cap}(E \cap Q_k) < \varepsilon,$$

with k_n being the least integer k_n such that $Q_k \subset \{x \in D : \delta(x) \leq 1/n\}$ for $k \geq k_n$. Thus $W(\xi)$ is uniformly convergent.

On the other hand, let us assume that $W(\xi)$ is uniformly convergent. Take an arbitrary $\varepsilon > 0$. Then there is k_0 such that

$$\sup_{\xi \in \partial D} \sum_{k > k_0} \frac{q_k^2}{\rho_k(\xi)^{\mathrm{d}}} \operatorname{cap}(E \cap Q_k) \leq M_6 \varepsilon, \tag{28}$$

where M_6 is the constant in Theorem V.15. We find $n = n(k_0)$ such that

$$\{x \in D : \delta(x) \leq \frac{1}{n}\} \subset \bigcup_{k > k_0} Q_k. \tag{29}$$

Therefore,

$$\sup_{\xi \in \partial D} W(\xi, E_n) < M_6 \varepsilon.$$

Theorem V.15 (i) gives us that $\Omega_n = D \setminus E_n$ is $(1-\varepsilon)$-boundary layer. Thus, by definition, $\Omega = D \setminus E$ is a good boundary layer. $\square$

Let us note that a good boundary layer is always a boundary layer. This property does not seem to follow from the definition directly. For the classical boundary layers this was proved by Essén [20, Theorem 3 (b)]. Our proof heavily depends on Theorem V.15.

THEOREM V.21. *If $\Omega = D \setminus E$ is a good boundary layer, then Ω is a boundary layer.*

PROOF. For simplicity we prove the theorem only for d ≥ 3. The case when d $= 2$ is similar. Let us prove the theorem by contradiction. Let $\Omega = D \setminus E$ be a good boundary layer and suppose it is not a boundary layer. By Proposition V.4 we find $\xi_i \in \partial D$ such that

$$\widehat{R}^E_{K_{\xi_i}}(x_0) \to 1 \quad \text{as } i \to \infty. \tag{30}$$

Taking a subsequence, if necessary, we may assume that ξ_i converges to $\xi_0 \in \partial D$. Since $W(\xi_0) < \infty$, it follows from Corollary V.18 that E is minimally thin at ξ_0, and hence from Proposition V.12 that $\widehat{R}^E_{K_{\xi_0}}(x_0) < 1$. Let

$$\varepsilon = \frac{1 - \widehat{R}^E_{K_{\xi_0}}(x_0)}{2 + \widehat{R}^E_{K_{\xi_0}}(x_0)} > 0. \tag{31}$$

By Proposition V.20 $W(\xi)$ is uniformly convergent and we can find k_0 such that (28) holds. Let $n = n(k_0)$ be such that (29) holds. By Theorem V.15 we have

$$\sup_{\xi \in \partial D} \widehat{R}^{E_n}_{K_\xi}(x_0) < \varepsilon. \tag{32}$$

By the Hölder continuity of the kernel functions [22, Theorem 7.1], we see that

$$K_{\xi_i}/K_{\xi_0} \to 1 \quad \text{uniformly on } F_n = \bigcup_{Q_k \cap \{x \in D : \delta(x) \geq 1/n\} \neq \emptyset} E \cap Q_k.$$

Hence we may assume that $K_{\xi_i} \leq (1+\varepsilon) K_{\xi_0}$ on F_n. This implies

$$\widehat{R}^{F_n}_{K_{\xi_i}} \leq (1+\varepsilon) \widehat{R}^{F_n}_{K_{\xi_0}} \leq (1+\varepsilon) \widehat{R}^E_{K_{\xi_0}} \quad \text{on } D,$$

and in particular

$$\widehat{R}^{F_n}_{K_{\xi_i}}(x_0) \leq (1+\varepsilon) \widehat{R}^E_{K_{\xi_0}}(x_0). \tag{33}$$

Now, (30), (31), (32) and (33) altogether and the subadditivity of reduced functions yield

$$\begin{aligned} 1 = \lim_{i\to\infty} \widehat{R}^E_{K_{\xi_i}}(x_0) &\leq \limsup_{i\to\infty} \widehat{R}^{E_n}_{K_{\xi_i}}(x_0) + \limsup_{i\to\infty} \widehat{R}^{F_n}_{K_{\xi_i}}(x_0) \\ &\leq \varepsilon + (1+\varepsilon) \widehat{R}^E_{K_{\xi_0}}(x_0) = \frac{1 + 2\widehat{R}^E_{K_{\xi_0}}(x_0)}{2 + \widehat{R}^E_{K_{\xi_0}}(x_0)} < 1. \end{aligned}$$

Thus a contradiction arises. The theorem is proved. $\square$

6. Weak boundary layers

In the original definition of boundary layers, we take the harmonic measure in the origin. In Definition V.2 we put x_0 in that position. How important is the choice of reference point? We will in this section investigate that question.

Let D be an arbitrary NTA domain in $\mathbb{R}^d$, as in section 2. In order to simplify the notation, we will introduce an auxiliary function. Let

$$H_\xi(x) := \frac{1}{K_\xi(x)} \widehat{R}^E_{K_\xi}(x).$$

From Proposition V.4 (ii) we see that Ω is a boundary layer at x_0 if and only if $H_\xi(x_0) \le q < 1$ for all $\xi \in \partial D$. (Recall that $K_\xi(x_0) = 1$.)

Let us now choose the "best" reference point for our purpose instead of x_0 to get a slightly weaker assumption on Ω, i.e. let

$$\inf_{x\in\Omega} \sup_{\xi\in\partial D} H_\xi(x) < 1. \tag{34}$$

It turns out that this weakening does not make any essential difference.

Proposition V.22. *Ω is a boundary layer at x_0 if and only if* (34) *holds.*

Proof. It suffices to show the 'if' part. Suppose that (34) holds. Then there exist q, $0 < q < 1$, and $x_1 \in \Omega$ such that $\sup_{\xi\in\partial D} H_\xi(x_1) \le q$. Let $q < q' < 1$. Since both K_ξ and $\widehat{R}^E_{K_\xi}$ are positive and harmonic in Ω, it follows from the Harnack principle that there is $\varepsilon > 0$ such that $\overline{B_\varepsilon} \subset \Omega$ and

$$\sup_{\xi\in\partial D} H_\xi(x) \le q' \quad \text{for } x \in \overline{B_\varepsilon},$$

where $B_\varepsilon = B(x_1, \varepsilon)$. In view of Proposition V.4, we see that Ω is a $(1-q')$-boundary layer at $x_2 \in \overline{B_\varepsilon}$ i.e.

$$\omega(x_2, I) \ge (1-q')\tilde{\omega}(x_2, I) \tag{35}$$

for every Borel subset $I \subset \partial D$. For a moment we fix the Borel set $I \subset \partial D$. By the minimum principle

$$\omega(x, I) \ge \omega(x, \partial B_\varepsilon, \Omega \setminus \overline{B_\varepsilon}) \min_{x_2\in\partial B_\varepsilon} \omega(x_2, I)$$

for $x \in \Omega \setminus \overline{B_\varepsilon}$. Using (35), we evaluate the above inequality at $x = x_0$ to obtain

$$\omega(x_0, I) \ge \omega(x_0, \partial B_\varepsilon, \Omega \setminus \overline{B_\varepsilon})(1-q') \min_{x_2\in\partial B_\varepsilon} \tilde{\omega}(x_2, I).$$

By the Harnack principle again

$$\tilde{\omega}(x_2, I) \approx \tilde{\omega}(x_0, I) \quad \text{for } x_2 \in \partial B_\varepsilon,$$

and hence

$$\omega(x_0, I) \ge M_\varepsilon(1-q')M\tilde{\omega}(x_0, I),$$

where $M_\varepsilon = \omega(x_0, \partial B_\varepsilon, \Omega \setminus \overline{B_\varepsilon}) > 0$. Since I is an arbitrary Borel subset in ∂D, this implies that Ω is a $M_\varepsilon(1-q')M$-boundary layer at x_0. □

The following chain of inequalities encourage us to define another variant of boundary layers.

$$\sup_{\xi\in\partial D} \inf_{x\in\Omega} H_\xi(x) \le \inf_{x\in\Omega} \sup_{\xi\in\partial D} H_\xi(x) \le \sup_{\xi\in\partial D} H_\xi(x_0). \tag{36}$$

DEFINITION V.23. *We say that* Ω *is a weak boundary layer if*

$$\sup_{\xi\in\partial D}\inf_{x\in\Omega} H_\xi(x) < 1.$$

In view of Definition V.5 we introduce

$$\Phi(\xi,x) := \sum_{j=1}^{\infty}\frac{1}{K_\xi(x)}\widehat{R}_{K_\xi}^{E_j(\xi)}(x),$$
$$\Phi_w(\xi) := \inf_{x\in\Omega}\Phi(\xi,x).$$

We have the following proposition (cf. Proposition V.4).

PROPOSITION V.24. *The following statements are equivalent:*

(i) Ω *is a weak boundary layer.*
(ii) $\inf_x H_\xi(x) < 1$ *for every* $\xi\in\partial D$.
(iii) $\inf_x H_\xi(x) = 0$ *for every* $\xi\in\partial D$.
(iv) $\Phi_w(\xi) = 0$ *for every* $\xi\in\partial D$.
(v) E *is minimally thin at every* $\xi\in\partial D$.
(vi) $\inf_x \dfrac{1}{h(x)}\widehat{R}_h^E(x) < 1$ *for every positive harmonic function* h.
(vii) $\inf_x \dfrac{1}{h(x)}\widehat{R}_h^E(x) = 0$ *for every positive harmonic function* h.

This proposition is an easy consequence of the following pointwise result, which can be shown by the well-known minimal fine limit theorem (e.g. [18, 1.XII.18]).

THEOREM G. *Let* $h = K\mu_h$ *be a positive harmonic function on* D *and let* u *be a Green potential. Then, for* μ_h *almost every boundary point* ξ, *there is a set* F_ξ *which is minimally thin at* ξ *such that*

$$\lim_{\substack{x\to\xi\\ x\in D\setminus F_\xi}}\frac{u(x)}{h(x)} = 0.$$

PROPOSITION V.25. *Let* $\xi\in\partial D$. *Then the following statements are equivalent:*

(i) $\inf_x H_\xi(x) < 1$.
(ii) $\inf_x H_\xi(x) = 0$.
(iii) $\Phi_w(\xi) = 0$.
(iv) E *is minimally thin at* ξ

PROOF. By the countable subadditivity of reduced functions and the definition of minimal thinness we readily have (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (iv). Suppose (iv) holds. By Proposition V.12 we see that $\sum_{j=1}^{\infty}\widehat{R}_{K_\xi}^{E_j(\xi)}$ is a Green potential. By Theorem G there is a set F_ξ minimally thin at ξ such that

$$\lim_{\substack{x\to\xi\\ x\in D\setminus F_\xi}}\Phi(\xi,x) = 0.$$

In particular (iii) holds. $\square$

PROOF OF PROPOSITION V.24. The equivalence (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v) readily follows from Proposition V.25. Obviously, (vii) $\Longrightarrow$ (vi). Since K_ξ is a positive harmonic function, it is obvious that (vi) $\Longrightarrow$ (ii). Let us show (v) $\Longrightarrow$ (vii). Suppose E is minimally thin at every $\xi\in\partial D$. Let $h = K\mu_h$ be a positive harmonic function. Since E is minimally thin at every $\xi\in\partial D$, it follows that $\widehat{R}_h^E$ is a Green potential (see e.g. [18, 1.XII.17 Example]). Hence Theorem G says that for μ_h-a.e. $\xi \in \partial D$, and hence at least one $\xi \in \partial D$, there is a set F_ξ minimally thin at ξ such that

$$\lim_{\substack{x\to\xi \\ x\in D\setminus F_\xi}} \frac{1}{h(x)} \widehat{R}_h^E(x) = 0.$$

In particular, (vii) holds. $\square$

7. Relationship among various boundary layers

We conclude with a list of implications between the different types of boundary layers. In this section we let D be a Liapunov or $C^{1,\alpha}$ domain. We have

(i) Ω is a good boundary layer $\Longrightarrow$ Ω is a boundary layer.
(ii) Ω is a boundary layer $\Longrightarrow$ Ω is a weak boundary layer.
(iii) For $\xi_0 \in \partial D$ and $\alpha > 0$ let $\Gamma(\xi_0) = \Gamma_\alpha(\xi_0) = \{x \in D : \delta(x) > \alpha|x - \xi_0|\}$ be a non-tangential cone or "Stoltz cone" with vertex at ξ_0. If $E \subset \Gamma(\xi_0)$, then the three types of boundary layers coincide.

In Theorem V.21 we have observed (i); in view of (36) and Proposition V.22, (ii) is obvious. These implications can not be turned around as seen from examples [42, Ex. 5.1] and [20, Theorem 3 (a)] combined with Proposition V.24. The coincidence (iii) follows immediately from the following proposition.

PROPOSITION V.26. *Let $\xi_0 \in \partial D$ and $\alpha > 0$. Suppose $E \subset \Gamma(\xi_0) = \Gamma_\alpha(\xi_0)$. Then $\Omega = D \setminus E$ is a weak boundary layer if and only if Ω is a good boundary layer.*

PROOF. Let us assume that Ω is a weak boundary layer. Then we have from Proposition V.24 that E is minimally thin at ξ_0, or equivalently $W(\xi_0) < \infty$. For every Whitney cube Q_k intersecting $\Gamma(\xi_0)$ we have $q_k \approx \rho_k(\xi_0)$. Therefore we have that the convergence of $W(\xi_0)$ is equivalent to

$$\sum_k \left(\log \frac{4q_k}{\operatorname{cap}(E_k)}\right)^{-1} < \infty \qquad \text{if d} = 2,$$
$$\sum_k q_k^{2-\mathrm{d}} \operatorname{cap}(E \cap Q_k) < \infty \qquad \text{if d} \geq 3.$$

Since $q_k \leq \rho_k(\xi)$ for every $\xi \in \partial D$, we conclude that $W(\xi)$ is uniformly convergent for $\xi \in \partial D$ in both cases. Hence, due to Proposition V.20, Ω is a good boundary layer. The opposite implication is trivial. $\square$

CHAPTER VI

Boundary layers that are the complements of Kleinian Archipelagoes

We will show that the statement about necessary conditions for boundary layers defined as the complement of the "fattened" orbit of a Fuchsian group in [31] can be sharpened and generalized. We will also see that the statements are meaningful. In other words we will show that there is a Kleinian group Γ such that the complement of the fattened orbit of Γ is a boundary layer.

Furthermore, we will give an exact description of those Kleinian groups which have the property that the complement of their Kleinian Archipelago is a boundary layer, see Proposition VI.7.

We will also show that a complement of a Kleinian Archipelago is a good boundary layer only in the trivial case.

1. Two necessary conditions

Let us first recall some definitions from Section 1 on page 21. Let Γ be a Kleinian group. By the fact that Γ is discontinuous it is possible to find an $r_\Gamma > 0$ such that the balls B_j do not intersect each other, where $B_j := \{z \in D : d(z, \gamma_i 0) < r_\Gamma, \gamma_i \in \Gamma \setminus \{I\}\}$. Let $E := \bigcup_j B_j$ and $\Omega := D \setminus E$. That is, E is the fattened orbit of Γ.

First we show a tuned up version of Proposition 5.7 in [31].

Proposition VI.1. *Let Ω be the complement of a Fuchsian Archipelago. If Ω is a boundary layer then Γ is of convergence type.*

Proof. Let us use the following notation.

$$\Omega_k := U \setminus \cup_{j \neq k} B_j = \Omega \cup B_k.$$

Since Ω is a boundary layer we trivially also have that Ω_0 is a boundary layer, i.e. there is a $c' > 0$ such that

$$\omega(\Omega_0, I, 0) \geq c'|I|, \text{ for all arcs } I \in \mathbb{T}.$$

Let us then choose I to be the whole boundary $\mathbb{T}$. We then get that

$$\omega(\Omega_0, \mathbb{T}, 0) \geq c, \quad \text{where } c = 2\pi c'. \tag{37}$$

Since $\gamma_k(\Omega) = \Omega$ and

$$\gamma_k(B_0) = \{\gamma_k z : d(z, 0) < r_\Gamma\} = \{z : d(\gamma_k^{-1} z, 0) < r_\Gamma\} =$$

$$= \{z : d(z, \gamma_k 0) < r_\Gamma\} = B_k,$$

we note that
$$\gamma_k(\Omega_0) = \gamma_k(\Omega) \cup \gamma_k(B_0) = \Omega \cup B_k = \Omega_k,$$
we also recall the basic fact that the harmonic measure is invariant under Möbius transforms, which, together with the invariance $\gamma_k(\mathbb{T}) = \mathbb{T}$, implies that
$$\omega(\Omega_0, \mathbb{T}, 0) = \omega(\gamma_k(\Omega_0), \gamma_k(\mathbb{T}), \gamma_k 0) = \omega(\Omega_k, \mathbb{T}, \gamma_k 0).$$

That together with (37) implies that for every k
$$\omega(\Omega_k, \mathbb{T}, \gamma_k 0) \geq c > 0.$$
We are then in the position where we can use the result [24, Theorem 2] to reach the conclusion that $\sum_k t_k$ converges. That is, Γ is of convergence type, see for example page 90. $\square$

Let us use a result in the previous chapter to generalize the situation above (to higher dimensions and weaker assumptions) and give this stronger statement.

PROPOSITION VI.2. *Let Ω be the complement of a Kleinian Archipelagoas above.*

If Ω is a weak boundary layer then the set of non-tangential limit points, Λ_c, is empty.

PROOF. We know from Proposition V.24 that Ω is a weak boundary layer if and only if E minimally thin everywhere. Hence by Proposition III.12 we see that $\Lambda_c = \emptyset$. $\square$

REMARK VI.3. Proposition VI.2 implies in fact Proposition VI.1, see page 66. A natural question to ask is: Is there really an example of a Γ such that the generated Ω is a boundary layer? We will in the next section study that question closer.

2. An orbit projection

For simplicity, and since the second statement in Proposition VI.2 above was a planar statement, we will consider the Fuchsian case first. We will in this section show that if Γ is just generated by one element, which is parabolic, then Ω, defined as above[1], is a boundary layer.

Let us first for simplicity and without loss of generality transform the situation to the upper half-plane and let the orbit of Γ be situated on the boundary of a horocycle with center at $\frac{1}{2}I$ and radius $\frac{1}{2}$ (in the Euclidean sense) as depicted in Figure VI.9.

From the picture it is easy to see that E is minimally thin everywhere, since the only point in question is the origin but since $0 \notin \mathcal{L}_s(\frac{1}{2})$ we can use Theorem IV.12 and obtain that E is minimally thin at the origin. Hence we have from Proposition V.24 that Ω is a weak boundary layer.

However, as we will later see, Ω is not a good boundary layer.

That leaves us out in the shadow-land, where we have to be more precise to analyze the question if Ω is a boundary layer. (We can compare this situation with Vasyunin's example on page 160 in [42] and the example on page 100 in [20].) The plan is as follows. First we will show that W_0 (which was defined on page 23) is not only finite but uniformly bounded. Then, by adjusting the hyperbolic radius in the definition of the covering balls B_j, we can get a suitable

[1]If we allow ourselves to choose the hyperbolic radius r_Γ for the balls B_j small enough.

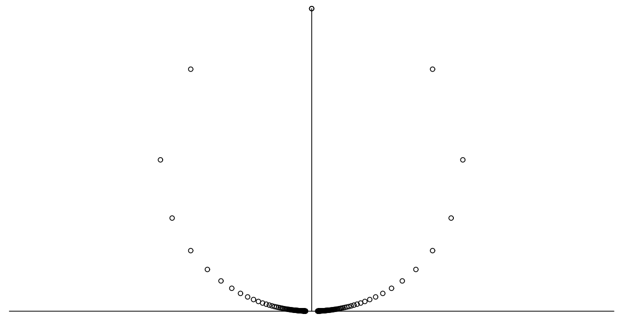

FIGURE VI.9. The upper half-plane with the orbit of Γ with its parabolic fixed point at the origin. Note that we have, for plotting reasons, put Euclidean disks on the orbit points instead of hyperbolic ones.

bound of the series W (see Definition B.2). By suitable we mean that we can now use (i) in Theorem V.15 to draw the conclusion that Ω is a boundary layer.

In Theorem B.2 on page 94 we learn that E is minimally thin at ξ if and only if $W(\xi) < \infty$. From Lemma III.5 we conclude that the same condition holds for W_0, i.e. E is minimally thin at ξ if and only if $W_0(\xi) < \infty$. Hence $W_0(\xi) < \infty$ for all ξ.

3. The essentially unique boundary layer

Let us concentrate on the right side of the orbit. We will use a geometric transform to study the fraction $\dfrac{q_k^2}{\rho_k(\xi)^2}$. Let us from the origin project orbit points $\gamma_k(i)$ onto the line $\{z : \mathrm{Im}(z) = 1\}$, see Figure VI.10. Note that the intersections of the projection lines with the horizontal line are at constant distance, c, from each other. We will get the same situation by transforming the upper half-plane by a Möbius transformation that takes the origin to infinity. Recall that every parabolic element is conjugated to a mapping of the form $z \to z + c$. (In our example $c = \frac{1}{2}$.)

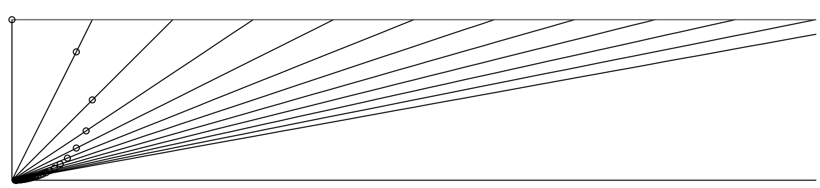

FIGURE VI.10. The normalization transform—or transforming the parabolic fixed point to infinity.

Let us define

$$\mathcal{S} := \sum_{k=0}^{\infty} \frac{1}{1 + (kc)^2}.$$

We see that

$$2\mathcal{S} - 1 = W_0(0). \tag{38}$$

It is also possible to give, with the help of Maple, an exact expression of $\mathcal{S}$ in terms of the *digamma function*, $\Psi(x) := \frac{d}{dx}\log(\Gamma(x))$, as follows.

$$\mathcal{S} = \frac{i}{2c}(\Psi(-\frac{i}{c}) - \Psi(\frac{i}{c})).$$

REMARK VI.4. In our concrete example, we can then give an approximate value of $\mathcal{S}$.

$$\mathcal{S} = i(\Psi(-2i) + \Psi(2i)) \approx 3.64.$$

To show that W_0 is uniformly bounded, we have to consider $\limsup_{x\to 0} W_0(x)$. We will do that by letting $x > 0$ be sufficiently small but fixed and study the series $W_0(x)$ in the same fashion as we did for $W_0(0)$ above, we will in fact repeatedly use the comparison with $\mathcal{S}$. The summation $W_0(x)$ will be chopped up into four different pieces and analyzed in a basic geometric way.

The dividing comes naturally into play if we are repeating the projective construction above. We will now, for $x > 0$, not get a constant distance between the intersection of the projection lines and $l = \{z : \mathrm{Im}(z) = 1\}$ but something in the following manner. Let us suppose we are plotting the intersection points, one each second, we will then watch the points march on the line l.

First the intersection points will go to the right in a rather even speed — like the points in the $W_0(0)$ situation — but the speed will eventually decrease and even change direction. That sequence is studied in Part I below.

After the points have very slowly turned around they will increase their velocity and at some time pass above the base point x. This is the Part II sequence.

Part III is the situation where the points goes steady to the left until they pass above the "half way stop" $\frac{x}{2}$.

Finally, the points goes out to the horizon to $-\infty$ in Part IV.

In order to give a more concrete picture, we have made a series of Maple plottings of an example where Γ is as above and $x = 0.19$.

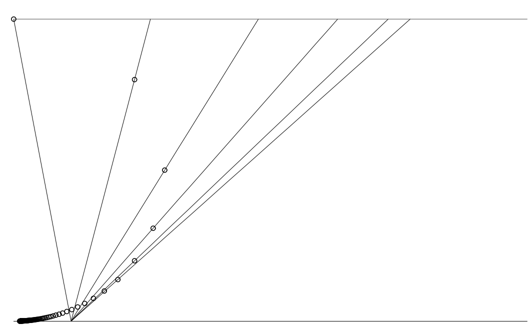

FIGURE VI.11. Part I for our example where $x = 0.19$.

3.1. Part I. The "march of the points" are changing direction when the projecting lines are approximately the tangent of the horocycle that passes through the base point x. Since we have that x is small we can approximate the horocycle with the graph of the squared function. That is, let us suppose that the tangent point is above x_0. We can then of course easily compute $x_0 = 2x$ from the tangent relation $x_0^2 - 2x_0(x_0 - x) = 0$. See Figure VI.12.

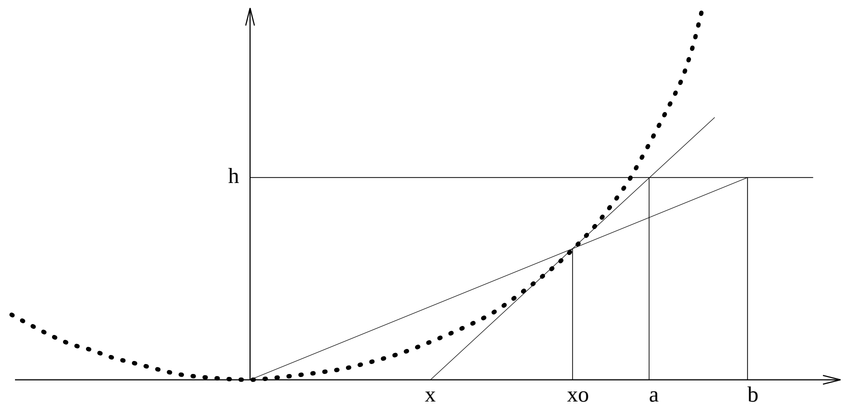

FIGURE VI.12. x_0 is the tangent point.

Let us for the sake of notation use $a' = a - x$. We then get from Figure VI.12 that

$$a' = \frac{1}{2x_0} = \frac{1}{4x}$$

and that

$$b = \frac{x_0}{x_0^2} = \frac{1}{2x}.$$

Note that we, unlike the $W_0(0)$ situation where the first intersecting point is right above the origin, get at least one point to the left of the base x; but since x is small we can without loss of generality assume that there is only one point in the beginning that lies above to the left of x.

Furthermore, due to the fact that the point "slows down" before the turning point, we have that the points are gathered more to the right than equally spread out. In other words, we can get an upper estimate of this first part of the series by assuming that the intersection points are equally spread out. To see what the step length, c_I, would be for this equally spread out situation we just note that the number of intersection points between the origin and b for the original $W_0(0)$ situation is approximately $\frac{b}{c}$. That give us that the step length c_I is $\frac{a'c}{b}$. Then we can do the following rough estimate for the even distribution.

$$\sum_I \frac{1}{1+k^2c_I^2} \le \sum_{k=0}^{\infty} \frac{1}{1+k^2c_I^2} \le 1 + \frac{1}{c_I}\int_0^\infty \frac{1}{1+t^2}\,dt =$$
$$= 1 + \frac{c}{c_I}\frac{1}{c}\int_0^\infty \frac{1}{1+t^2}\,dt \le \frac{c}{c_I}\mathcal{S} = 1 + \frac{b}{a'}\mathcal{S}.$$

That, together with the above considerations give us the following estimate of the first part of the series $W_0(x)$.

$$\mathcal{S}_I := \sum_I \frac{q_k^2}{\rho_k(x)^2} \le 1 + 1 + \frac{b}{a-x}\mathcal{S} = 2 + 2\mathcal{S},$$

where a and b are defined in Figure VI.12.

3.2. Part II. In this part we consider the points in the orbit from the tangent point, x_0, to the point roughly above x. Let b_2 be the first coordinate of the intersecting point of the line l and the line from origin through (x, x^2). That is, $b_2 = \frac{1}{x}$. We can then do as in Part I above to get an estimate by first noting that the number of intersections in Part II is approximately $\frac{b_2-b}{c}$ and then, due to the fact that the intersection points are gathered to the right, get an upper estimate

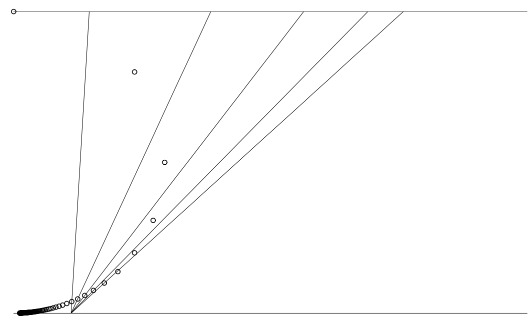

FIGURE VI.13. In part II, the intersection points with l are gathered to the right similar to the picture in part I.

by computing an estimate for the *even* point distribution. The even step length will be $c_{II} := \frac{a'c}{b_2 - b}$ and as in Part I, we can estimate in the following way.

$$\sum_{II} \frac{1}{1+k^2 c_{II}^2} \leq \sum_{k=0}^{\infty} \frac{1}{1+k^2 c_{II}^2} \leq 1 + \frac{c}{c_{II}} \mathcal{S} = 1 + \frac{b_2 - b}{a'} \mathcal{S}.$$

We conclude that

$$\mathcal{S}_{II} \leq 1 + \frac{b_2 - b}{a'} \mathcal{S} = 1 + 2\mathcal{S}.$$

3.3. Part III. Let us now turn to the S_{III} part. Let $t \in [\frac{x}{2}, x]$ be the first coordinate of a point of the parabolic orbit for part III. Furthermore, let $v(t)$ be the first coordinate of the neighbor point to the left of the point above t.

Since we know that the projected sequence seen from the origin onto the line $l = \{z : Im(z) = 1\}$ is even distributed with distance c we can give an explicit formula for $v(t)$. Since

$$c = \frac{v}{v^2} - \frac{t}{t^2}$$

we have immediately that

$$v(t) = \frac{t}{ct+1}.$$

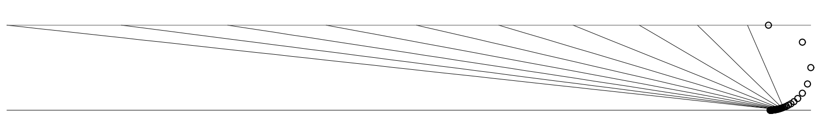

FIGURE VI.14. The projected step size is is increasing as one goes to the left.

Let us denote by $s(t)$ the step-size of the projection from x to l of the points above t and $v(t)$. (See Figure VI.14.) We have that

$$s(t) = \frac{x-v}{v^2} - \frac{x-t}{t^2} = x\left(\frac{(ct+1)^2}{t^2} - \frac{1}{t^2}\right) - c = c^2x + \frac{2cx}{t} - c.$$

Since $\frac{\partial s(t)}{\partial t} < 0$, the smallest step is the most right one which is $s(x) = c^2x + c \geq c$. (Which is indeed the case in our example, see Figure VI.14.)

We can therefore do the following estimate

$$\mathcal{S}_{III} \leq \mathcal{S}.$$

3.4. Part IV — the tail. Let us now consider the points that starts above $\frac{x}{2}$ and goes off toward the origin, i.e. the tail. The tail is very easy to take care of. We just note that $\rho_k(x) \geq \rho_k(0)$ for every k in the tail. We end up with the estimate $\mathcal{S}_{IV} \leq \mathcal{S}$.

3.5. Adding up. Let us now put all this together. Since the contribution of the terms from the points on the left side of the imaginary axis in the $W_0(x)$ series is less than $\mathcal{S}$, we have finally that

$$\limsup_{x\to 0} W_0(x) \leq (2+2\mathcal{S}) + (1+2\mathcal{S}) + \mathcal{S} + \mathcal{S} + \mathcal{S} = 7\mathcal{S} + 3.$$

Note that from equation (38) we know that $W_0(0) = 2\mathcal{S} - 1$ giving us that

$$\sup_x W_0(x) \leq 7\mathcal{S} + 3,$$

where the supremum is taken over the real line. The same estimate will hold if we transform the upper half-plane back to the unit disk U.

3.6. An analogue for the higher dimensions. We will see that a similar reasoning, as above, holds also for the general, higher dimensional case. Let the dimension be d and consider a point x close to the origin but lying on the hyper-surface of dimension $\mathrm{d}-1$ which is the boundary of the upper-half-space model of the d dimensional hyperbolic space. Let us also consider a horoball at 0 whose surface includes the orbit point of the single parabolic group. Note that the group can be of different ranks, i.e. 1 to $\mathrm{d}-1$. If we can get a uniform estimate as above for a rank $\mathrm{d}-1$ situation, the other cases will automatically follow. Thus, let us assume that Γ is generated by a single parabolic element of rank $\mathrm{d}-1$.

Let us now consider the unique hyper-surface (of dimension $\mathrm{d}-1$) whose intersection with the boundary $\partial\mathbb{H}$ is perpendicular to the vector x and is tangent to the horoball. Furthermore, let x_0 be the projected point on $\partial\mathbb{H}$ of the above tangent point. We will also need the point that is halfway to the origin from x, i.e. $\frac{|x|}{2}x$. By using the above points x, x_0 and $\frac{|x|}{2}x$ and the hyper-surface through the origin and perpendicular to x, we can, analogously to the above, split the series $W_0(x)$ into five parts: the part connected to the "opposite-half-horoball" and the four parts as in the subsections above I, II, III and IV.

We can think of the line l as the hyper-surface l_{d} parallel to $\partial\mathbb{H}$ with distance 1 to $\partial\mathbb{H}$. Furthermore, we can think of the constant step-size c (introduced on page 59) as the reciprocal point density of the projection of the orbit points on the hyper-surface l_{d} seen from the origin. The step-size $s(t)$ at page 63, is now to be thought of as the (non-constant) reciprocal point density of the projection on l_{d} seen from x.

Using the above analogue we see that we can get similar estimates for this general case. That is

$$\sup_x W_0(x) \le C_1(\mathrm{d})W_0(0) + C_2(\mathrm{d}).$$

3.7. Conclusion. We can for every $m > 0$ choose an r_Γ such that for $\mathrm{d} = 2$

$$\left(\log \frac{4t_k}{\operatorname{cap}(E \cap Q_k)}\right)^{-1} \le m \text{ for every } k,$$

and for $\mathrm{d} \ge 3$,

$$\frac{\operatorname{cap}(E \cap Q_k)}{t_k^{\mathrm{d}-2}} \le m \text{ for every } k.$$

Recall that r_Γ is the hyperbolic radius of the disks or spheres whose union is the set E.

Let us now choose m to be $\frac{M_6 q}{7\mathcal{S}+3}$ in the case $\mathrm{d} = 2$ and for $\mathrm{d} \ge 3$, let $m = \frac{M_6 q}{C_1(\mathrm{d})W_0(0)+C_2(\mathrm{d})}$, where M_6 is the constant in Theorem V.15 on page 50 and $q \in (0,1)$. Then we have

$$\sup_{\xi \in \partial B} W(\xi) \le \sup_{\xi \in \partial B} mW_0(\xi) \le M_6 q.$$

Now (i) in Theorem V.15 gives us that Ω is a $(1-q)$-boundary layer.

Thus we have the following result.

Lemma VI.5. *Let Γ be a Kleinian group generated by a single parabolic element. For every c in $(0,1)$ there is an $r_\Gamma > 0$ such that for $E = \cup B_i(r_\Gamma)$ we have that $\Omega(r_\Gamma)$=$U \setminus E$ is a c-boundary layer.*

4. Ω is not a good boundary layer

Let us start with the planar case and let Ω be the complement of the Fuchsian Archipelago of a group generated by a single parabolic element. We will show that Ω is never a good boundary layer. As above, let us consider the upper-half-plane $\mathbb{H}$.

Let $\Omega_n = \Omega \cup H_n$ where $H_n = \{z \in \mathbb{H} : \operatorname{Im} z > \frac{1}{n}\}$. We are going to show that there exists a sequence $\{x_n\}$ of points on the real line such that $\lim_{n\to\infty} W(x_n) >$ Const. > 0. If we can find such a sequence we can conclude that Ω_n is not a $(1-q)$-boundary layer such that $q \to 0$ as $n \to \infty$ (see Corollary V.16). Hence Ω_n would not be a good boundary layer.

Let us for a moment fix n and choose $x_n = x = \frac{1}{2\sqrt{n}}$. Asymptotically, the situation will be as in Figure VI.12 with $x_0 = \frac{1}{\sqrt{n}}$ and every island in the Fuchsian Archipelago removed if their distance to the boundary is greater than $\frac{1}{n}$.

We aim at a lower estimate of $\mathcal{S}_{III}$. How many terms are there in $\mathcal{S}_{III}$? Let us use the fact that we have an even distribution of intersecting points projected from the origin, see Figure VI.10. We let α and β be the first coordinates of the intersecting points on the line l of the projection of (x, x^2) and $(\frac{x}{2}, \frac{x^2}{2^2})$ seen from the origin. Then the number of terms in $\mathcal{S}_{III}$ will approximately be $\frac{\beta-\alpha}{c}$, where c is the step length of the parabolic map after the transformation according to Figure VI.10. Let us estimate this number by the following easy geometrical observation.

$$\beta - \alpha = \frac{x/2}{(x/2)^2} - \frac{x}{x^2} = \frac{1}{x}.$$

Let us by N denote the integer part of $\frac{1}{xc}$.

Let us now project onto l from x instead of the origin and let l_{III} be the smallest connected sub-interval of l such that the projected points of $\mathcal{S}_{III}$ are in l_{III} . We obtain that the length of the interval l_{III} is β, due to a similar geometric construction as above. We know from Part III above that the smallest step, when we project from x, is the rightmost on the interval l_{III}. Hence, if we rearrange the intersecting points so that we have an even step-size, we will get a smaller sum than $\mathcal{S}_{III}$.

Let us denote the even step length by $c_e = \frac{\beta}{N}$. We have immediately that

$$c_e \leq \frac{\beta}{\frac{1}{xc} - 1} = \frac{\beta xc}{1 - xc} = \frac{2c}{1 - xc} \to 2c \text{ as } x \to 0.$$

Let us therefore suppose that x is small enough such that $c_e \leq 3c$ If we put this together we obtain the following.

$$\mathcal{S}_{III} = \sum_{III} \left(\frac{t_k}{\rho_k(x)} \right)^2 > \sum_{k=1}^{N} \frac{1}{1 + k^2 c_e^2} \geq \sum_{k=1}^{N} \frac{1}{1 + 9k^2 c^2}.$$

Let us now free n. As $n \to \infty$ $x = x_n \to 0$ and $N \to \infty$. Hence

$$\liminf_{n\to\infty} \mathcal{S}_{III} > \sum_{k=1}^{\infty} \frac{1}{1 + 9k^2 c^2} = M_{III} > 0. \tag{39}$$

We can give an exact expression for the constant M_{III} by the use of the digamma function again.

$$M_{III} = \frac{i}{6c} \left(\psi(\frac{-i}{3c}) - \psi(\frac{i}{3c}) \right) - 1.$$

For the W-series, we have, thanks to Lemma III.5, $W_0(\xi) \leq c_W W(\xi)$ and thus,

$$\sup_{\xi \in \partial\mathbb{H}} W(\xi) \geq W(\frac{1}{2\sqrt{n}}) \geq \frac{1}{c_W} W_0(\frac{1}{2\sqrt{n}}) > \frac{1}{c_W} \mathcal{S}_{III}.$$

Due to Equation (39) above, we have that $\mathcal{S}_{III}$ tends to something greater than the strictly positive constant M_{III} as n tends to ∞. That gives us now

$$\sup_{\xi \in \partial\mathbb{H}} W(\xi) \not\to 0 \text{ as } n \to \infty.$$

We conclude that Ω is not a good boundary layer. This holds for every strictly positive choice of r_Γ.

Now, since we studied the most promising case of Ω to be a good boundary layer (i.e. the simple parabolic case), we have the result in Lemma VI.6 below for Fuchsian groups.

For the higher dimensional case, we argue as in the proof of Lemma VI.5 and obtain that part III will always contribute with a strictly positive amount, although we do not explicitly compute an estimate as in the Fuchsian case. Hence we see that Ω can not be a boundary layer and we have proved the following lemma.

LEMMA VI.6. *The complement of a Kleinian Archipelago is a good boundary layer if and only if the Kleinian group is trivial, i.e.* $\Gamma = \{I\}$.

5. The main result

Thanks to the first result in Proposition VI.2 we have that for Ω in general, it is necessary that Γ is such that Λ_c is empty. This is a rather strong condition as we shall see now. Let us suppose that Γ is generated by two or more parabolic elements with different fixed points, and let us again go to the upper half-space model. We can without loss of generality assume that $z \rightarrowtail z + c$ is one of the generators. We see that it will have its fixed point at ∞. Let $\tau \in \partial\mathbb{H}$ be the fixed point for another parabolic generator, then $\tau + c$ is in Λ and $\tau + 2c$ etc. We will have the limit set Λ to be of infinite cardinality, i.e. Γ is non-elementary. Then we can use Theorem 1.1 in [13], which says "if Γ is non-elementary then $\delta(\Gamma) = \dim(\Lambda_c)$". We also know from page 28 that $\delta(\Gamma) \geq \frac{1}{2}$. Hence, if Γ is generated by more than one parabolic element, the non-tangential limit set Λ_c is not only non-empty, but has Hausdorff dimension greater than $\frac{1}{2}$. That implies that Γ must be of the form studied above, i.e. generated by a single parabolic element. Thus, by the Lemmas VI.5 and VI.6, we have a complete description of the situation when a Kleinian group generates a boundary layer.

THEOREM VI.7. *Let $\Gamma \neq \{I\}$ be a Kleinian group and let Ω be the complement of the Archipelago of Γ. Then Ω is a boundary layer if and only if Γ is generated by a single parabolic element and r_Γ is chosen small enough. Furthermore, Ω is never a good boundary layer.*

CHAPTER VII

On Discrete Potential Theory and Discrete Groups

We will investigate some relations between the theory of Kleinian groups and potential theory in the language of discrete potential theory. To be able to do that we need to define some concepts such as *minimal thinness* and *boundary layers* from the continuous potential theory for a discrete setting.

1. Introduction

Discrete groups are, by the very construction, and as their name indicates, discrete. If we, on the other hand, study the classical potential theory, we are dealing with the continuous world.

We tried in Chapter III to compare thin sets from the continuous potential theory with the theory of Kleinian groups. We did that by forcing the discrete group to the continuous playground by putting a hyperbolic disk around each point in an orbit of the group to derive a set with strictly positive capacity.

In this present chapter we will try to stay in the "discrete park" as far as it is possible, to let the concepts meet there. The theory of discrete potential theory with its natural connections to Markov chains and random walk is not only a nice variant of the classical continuous potential theory, but also interesting for the applications (e.g. electrical nets, c.f. [46]) and its adaptedness to concrete numerical calculations. It should, for historical reasons, also be mentioned that the very start of the potential theory was rather discrete when Isaac Newton studied point particles.

This chapter is organized as follows. In Section 2 we recall the notation and definition of some basic concepts in the theory of Markov chains, the transition operator matrix, the Green kernel, the Martin kernel and the Martin boundary. We also discuss absorption and give a definition of minimal thinness in a discrete meaning, see Definition VII.6.

Section 3 is devoted to results about minimal thinness. We will show that minimal thinness is a local property (Proposition VII.7). We will also get some equivalent formulations of minimal thinness using the notion of conditioned random processes in Theorem VII.11 and Proposition VII.14. As a spin-off effect of Theorem VII.11, we will give a result in Corollary VII.12 about ordinary continuous minimal thinness.

By choosing a special capacity, and using a result in [11] together with Proposition VII.14, we get an equivalent formulation for minimal thinness in terms of capacity in Corollary VII.17.

The last part of Section 3 deals with a discrete semi-analogue of a result of A. Beurling, Theorem I, which already in its original formulation bears a strong flavor of discreteness as it contains a separated point sequence. We give in fact two results in the spirit of the first half of Theorem I; Proposition VII.19, that stresses the connection to minimal thinness, and Corollary VII.21, where the corresponding condition (57) in Theorem I is given a probabilistic interpretation.

Minimal thinness is related to the concept *boundary layer* introduced by A. Volberg in [42] and studied in Chapter V above. In section 4 we give a discrete definition and some results giving connections with the absorption matrix, B^E, which was used in the definition of minimal thinness. We will also give a discrete definition of a variant of boundary layers, which is even closer connected to minimal thinness, called *weak boundary layer* and was introduced on page 55. Proposition VII.27 gives the strong complementary connection between weak boundary layers and minimally thin sets.

In Section 5 finally, we turn to the theory of discrete or Kleinian groups. We start by stating and proving a result, Theorem VII.31, which is a discrete analogue to Theorem III.23 above.

At last, we get very concrete by studying a special case of Fuchsian groups, the Schottky groups. For simplicity, we will then also let the transition operator be of nearest neighbor type and uniformly distributed. We will then see, among other things, that the complement of a horocycle is minimally thin at the "tangent point", see Proposition VII.34.

2. The discrete setting

Our universe in this and the following sections will be the countable set X.

2.1. Basic definitions. Let us begin by recalling some definitions and notions. The stochastic transition operator matrix will be denoted by

$$P = \Big(p(x,y)\Big)_{x,y\in X},$$

with non-negative elements and row sums equal to one. We will assume that P is *irreducible*, i.e.

$$\forall x,y \in X \quad \text{there is an } n \text{ such that } p^{(n)}(x,y) > 0,$$

where $p^{(n)}(x,y)$ is the x,y-element in the matrix which is the product of n copies of the matrix P and can be interpreted as the probability that after n steps in a P-random walk, we have gone from the Markov state x to the state y, i.e. $\Pr_x[X_n = y]$.

Furthermore, we will assume that P is such that the Markov chain $\{X_n\}$ is *transient*. That is, there is an $x \in X$ such that

$$\Pr_x[\exists n : X_n = x] < 1.$$

For equivalent formulations, see [44, Theorem 2.1].

Definition VII.1. *The discrete Laplacian is $P - I$, i.e. h is P-**harmonic**, or simply **harmonic**, if $Ph(x) = h(x)$, where $Ph(x)$ is defined as*

$$Ph(x) = \sum_{y\in X} p(x,y)h(y).$$

(The harmonic functions are sometimes called **regular**, *see for example* [28].*)*

DEFINITION VII.2. *We define the* ***Green kernel*** *by*

$$G(x,y)=\sum_{n=0}^{\infty} p^{(n)}(x,y).$$

The Green kernel can be probabilistically thought of as the expected number of visits in y for a Markov process, X_n, following the law of P, i.e. $p(z,y)=\Pr[X_{n+1}=y|X_n=z]$, and starting at x.

DEFINITION VII.3. *We make the following definition*

$$F(x,y)=Pr_x[\exists n\geq 0 \textit{ such that } X_n=y].$$

That is, $F(x,y)$ is the probability that a process started at state x reaches state y.

Let x_0 be a fixed reference point in X.

DEFINITION VII.4. *The* ***Martin kernel***

$$K(x,y)=\frac{G(x,y)}{G(x_0,y)} \quad \textit{for } x,y \textit{ and } x_0\in X,$$

Once we have defined the Martin kernel, we can define the so called *Martin compactification* X^* to be the minimal completion of X such that, for every $x\in X$, $K(x,\cdot)$ extends continuously. See for example [17] or [28, Chapter 10] how this is done.

The resulting limit set, $\partial X=X^*\setminus X$, will be called the **Martin boundary** of X, or **Martin exit boundary** as in [17] and [28]. Finally, we extend the Martin kernel to that boundary by

$$K(x,\xi)=\lim_{y\to\xi}\frac{G(x,y)}{G(x_0,y)} \quad \text{for } \xi\in\partial X.$$

(The limit exists, due to the definition of the boundary.)

REMARK VII.5. Note that $G(x,y)=F(x,y)G(y,y)$, and thus

$$K(x,y)=\frac{F(x,y)}{F(x_0,y)},$$

i.e. the Martin kernel $K(x,y)$ is the fraction of the probability to reach state y from x over the probability to reach y from x_0. We will later use the notation $K_\xi(\cdot)$ for $K(\cdot,\xi)$.

2.2. Absorption and minimal thinness. To be able to accomplish a discrete analysis we will have to convert some definitions and results from the continuous settings concerning small sets at the boundary (such as minimally thin sets) to the discrete setting.

B^E is called the absorption matrix for E (see [28, p. 109]) and is defined as

$$(B^E)_{xy}=\Pr_x[\,\exists\, n\geq 0: X_n=y\in E, \text{ the first entry in } E].$$

In the finite case (i.e. when X is finite) we can view the operator B^E as a square matrix with X as the index set. If we renumber the states in X such that we "start" with the states in E we can divide the matrix into four parts as indicated below,

$$B^E = \begin{pmatrix} I & 0 \\ B & 0 \end{pmatrix}$$

where I is the identity matrix, indicating that if we start with a state in E we will for sure be absorbed, and B the matrix with elements

$$B_{x,y} = \Pr_x[X_n \text{ is absorbed in } y] \quad \text{for } x \in X \setminus E \text{ and } y \in E.$$

All of the above notations and definitions are standard, see for example [28], [17] or [39]. The similarity to the classical situation (see Proposition V.12) encourages us to make the following definition.

DEFINITION VII.6. *E is* ***minimally thin*** *at ξ if there is an x in $X \setminus E$ such that*

$$B^E K_\xi(x) < K_\xi(x).$$

3. Equivalent conditions for minimal thinness

We will in this section study the concept of minimal thinness defined above and give some equivalent formulations.

3.1. Minimal thinness is a local property. Using the above definition for the operator B^E we immediately obtain the sub-additivity for $B^E K_\xi$. Let $x \in X \setminus E$.

$$B^{E_1 \cup E_2} K_\xi(x) = \sum_{x_j \in E_1 \cup E_2} B_{x,x_j} K_\xi(x_j) = \sum_{x_j \in E_1} B_{x,x_j} K_\xi(x_j) + \sum_{x_j \in E_2} B_{x,x_j} K_\xi(x_j).$$

By the definition it follows that

$$B_{x,x_j} = \Pr_x[\exists n \geq 0 : X_n = x_j \in E_1, \text{ the first entry in } E_1 \cup E_2] \leq$$

$$\leq \Pr_x[\exists n \geq 0 : X_n = x_j \in E_1, \text{ the first entry in } E_1],$$

and similar for E_2. Thus we have that

$$B^{E_1 \cup E_2} K_\xi(x) \leq B^{E_1} K_\xi(x) + B^{E_2} K_\xi(x). \tag{40}$$

We say that $E^c = X \setminus E$ is *irreducible* if for all $x, y \in E^c$ there is an n such that

$$\Pr_x[X_n = y \text{ and } X_1, X_2, \ldots, X_{n-1} \in E^c] > 0.$$

PROPOSITION VII.7. *The following two properties holds for minimal thinness.*

- *Minimal thinness is a local property. That is, for every neighborhood $\mathcal{O}$ of $\xi \in \partial X$, where $\mathcal{O}$ is a subset of the Martin compactification X^*, we have the following. E is minimally thin at ξ if and only if $E \cap \mathcal{O}$ is minimally thin at ξ.*
- *If we suppose that E^c is irreducible, $x_0 \in E^c$ and that E is minimally thin at $\xi \in \partial X$ then*

  $$B^E K_\xi(x) < K_\xi(x) \; \forall x \in E^c.$$

  We have especially that

  $$B^E K_\xi(x_0) < 1.$$

PROOF. We know, c.f. [28], that K_ξ is P-harmonic and that $B^E K_\xi$ is P- superharmonic, hence $g := B^E K_\xi - K_\xi$ is superharmonic. Since E is minimally thin at ξ we know that there are points (maybe only one) in $X \setminus E$ such that $B^E K_\xi(x) < K_\xi(x)$, i.e. $g(x) < 0$. Suppose now that $x_1 \in X$ is a minimizing point of g. Then the minimum principle (see for example [44] or [38] p. 20) tells us that g must be constant, which contradicts the fact that $g = 0$ on E and $g(x_1) < 0$. We conclude that there is a minimizing sequence for g of points in E^c tending to the boundary. We can say more than that. Since $g = 0$ on the boundary outside every neighborhood of the limit point ξ we know that the minimizing sequence must tend to $\xi \in \partial X$.

Let $E_1 = E \setminus \mathcal{O}$ and $E_2 = E \cap \mathcal{O}$, and suppose that E_2 is minimally thin at ξ. Then we know from the above, that there is an x_i in $\mathcal{O} \cap X$ such that $B^{E_2} K_\xi(x_i) < K_\xi(x_i)$. Let us choose an ε such that

$$B^{E_2} K_\xi(x_i) < (1-\varepsilon) K_\xi(x_i).$$

Let us now study the whole set E. We have due to the sub-additivity in (40), that

$$(41) \qquad B^E K_\xi(x_i) \le B^{E_1} K_\xi(x_i) + B^{E_2} K_\xi(x_i) < B^{E_1} K_\xi(x_i) + (1-\varepsilon) K_\xi(x_i).$$

Let us study the first term closer. We see from the definition of the absorption matrix that we can make the following estimate for $x \in \mathcal{O} \cap X$.

$$B^{E_1} K_\xi(x) \le \sup_{y \in \partial \mathcal{O}} K_\xi(y) =: M < \infty.$$

Let us now, if necessary, choose a new point x_j closer to ξ such that $K_\xi(x_j) > M/\varepsilon$ and such that $B^{E_2} K_\xi(x_j) < (1-\varepsilon) K_\xi(x_j)$ still holds. This can always be done since $K_\xi(x) \to \infty$ as $x \to \xi$, and since there exists a minimizing sequence tending to ξ.

Let us use (41) to obtain the following.

$$B^E K_\xi(x_j) < B^{E_1} K_\xi(x_j) + (1-\varepsilon) K_\xi(x_j) \le M + (1-\varepsilon) K_\xi(x_j) <$$
$$< \varepsilon K_\xi(x_j) + (1-\varepsilon) K_\xi(x_j) = K_\xi(x_j).$$

That is, E is minimally thin at ξ. The opposite implication is immediate thanks to the trivial estimate $B^{E_1 \cup E_2} K_\xi \ge B^{E_1} K_\xi$.

We conclude that minimal thinness is a *local property.* We have finished the first part of the proof.

For the second part, let us assume that E^c is irreducible. From Lemma VII.9 below we see that g is in fact harmonic, seen as a function on E^c. Therefore, we have also the maximum principle, which we will now use. Suppose that x_2 is a point inside E^c such that $g(x_2) = 0$. Then we see that the harmonic function g attains a maximum inside E^c and we obtain a contradiction from the fact that the maximum principle forces g to be constant in E^c. □

Let us now introduce the normalized restricted transition matrix $\widetilde{P}$ as the restriction of P to the states in E^c normalized so that the series $\sum_{y \in E^c} \widetilde{p}(x,y) = 1$.

REMARK VII.8. We can use the substochastic operator $\dot{P}$, defined in equation (46) below, to describe $\widetilde{P}$.

$$\widetilde{p}(x,y) = \frac{\dot{p}(x,y)}{\sum_{z \in E^c} \dot{p}(x,z)} \text{ for } x, y \in E^c.$$

LEMMA VII.9. *If E^c is irreducible then $B^E K_\xi(x)$ is a $\widetilde{P}$-harmonic function in E^c.*

PROOF. We will just show that $\widetilde{P} B^E K_\xi(x) = B^E K_\xi(x)$ for $x \in E^c$.

$$\widetilde{P} B^E K_\xi(x) = \sum_{y \in E^c} \widetilde{p}(x,y) B^E K_\xi(y) = \sum_{y \in E^c} \widetilde{p}(x,y) \sum_{x_j \in E} B_{y,x_j} K_\xi(x_j) =$$

$$= \sum_{x_j \in E} K_\xi(x_j) \sum_{y \in E^c} \widetilde{p}(x,y) B_{y,x_j} = \sum_{x_j \in E} K_\xi(x_j) B_{x,x_j} = B^E K_\xi(x),$$

since we have by definition that $\sum_{y \in E^c} \widetilde{p}(x,y) = 1$ which means that we have an expansion equality in the second last equality above. □

3.2. Conditioned processes. Let Z_n be a process that started at the state x_0 and is conditioned to go to the boundary point $\xi \in \partial X$ and let E be a subset of X. More precisely, we will by Z_n mean a process on X acting under Doob's conditioned transition operator, which is a re-normalization of P.

Thanks to this conditioned transition operator we can view the conditioned process as an ordinary, unconditioned, random process with an adjusted transition law (see [18, p. 566]),

$$(P_h)_{x,y} = p_h(x,y) = \frac{p(x,y)h(y)}{h(x)}, \tag{42}$$

where $h(\cdot)$ is the Martin kernel at ξ.

We immediately get that

$$p_h(x_0,y) = \frac{p(x_0,y)K_\xi(y)}{K_\xi(x_0)} = p(x_0,y)K_\xi(y), \quad \text{since } K_\xi(x_0) = 1.$$

REMARK VII.10. Note that P_h is a new stochastic operator since P_h has row-sums equal to one. (That follows easily from the P-harmonicity of the Martin kernel).

THEOREM VII.11. *E is minimally thin at ξ if and only if there exists an x in X such that the conditioned process Z_n, with starting point at x conditioned to go to ξ, avoids E with non-zero probability.*

PROOF. Let us prove the complementary statement, i.e. let us prove the following. E is *not* minimally thin at ξ if and only if for every $x \in X$ the conditioned process Z_n with starting point at x hits E on its way to ξ almost surely.

We start with a remark of the iterated transition operator. Analogously to equation (42) above, we get that

$$p_h^{(n)}(x,y) = \frac{p^{(n)}(x,y)K_\xi(y)}{K_\xi(x)} \tag{43}$$

since we can view $P^{(n)}$ as another transition law and study its conditioned matrix $P_h^{(n)}$ in the same way as for P.

Next, by Definition VII.6 we have immediately that

$$E \text{ is } \textit{not} \text{ minimally thin at } \xi \ \Leftrightarrow \frac{1}{K_\xi(x)} B^E K_\xi(x) = 1, \ \forall x \in X. \tag{44}$$

Let us now show that the right-hand side of the above equality is equivalent to the second statement in the theorem.

We then have as above that the probability of the conditioned process Z_n with starting point at x to hit E on its way to ξ can be expressed in the following way.

$$\Pr[\{Z_n\} \text{ hits } E] = \sum_{y \in E} \Pr[\{Z_n\} \text{ is absorbed in } y]. \tag{45}$$

We will now study the absorption case closer. Let us now study a variant $\dot{P}$ of the transition operator P by letting E be an absorption set such that

$$\dot{p}(x,y) = \begin{cases} p(x,y) & \text{if } x \notin E, \\ 0 & \text{if } x \in E. \end{cases} \tag{46}$$

Thus $\dot{P}$ is a substochastic transition operator, i.e. the row-sums $\sum_{y \in E} \dot{p}(\cdot, y) \leq 1$. Furthermore, it is easy to see that $\dot{F}(x,y) = B_{x,y}$.

The construction of the conditioned operator is as before as seen at page 566 in [18], i.e.

$$\dot{p}_h(x,y) = \frac{\dot{p}(x,y)K_\xi(y)}{K_\xi(x)} \tag{47}$$

and as in equation 43 above

$$\dot{p}_h^{(n)}(x,y) = \frac{\dot{p}^{(n)}(x,y)K_\xi(y)}{K_\xi(x)}. \tag{48}$$

Let us now study the conditioned variant of $F(x,y)$ when $x \notin E$ and $y \in E$ to see that

$$\dot{F}_h(x,y) = \Pr[\{Z_n\} \text{ is absorbed in } y], \tag{49}$$

where, as above Z_n is a random process that started at x conditioned to exit at ξ. That is, $\Pr[\{Z_n\} \text{ hits } E] = \dot{F}_h(x,y)$.

Now, since in Remark VII.5, $G(x,y) = F(x,y)G(y,y)$, we have that for the conditioned analogue

$$F_h(x,y) = \frac{G_h(x,y)}{G_h(y,y)}, \tag{50}$$

where we define the *conditioned Green kernel* as

$$G_h(\cdot,\cdot) = \sum_{n=0}^{\infty} p_h^{(n)}(\cdot,\cdot).$$

Hence, by the use of equation (43), we have that

$$G_h(x,y) = G(x,y)\frac{K_\xi(y)}{K_\xi(x)}. \tag{51}$$

From that we immediately get that $G_h(y,y) = G(y,y)$ which we can use in equation (50) above to obtain

$$F_h(x,y) = \frac{G_h(x,y)}{G(y,y)}$$

and by equation (51) we get

$$F_h(x,y) = \frac{G(x,y)}{G(y,y)}\frac{K_\xi(y)}{K_\xi(x)} = F(x,y)\frac{K_\xi(y)}{K_\xi(x)}.$$

Since we can think of $\dot{P}$ as just another substochastic transition operator, we analogously have that

$$\dot{F}_h(x,y) = \dot{F}(x,y)\frac{K_\xi(y)}{K_\xi(x)}. \tag{52}$$

Thus we have from equation (49) that

$$\sum_{y\in E} \Pr[\{Z_n\} \text{ is absorbed in } y] = \sum_{y\in E} B_{x,y}\frac{K_\xi(y)}{K_\xi(x)} = \frac{1}{K_\xi(x)}B^E K_\xi(x).$$

Hence by equations (45) and (44) we conclude that the probability of the conditioned process Z_n with starting point at x to hit E on its way to ξ is one for every $x \in X$ if and only if E is not minimally thin at ξ.

□

3.3. A continuous analogue. The above Theorem VII.11 give us a nice probabilistic interpretation of the discrete minimal thinness for an arbitrary discrete irreducible stochastic operator.

Is the analogue true for the classical (continuous) case? That is, can we give a probabilistic interpretation of classical minimal thinness with the help of conditioned Brownian motion?

Yes we can, and it is rather easy thanks to the rich book of Doob [18]. (The conditioned Brownian motion is defied at page 668 in [18].) Let us show the following.

COROLLARY VII.12. *Let D be a Greenian set*[1]*. Furthermore, let B_t be a conditioned Brownian motion in D, conditioned by the Martin kernel K_τ to exit at $\tau \in \partial D$. Then E is minimally thin at τ if and only if there exists a point $x \in D$ such that*

$$Pr_x[B_t \text{ avoids } E] > 0.$$

PROOF. If we translate (b1) in Theorem 1.XII.11 on p. 208 in [18] and combine it with equation (7.3sm) and the equation below on page 686, where we put $v = K_\tau$, $\xi = x$, $A = E$, and note that $K_\tau < \infty$ for all $x \in D$, we get that E is minimally thin at τ if and only if

$$P^{K_\tau}\{T_x^E < T_x^{\partial D}\} < 1,$$

which we translate in the following way.

E is minimally thin at τ if and only if the probability for a K_τ–conditioned Brownian motion to hit E before hitting the boundary ∂D, when started at $x \in D$, is strictly less than one. □

REMARK VII.13. This corollary can be useful to heuristically describe ordinary minimal thinness. Furthermore, it is a good example how the discrete situation can generate conjectures about the original classical case.

[1]An open subset of $\mathbb{R}^d$ which supports a positive non-constant superharmonic function.

3.4. A zero-one law. Let us return to the discrete case. The following proposition is inspired by [18, 3.III.3] which deals with Brownian motion. Set L_E to be the last hitting time of E.

PROPOSITION VII.14.

$$Pr[L_E = \infty] = \begin{cases} 0 & \textit{if } E \textit{ is minimally thin at } \xi \in \partial X, \\ 1 & \textit{if } E \textit{ is not minimally thin at } \xi \in \partial X. \end{cases}$$

REMARK VII.15. It is sometimes useful to do the basic reformulation $\Pr[L_E = \infty] = \Pr[Z_n \in E \text{ infinitely often}]$ to get a slightly different statement[2].

PROOF. Let us consider the Martin compactification of X under the conditioned transition operator P_h. Since any random walk is forced to go to the one and only point ξ which lies on the underlying Martin boundary for the original transition operator P, we see that we have a one point compactification, i.e. the Martin boundary for the law P_h is just the lonely point ξ.

Hence we can only get constant solutions to the trivial Dirichlet problem thanks to the maximum- and minimum principles. In other words — every bounded P_h-harmonic function is constant.

Let us use this fact together with an argument from the proof of Proposition 3.5 in [11] where the following auxiliary function was presented.

$$u(x) := \Pr_x[Z_n \in E \text{ infinitely often }]$$

(recall that Z_n is conditioned to exit at ξ).

We have immediately that u is bounded and P_h-harmonic. Hence it is constant.

The Markov property gives us now that

$$u(Z_n) = \Pr[\{Z_k : k \geq 0\} \text{ hits } E \text{ i.o. } | Z_1, Z_2, \dots, Z_n] =$$
$$= E(\chi_{\{Z_k \text{ hits } E \text{ i.o.}\}} | Z_1, Z_2, \dots, Z_n),$$

where χ stands for the characteristic function. Due to the martingale convergence theorem we have the following.

$$E(\chi_{\{Z_k \text{ hits } E \text{ i.o.}\}} | Z_1, Z_2, \dots, Z_n) \to \chi_{\{Z_k \text{ hits } E \text{ i.o.}\}}.$$

By the fact that u is constant we will obtain the following zero-one situation.

$$u = \Pr[Z_n \in E \text{ infinitely often}] = \begin{cases} 0 \\ 1. \end{cases}$$

By the remark above, we see that $\Pr[L_E = \infty]$ is either zero or one.

We will now couple this dichotomy to the minimal thinness of E. First, let us suppose that E is not minimally thin at the conditioned exit point ξ. Then we have by Theorem VII.11 that

$$v(x) := \Pr_x[\{Z_n\}_{n \geq 1} \text{ hits } E] = 1 \; \forall x \in X, \tag{53}$$

Let now T_k be the hitting time for the k:th visit of Z_n at E. Due to Equation (53) we see that $T_k < \infty$ for all $k \geq 1$ almost surely. We have that T_k is a stopping time and we can consider the new stochastic variable

$$W_k := Z_{T_k}.$$

[2] as in the proof below

The strong Markov property give us then that

$$\Pr[\{Z_n\} \text{ hits } E\ K \text{ times or more}] = \prod_{k=1}^{K-1} v(W_k) = 1$$

since we had that $v(x) = 1$ for all $x \in X$. Going to the limit we will then obtain the following.

$$\Pr[Z_n \in E \text{ infinitely often}] = \prod_{k=1}^{\infty} v(W_k) = 1.$$

That is, if E is not minimally thin at ξ then $\Pr[L_E = \infty] = 1$.

On the other hand, let E be minimally thin at the boundary. Then we know from Theorem VII.11 that there is an $x \in X$ such that

$$\Pr_x[Z_n \text{ hits } E] < 1.$$

Thus we can write

$$\Pr[L_E = \infty] = \Pr[Z_n \in E \text{ infinitely often}] =$$

$$\Pr[Z_n \text{ hits } E \text{ i.o. and } Z_n \text{ hits } x] + \Pr[Z_n \text{ hits } E \text{ i.o. and } Z_n \text{ does not hit } x] \leq$$

$$\leq F_h(x_0, x)\Pr_x[Z_n \text{ hits } E \text{ i.o.}] + (1 - F_h(x_0, x)) < 1,$$

where we used the fact that the conditioned variant of $F(\cdot, \cdot)$,

$$F_h(x_0, x) = F(x_0, x)K_\xi(x),$$

is strictly positive due to the irreducibility of P. Hence we see that if E is minimally thin at a boundary point ξ, then $\Pr[L_E = \infty]$ is strictly less than one which, due to the zero-one situation, implies that $\Pr[L_E = \infty] = 0$ □

3.5. Capacity. In this subsection we will present an equivalent condition of minimal thinness. By a special choice of capacity one can get both a necessary condition and a sufficient condition for minimal thinness. We will obtain that in Corollary VII.17 below.

Let us cite some definitions made in [11].

DEFINITION VII.16. *Let E be a set and $\mathcal{B}$ a σ-field of subsets of E. Given a measurable function $H : E \times E \to [0, \infty]$ and a finite measure μ on $(E, \mathcal{B})$, the H-energy of μ is*

$$I_H(\mu) = \sum_i \sum_j H(x_i, x_j)\, \mu_i\, \mu_j,$$

where $x_i \in E$ and $\mu(x_i) = \mu_i$. The capacity of E in the kernel H is

$$\mathrm{Cap}_H(E) = \left[\inf_\mu I_H(\mu)\right]^{-1}$$

where the infimum is over probability measures μ on $(E, \mathcal{B})$ and, by convention, $\infty^{-1} = 0$.

We also define the *asymptotic capacity* of E in the kernel H as

$$\mathrm{Cap}_H^{(\infty)}(E) = \inf_{\{E_0 \text{ finite}\}} \mathrm{Cap}_H(E \setminus E_0). \tag{54}$$

Classically, the choice of the kernel function H has been the Green kernel $G(x, y)$.

In [11], the authors used another kernel, namely the Martin kernel $K(x,y)$, to obtain the following second half of Theorem 2.2 in [11] which we cite here for the convenience of the reader.

THEOREM H. *Let $\{X_n\}$ be a transient Markov chain on the countable state space X with initial state x_0 and transition probabilities $p(x,y)$. For any subset E of X we have*

$$\frac{1}{2}\mathrm{Cap}_K^{(\infty)}(E) \leq P_{x_0}[X_n \in E \textit{ infinitely often }] \leq \mathrm{Cap}_K^{(\infty)}(E) \tag{55}$$

where K is the Martin kernel.

We will now study a variant of the Martin kernel, more suitable for our purposes. Let from now on the kernel function $H(x,y)$ be the *conditioned Martin kernel* at ξ, i.e.

$$H(x,y) := K(x,y)\frac{1}{K_\xi(x)}.$$

From Theorem H and Proposition VII.14 above, we get the following corollary.

COROLLARY VII.17. *Let $H(x,y) = K(x,y)\frac{1}{K_\xi(x)}$. Then the following holds.*

$$E \textit{ is minimally thin at } \xi \textit{ if and only if } \mathrm{Cap}_H^{(\infty)}(E) = 0.$$

PROOF. Let us once again consider the conditioned random walk Z_n on X conditioned to eventually go to ξ at the boundary. The transition probability is $p_h(x,y) = p(x,y)\frac{K_\xi(y)}{K_\xi(x)}$. (Note that we mark the dependence of the transition operator in the conditioned Martin kernel by the lower index h.)

Nevertheless, Theorem H still holds for that transition operator, i.e. equation (55) transforms into

$$\frac{1}{2}Cap_{K_h}^{(\infty)}(E) \leq P_{hx_0}[X_n \in E \text{ infinitely often }] \leq Cap_{K_h}^{(\infty)}(E). \tag{56}$$

From Proposition VII.14 we have that E is minimally thin at ξ if and only if

$$\mathrm{Pr}_{x_0}[Z_n \in E \text{ infinitely often }] = 0.$$

Hence we have from equation (56) that E is minimally thin at ξ if and only if $Cap_{K_h}^{(\infty)}(E) = 0$.

The only thing that remains to do is to actually compute the kernel K_h.

By definition,

$$K_h(x,y) = \frac{G_h(x,y)}{G_h(x_0,y)},$$

where we recall that the conditioned Green kernel

$$G_h(x,y) = G(x,y)\frac{K_\xi(y)}{K_\xi(x)},$$

from equation (51) in the proof of Theorem VII.11. Since $K_\xi(x_0) = 1$ we finally get that

$$K_h(x,y) = \frac{G(x,y)}{G(x_0,y)}\frac{1}{K_\xi(x)} = K(x,y)\frac{1}{K_\xi(x)}$$

ending the proof of the corollary. □

3.6. Beurling's equivalence sequences. Let us for a while leave the discrete world and recall a classic result from 1965. Beurling defines and characterizes the so called equivalence sequences in [12] in the following way. Let D be a simply connected domain and let $S = \{z_n\}_n^\infty \subset D$ be a sequence tending to the limit point $\xi \in \partial D$ with the property that for each positive harmonic function u in D and $\lambda > 0$,

$$u(z_n) \geq \lambda K_\xi(z_n)\ \forall n \implies u(z) \geq \lambda K_\xi(z)\ \forall z \in D.$$

S is called an equivalence sequence for ξ if the above property holds.

THEOREM I. *S is an equivalence sequence for ξ if and only if it contains a subsequence $\{z_\nu\}_{\nu=1}^\infty$ with the two properties*

$$\sup_{\mu \neq \nu} g(z_{n_\mu}, z_{n_\nu}) < \infty$$

$$\sum_{\nu=1}^{\infty} g(z, z_{n_\nu}) K_\xi(z_{n_\nu}) = \infty,\ z \in D, \tag{57}$$

where g is the Green function for D.

It is well known (c.f. [7] or [20]) that the concept of equivalence sequences is closely related to minimal thinness. If we let E be the union of hyperbolic disks[3] with centers at $z_n \in S$, we will have that S is an equivalence sequence at ξ if and only if E is not minimally thin at ξ.

We will see that the use of hyperbolic disks is the price we have to pay as we work with the continuous case. Let us now turn back to the discrete setting.

Following the definition above we define the following.

DEFINITION VII.18. *$E := \{x_n\}_n^\infty \subset X$ is an* ***equivalence sequence*** *of $\xi \in \partial X$ if for each positive super-harmonic[4] function h in X*

$$h(x_n) \geq K_\xi(x_n)\ \forall n \implies h(z) \geq K_\xi(x)\ \forall x \in X.$$

Let us now state and prove a result in the spirit of Theorem I.

PROPOSITION VII.19. *Let, as above, E be a subset of X, then*

E is an equivalence sequence at $\xi \in \partial X$

$$\Leftrightarrow$$

E is not minimally thin at $\xi \in \partial X$

$$\Rightarrow$$

$$\sum_{y \in E} G(x_0, y) K_\xi(y) = \infty.$$

[3]We have to put hyperbolic disks around the points due to the fact that a denumerable point sequence is always polar and thus always minimally thin.

[4]i.e. super-regular in the notion of [28].

PROOF. The equivalence is almost immediate since we know that $B^E K_\xi$ is the smallest positive superharmonic function such that $B^E K_\xi \geq K_\xi$ on E. Definition VII.6 then tells us that E is minimally thin if and only if E is *not* an equivalence sequence finishing the first part of the proof.

Let us now show that the first statement implies the third. Let Z_n be a process with transition law P that started in the state x_0 and conditioned to go to the boundary point $\xi \in \partial X$. Let us first compute the expected number of times that Z_n visits a point $y \in X$. A way to study this situation is given by the conditional random walk matrix (see (42) just above Theorem VII.11). We have that

$$p_h(x_0, y) = \frac{p(x_0, y) K_\xi(y)}{K_\xi(x_0)} = p(x_0, y) K_\xi(y).$$

We also get from equation (43) that

$$p_h^{(n)}(x_0, y) = p^{(n)}(x_0, y) K_\xi(y).$$

Let us recall the definition of the conditioned Green kernel introduced in the proof of Theorem VII.11: $G_h(\cdot, \cdot) = \sum_{n=0}^{\infty} p_h^{(n)}(\cdot, \cdot)$. Now, we are ready to compute the expectation value

$$E(\#\{Z_n \text{ visits } y\}) = G_h(x_0, y) = \sum_{n=0}^{\infty} p_h^{(n)}(x_0, y) = \tag{58}$$

$$= \sum_{n=0}^{\infty} p^{(n)}(x_0, y) K_\xi(y) = G(x_0, y) K_\xi(y).$$

Next, let us compute the expected number of times that Z_n visits the set $E \subset X$ on the way to ξ.

$$E(\#\{Z_n \text{ visits } E\}) = \sum_{y \in E} E(\#\{Z_n \text{ visits } y\}) = \sum_{y \in E} G(x_0, y) K_\xi(y). \tag{59}$$

(I am sorry for using the letter E in two different meanings on the same line above.)

Consider now the notion of exit times L_E introduced in section 3.2. Proposition VII.14 tells us that if E is *not* minimally thin at ξ we have that $L_E = \infty$ almost surely; or in other words: Z_n visits E infinitely many times almost surely. Equation (59) then gives us

$$\infty = E(\#\{Z_n \text{ visits } E\}) = \sum_{y \in E} G(x_0, y) K_\xi(y).$$

□

REMARK VII.20. As a byproduct of the proof we get the probabilistic interpretation of the series $\sum_{y \in E} G(x_0, y) K_\xi(y)$ to be the expected number of visits of E of a random process that starts in x_0 and is conditioned to exit at $\xi \in \partial X$.

The remark above encourages us to formulate the following straightforward half-side analogue to Beurling's Theorem I.

COROLLARY VII.21. *Let Z_n be a random process conditioned to eventually go to $\xi \in \partial X$. Then the expected number of times that Z_n visits E is infinite if E is an equivalence sequence at ξ.*

REMARK VII.22. In [7, Theorem 7.2 p. 18] A. Ancona gives in fact a discrete version of Theorem I in the framework of Gromovs theory of δ-hyperbolic graphs.

4. Boundary layers

4.1. The discrete definition. Since we have assumed that P is a transitive operator, i.e. it generates a transitive Markov chain, we can define the asymptotic stochastic variable X_∞, see [44]. Let us then study the class of harmonic measures $(\nu_x)_{x\in X}$, derived in [44, Chapter 2]. The measures are defined in the following way.

$$\nu_x(B) = \mathrm{Pr}_x[X_\infty \in B],$$

where B is a Borel set and a subset in ∂X. Let us do the same construction for the set $E^c = X \setminus E$, where we suppose that E^c is irreducible, and call the resulting measure class $(\dot{\nu}_x)_{x\in E^c}$. (We will in the proof of Proposition VII.25 below see that we indeed are dealing with a measure class.) We can now mimic the situation in Definition V.2 on page 45.

DEFINITION VII.23. *Let $c \in (0,1)$. We say that E^c is a c-boundary layer (at x_0) if*

$$\dot{\nu}_{x_0}(I) \geq c\nu_{x_0}(I) \quad \textit{for every Borel set } I \subset \partial X.$$

We sometimes drop the prefix "c-" if Ω is a c-boundary layer for some $c > 0$.

E^c is a boundary layer if one can, by a random walk, reach each subset of the boundary, without hitting the "taboo set" E on the way, with probability comparable to the size of the boundary. In other words, E should be small enough everywhere near the boundary ∂X.

The following is inspired by Proposition V.4.

PROPOSITION VII.24.

$$B^E K_\xi(x_0) \leq 1 - c \textit{ for all } \xi \in \partial X,$$

if and only if E^c is a c-boundary layer.

PROOF. We will closely follow the proof in [6] of Proposition V.4. Let us fix a Borel set I on ∂X. To simplify the notation, let $\dot{\nu} = \dot{\nu}_x(I)$ and $\nu = \nu_x(I)$.

We will have

$$\nu - \dot{\nu} = \begin{cases} 0 & \text{on } \partial X, \\ \nu & \text{on } E. \end{cases}$$

We also note that $\nu - \dot{\nu}$ is P-harmonic in E^c. From this we see , by the uniqueness of harmonic functions and Lemma VII.9, that

$$\nu - \dot{\nu} = B^E \nu \text{ on } E^c.$$

Hence E^c is a c-boundary layer if and only if

$$c\nu_{x_0} \leq \nu_{x_0} - B^E\nu(x_0),$$

or equivalently

$$B^E\nu(x_0) \leq (1-c)\nu_{x_0} \quad \text{for every Borel set } I \subset \partial X. \tag{60}$$

Let us rewrite this in the following form.

$$B^E \frac{\nu}{\nu_{x_0}}(x_0) \leq (1-c) \quad \text{for every Borel set } I \subset \partial X. \tag{61}$$

Proposition 10-21 in [28] tells us that for a Borel set A in ∂X, we have the following.

$$\nu_x(A) = \int_A K(x,z)\, d\nu_{x_0}(z). \tag{62}$$

Now, let $\nu^n = \nu(\cdot, B(\xi, r_n) \cap \partial X)$ and $r_n \to 0$ as $n \to \infty$, then we obtain

$$\lim_{n\to\infty} \frac{\nu_x^n}{\nu_{x_0}^n} = \lim_{n\to\infty} \frac{\int_{B(\xi,r_n)} K(x,z)\, d\nu_{x_0}^n(z)}{\int_{B(\xi,r_n)}\, d\nu_{x_0}^n(z)} = K(x,\xi) = K_\xi(x),$$

due to the continuity[5] of the second variable in the Martin kernel (c.f. p. 340 in [28]).

Hence (61) implies

$$B^E K_\xi(x_0) \leq 1 - c \text{ for every } \xi \in \partial X. \tag{63}$$

On the other hand, suppose that (63) holds and let us integrate both sides of the inequality with respect to the harmonic measure ν_{x_0} over the Borel set I. The right hand side will then be

$$\int_I (1-c)\, d\nu_{x_0}(\xi) = (1-c)\nu_{x_0}(I),$$

and for the left hand side we have

$$\int_I B^E K_\xi(x_0)\, d\nu_{x_0}(\xi) = \int_I \sum_{x_j\in E} B_{x_0,x_j} K_\xi(x_j)\, d\nu_{x_0}(\xi) =$$

$$= \sum_{x_j\in E} B_{x_0,x_j} \int_I K_\xi(x_j)\, d\nu_{x_0}(\xi) = (\text{Equation (62)}) =$$

$$= \sum_{x_j\in E} B_{x_0,x_j} \nu_{x_j}(I) = B^E\nu(x_0) \ \ (\text{with } I \text{ as the underlying set}).$$

Comparing the left and right sides we get that (63) implies (60) and we obtain that E^c is a c-boundary layer if and only if (63) holds. □

4.2. Weak boundary layers. We have put the reference point in a special position in the definition of boundary layers. Can we weaken that slightly to choose the "best" possible starting point for Z_n?

Yes, we can do that as is shown in the following proposition, which is a parallel to Proposition V.22 on page 54.

PROPOSITION VII.25. *E^c is a boundary layer if and only if*

$$\inf_{x\in X} \sup_{\xi\in\partial X} \frac{1}{K_\xi(x)} B^E K_\xi(x) < 1.$$

PROOF. Suppose that

$$\inf_{x\in X} \sup_{\xi\in\partial X} \frac{1}{K_\xi(x)} B^E K_\xi(x) < 1$$

holds. Then we know that there exist q, $0 < q < 1$, and $x_1 \in E^c$ such that $\sup_{\xi\in\partial X} \frac{1}{K_\xi(x_1)} B^E K_\xi(x_1) \leq q$. Due to Proposition VII.24 we see that E^c is a $(1-q)$-boundary layer at x_1. That is,

$$\dot{\nu}_{x_1}(I) \geq (1-q)\nu_{x_1}(I)$$

[5]The continuity follows from the construction of the Martin compactification.

for every Borel subset $I \subset \partial X$.

Now, thanks to the irreducibility of E^c, we have that

$$\dot{\nu}_{x_0}(I) \geq \tilde{F}(x_0, x_1)\dot{\nu}_{x_1}(I) \geq \tilde{F}(x_0, x_1)(1-q)\nu_{x_1}(I) \geq$$

$$\geq \tilde{F}(x_0, x_1)(1-q)F(x_1, x_0)\nu_{x_0}(I) \geq C(1-q)\nu_{x_0}(I).$$

($\tilde{F}$ is with respect to E^c not X.)

Since I is an arbitrary Borel subset in ∂X, this implies that E^c is a boundary layer at x_0.

The "only if" part is trivial. □

The above condition tells you heuristically that it is enough to study the probability of random walks to hit sets on the boundary starting at *good* starting points, i.e. not necessary the origin.

Let us now take this a step further and not only study sequences of good starting points but also to let the hitting point be decided in advance, i.e. a switch of the order of limiting sequences. Or in other words: exchange the order of the *supremum* and *infimum* above and define the following variant of boundary layers. (Compare with Definition V.23.)

DEFINITION VII.26. *E^c is a weak boundary layer if*

$$\sup_{\xi \in \partial X} \inf_{x \in X} \frac{1}{K_\xi(x)} B^E K_\xi(x) < 1.$$

Encouraged by Proposition V.24, we state the following.

PROPOSITION VII.27. *E^c is a weak boundary layer if and only if E is minimally thin everywhere at ∂X.*

PROOF. First, let us suppose that the statement "E is minimally thin everywhere at ∂X" is false, i.e. there is a $\xi \in \partial X$ such that

$$\frac{1}{K_\xi(x)} B^E K_\xi(x) = 1 \quad \text{for all } x \in X. \tag{64}$$

Let now $\{x_i\}$ be a sequence in X such that

$$\lim_{i \to \infty} \frac{1}{K_\xi(x_i)} B^E K_\xi(x_i) = \inf_x \frac{1}{K_\xi(x)} B^E K_\xi(x).$$

Since now (64) holds, we see that

$$\inf_x \frac{1}{K_\xi(x)} B^E K_\xi(x) = 1.$$

Hence E^c is not a weak boundary layer.

To show the other implication, we suppose that E is minimally thin everywhere. Proposition VII.14 tells us now that for every $\xi \in \partial X$ the conditional process Z_n (conditioned to eventually go to ξ) has a last exit time of E, L_E, such that $\Pr(L_E = \infty) = 0$. Thus we see that it is possible to choose a sequence of starting points $\{x_j\}$ tending to ξ such that the probability to hit the set E for Z_n will tend to zero. That is, by considering the proof of Theorem VII.11,

$$\frac{1}{K_\xi(x_j)} B^E K_\xi(x_j) \to 0,$$

which gives us

$$\inf_x \frac{1}{K_\xi(x)} B^E K_\xi(x) = 0.$$

Since this holds for every $\xi \in \partial X$ we conclude that

$$\sup_\xi \inf_x \frac{1}{K_\xi(x)} B^E K_\xi(x) = 0$$

which means, by the definition, that E^c is a weak boundary layer. □

REMARK VII.28. We note that the proof gives us that the expression in the definition of weak boundary layers above is either zero or one, i.e.

$$\sup_\xi \inf_x \frac{1}{K_\xi(x)} B^E K_\xi(x) = \begin{cases} 0 & \begin{array}{c} E^c \text{ is a weak boundary layer} \\ \text{or equivalently} \\ E \text{ is minimally thin everywhere,} \end{array} \\ & \\ 1 & \begin{array}{c} E^c \text{ is not a weak boundary layer} \\ \text{or equivalently} \\ E \text{ is not minimally thin everywhere.} \end{array} \end{cases} \tag{65}$$

5. Kleinian groups

In this section we will try to use the above definitions and statements to study the situation on the orbits of Kleinian groups, and especially the comparison of subgroups. Let Γ be a Kleinian group, and let X be the orbit set of the reference point x_0 which we for simplicity will take to be the origin.

C. Series has given some results concerning the limit set of Fuchsian groups and the Martin boundary of X. See [37, Corollary 1.4]. Before we plunge into the discreteness, let us recall a result from the continuous setting. Theorem III.23 says that Γ is of convergence type if and only if E is minimally thin a.e.

5.1. Subgroups of Kleinian groups. Where we in the continuous situation use the unit ball as the implicit universe for the set of orbit points, we introduce now super-groups, $\widehat{\Gamma}$, to be able study the orbit of Γ as a subset of the orbit of $\widehat{\Gamma}$ and thus be able to use our results about discrete minimal thinness and discrete boundary layers.

Let as above $\widehat{\Gamma}$ be a Kleinian group and $\widehat{X}$ its orbit. Let furthermore, Γ be a subgroup of $\widehat{\Gamma}$ which generates the orbit set X.

We see immediately that X is a subset of $\widehat{X}$ and that there are two questions we could ask.

- How big is the set on $\partial\widehat{X}$ where X is minimally thin?
- Is $\widehat{X} \setminus X$ a boundary layer?

The answer of the two questions above depends of course on the choice of the transition matrix P.

REMARK VII.29. Note that we can do a similar set up for the slightly more abstract situation where we pick the elements in Γ to be our denumerable set X instead of first taking the orbit.

PROPOSITION VII.30.

$$\xi \in \partial\widehat{X} \setminus \partial X \implies X \textit{ is minimally thin at } \xi.$$

PROOF. Suppose that X is **not** minimally thin at ξ, then by the above Proposition VII.14, we see that $\Pr[L_X = \infty] = 1$. Thus ξ must be in ∂X. □

We can also obtain something in the spirit of Theorem III.23 to answer the first question above. Note that Theorem III.23 can be viewed in the following way.

$$\Gamma \text{ is of convergence type if and only if}$$

$$|\{\xi \in \partial B : \ E \text{ is minimally thin at } \xi\}| = |\partial B|.$$

THEOREM VII.31. *Suppose that $\widehat{\Gamma}$ has no parabolic elements. Then there exists a transition matrix P on $\widehat{X}$ such that the following holds.*

If Γ has a critical exponent strictly less than the critical exponent of $\widehat{\Gamma}$ then

$$\widehat{\nu}_{x_0}(\{\xi \in \partial\widehat{X} : \ X \textit{ is minimally thin at } \xi\}) = 1,$$

where $\widehat{\nu}_{x_0}(\cdot)$ is the harmonic measure with respect to $\widehat{X}$.

PROOF. S.P. Lalley's Theorem[6] 14 in [29] tells us that there exists a P matrix[7] on $\widehat{X}$ such that the "exit measure" $\widehat{\nu}_{x_0}$ equals the Patterson measure on $\partial\widehat{X}$.

Furthermore, it is well known that for the conical limit set, the Patterson measure is comparable to the Hausdorff measure of dimension equal to the critical exponent of $\widehat{\Gamma}$. Since we know that $\widehat{\Gamma}$ has no parabolic elements, it is clear that the conical limit set coincides with the limit set $\partial\widehat{X}$.

Thus we can conclude that $\widehat{\nu}_{x_0} \approx H_{\hat{\delta}}$, where $\hat{\delta}$ and δ are the critical exponents of $\widehat{\Gamma}$ and Γ respectively.

We immediately get that

$$\widehat{\nu}_{x_0}(\partial X) = 0$$

since the Hausdorff dimension of ∂X is δ which is strictly less than $\hat{\delta}$.

It is now easy to do the following estimation by the help of Proposition VII.30.

$$\widehat{\nu}_{x_0}(\{\xi \in \partial\widehat{X} : \ X \text{ is minimally thin at } \xi\} \geq$$

$$\widehat{\nu}_{x_0}(\partial\widehat{X} \setminus \partial X) \geq \widehat{\nu}_{x_0}(\partial\widehat{X}) - \widehat{\nu}_{x_0}(\partial X) = 1 - 0.$$

□

5.2. The Schottky group situation. For a so called Schottky group (see Definition A.5), we have a tree like situation, see for example [36, p. 337]. We will for this section suppose that P is of nearest neighbor type[8] and uniformly distributed, i.e. a simple random walk (SRW).

Let us first give an answer to the two questions in the beginning of this subsection.

PROPOSITION VII.32. *Let $\widehat{\Gamma}$ be a Schottky group. Then*

$$X \textit{ is minimally thin at } \xi \textit{ if and only if } \xi \in \partial\widehat{X} \setminus \partial X.$$

[6] Y. Peres kindly informed me that Furstenberg had a similar result 1971 in [23].

[7] Note that P will have infinite range.

[8] I.e. X_n goes, with probability one, to a neighbor point in the graph.

PROOF. From Proposition VII.30 we have the "if" part. To prove the other implication let us suppose that $\xi \in \partial X$. Since we know that the Schottky group gives us a tree[9], see for example [36], and we know that there is a unique branch that leads to a boundary point[10], ξ, in a tree, we have, by considering the conditioned process Z_n introduced in section 3.2, that Z_n can not have a last hitting time of E, otherwise $\xi \notin \partial X$. Hence by Proposition VII.14 we conclude that X is minimally thin at ξ. □

PROPOSITION VII.33. *Let $\widehat{\Gamma}$ be a Schottky group, then*

$$\widehat{X} \setminus X \text{ is a boundary layer if and only if } A = \{I\}.$$

PROOF. It is trivial to see that $\Omega = \widehat{X} \setminus X$ is a boundary layer if $A = \{I\}$. On the other hand if Ω is a boundary layer it has to be a weak boundary layer. Ω is a weak boundary layer if and only if X is minimally thin everywhere on $\partial \widehat{X}$ due to Proposition VII.27, which is equivalent to say that ∂X is empty by Proposition VII.32. Therefore, we can only have the trivial subgroup $A = \{I\}$. □

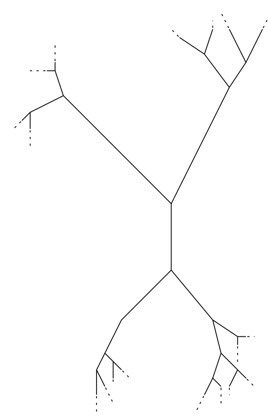

FIGURE VII.15. Here is a tree.

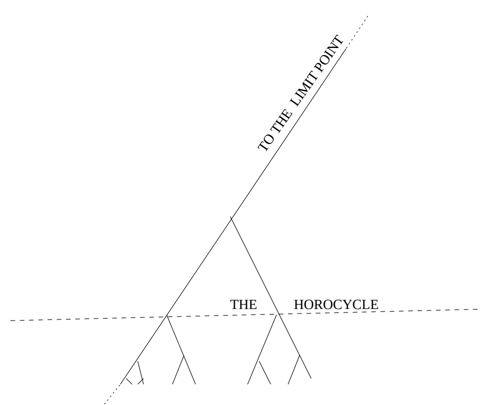

FIGURE VII.16. Grab a leaf in the tree in the above picture VII.15 and shake it well. This will be the resulting picture.

In continuous potential theory we know that the set in the unit disk outside a internal horocycle that touches the boundary at the boundary point ξ is minimally thin at ξ. The same turns out to be true in our discrete Schottky case.

[9]A tree is a graph that is connected, but has no circuits.
[10]Also known as a leaf.

PROPOSITION VII.34. *Let $\widehat{\Gamma}$ be a finitely generated Schottky group. Let E be the set "outside" a fixed horocycle at $\tau \in \partial\widehat{X}$. Then E is minimally thin at τ.*

REMARK VII.35. By "outside" a fixed horocycle we heuristically mean that we take the tree that the free group generates and cut all its branches, except the root. The cut-off part will be the set E [11].

PROOF. Suppose first that we have a tree with degree $q+1$ where $q \geq 0$ and recall that we deal with the simple random walk situation. By drawing the tree in such a way that the horocycle is a horizontal line, H_{x_0}, and the "outside" is "below" we can get a good overview of the situation.

It is easy to see that the probability to go down in the tree is $q/(q+1)$ and the probability to go up is $1/(q+1)$.

We intend to use Proposition VII.14 later to show that E is minimally thin at τ. To be able to study the exit times of the conditioned random walk we will once more[12] use Doob's conditional matrix.

$$(P_h)_{x,y} = \frac{p(x,y)h(y)}{h(x)},$$

where $h(\cdot)$ is the harmonic function we know as the Martin kernel at τ

$$K(\cdot,\tau) = \lim_{z \to \tau} K(\cdot, z).$$

The Green function of a SRW tree of degree $q+1$ is known to be (cf. [16]).

$$G(\cdot, z) = \frac{q}{q-1} q^{-d(\cdot,z)},$$

where $d(x,z)$ is the distance function which is the smallest number of edges between x and z.

We use this to obtain the Martin kernel.

$$K(y,z) = \frac{\frac{q}{q-1} q^{-d(y,z)}}{\frac{q}{q-1} q^{-d(x_0,z)}} = q^{d(x_0,z)-d(y,z)}. \tag{66}$$

Hence,

$$\frac{h(y)}{h(x)} = \lim_{z \to \tau} q^{d(x,z)-d(y,z)}.$$

Thus the conditioned probability to go up in the tree will be

$$P_h(\text{“up”}) = \frac{1}{q+1} \cdot q,$$

and to go down

$$P_h(\text{“down”}) = \frac{q}{q+1} \cdot q^{-1} = \frac{1}{q+1}.$$

Since we are only interested in those two events of going up or down, we project the study of the tree to the simple one-dimensional case and a stochastic variable Z that takes one step up (+1) with probability $\frac{q}{q+1}$ and one step down (-1) with probability $\frac{1}{q+1}$.

[11] This concept is due to W. Woess.

[12] as we did in the proofs of Theorem VII.11 and Proposition VII.19

The expectation value $E(Z)$ is then

$$E(Z) = \frac{q}{q+1} - \frac{1}{q+1} = \frac{q-1}{q+1},$$

which is strictly larger than 0 since we supposed that $q \geq 2$. Since Z is transient we will have a last visit of the reference point x_0 which means that we will have a last visit time of the horocycle H_{x_0}.

Thus E will be minimally thin at τ. $\square$

Remark VII.36. Note that the level-lines of the Martin kernel in Equation (66) in the proof above are in fact the horocycles.

Remark VII.37. Observe that this method can not give an example of a subgroup of $\widehat{\Gamma}$ generating the set E as a vertex set X due to a result telling us that a parabolic fixed point has to be alone, see for example [9, Theorem 5.1.2].

APPENDIX A

Kleinian groups

We will in this Appendix give some basic facts about the discrete groups and their limit sets.

A good introduction to this area is given in Beardon's "An introduction to hyperbolic geometry" in [10, pp. 1-33].

1. Basic properties of discontinuous groups

Denote by $\mathcal{M}$ the group of Möbius transformations in $\mathbb{R}^{\mathrm{d}}$, that keep the unit sphere B^{d} invariant. In other words, if we consider the planar case, mappings of the form

$$\frac{az+\bar{c}}{cz+\bar{a}}, \text{ where } a \text{ and } c \text{ are complex numbers such that } |a|^2-|c|^2=1.$$

In higher dimensions $\mathcal{M}$ is taken to be the group of all finite compositions of reflections (in spheres or planes) that preserve the orientation.

The elements in $\mathcal{M}$ are for our purpose too "dense". We thus need to select a sparser subgroup. This idea translates to a discreteness or discontinuity condition.

Definition A.1 (Discrete group). *Let us view $\mathcal{M}$ as a topological group and Γ as a subgroup of $\mathcal{M}$. We say that Γ is* ***discrete*** *if each point is isolated. That is, if Γ is discrete and γ_n tends to the identity mapping I, then there is an N such that $\gamma_n = I$ for all $n \geq N$. Here $\{\gamma_n\}$ denotes the members of Γ.*

Definition A.2 (Discontinuously acting group). *Γ acts* ***discontinuously*** *on the unit ball B if Γ is a subgroup of $\mathcal{M}$ and if for every compact subset K of B we have that $\gamma_n(K) \cap K$ is non-empty for only finitely many $\gamma_n \in \Gamma$.*

Remark A.3. In our case discrete and discontinuous groups are equivalent which follows from the fact that the elements in Γ preserve the unit ball and Theorem 5.3.14 (i) in [9].

Remark A.4. We will distinguish between the planar case and the general higher dimensional case by calling discontinuously acting subgroups of $\mathcal{M}$ **Fuchsian** groups if d = 2, and **Kleinian** groups if d $\geq$ 2. The notion "Kleinian" is usually restricted to the case d = 3, but since our results are of the form that we sometimes separate out the planar case and otherwise deal with the general situation (d $\geq$ 2) we choose this variant of notation. We hope that our choice of notation will not create any misunderstandings.

DEFINITION A.5 (SCHOTTKY GROUP). *We say that a Kleinian group Γ is a* ***Schottky group*** *if it is generated by side-paring-mappings for an even number of non intersecting spheres orthogonal to the unit ball, such that no sphere is mapped to itself and such that the outside of an orthogonal sphere is mapped into the inside of its image. (The region in the unit ball that is outside all the orthogonal spheres will be a fundamental domain.)*

The natural metric when dealing with the Möbius group is the hyperbolic metric, since the members of $\mathcal{M}$ act as isometry mappings with respect to this metric.

DEFINITION A.6 (HYPERBOLIC DISTANCE). *We define the hyperbolic distance, $d(\cdot,\cdot)$, between x and y in B by*

$$d(x,y) = \inf_{\nu} \int_{\nu} \frac{2|dz|}{1-|z|^2}, \text{ where } \nu \text{ is a smooth arc joining } x \text{ and } y.$$

We will also need a measure of the density of the orbit of Γ with respect to the origin.

DEFINITION A.7 (CRITICAL EXPONENT, δ). *Let $n(r)$ be the* ***orbital counting function****, i.e. the number of elements γ_n in Γ such that $d(0,\gamma_n x_0) < r$, then the* ***critical exponent*** *is defined as*

$$\delta = \limsup_{r\to\infty} \frac{1}{r} \log(n(r)).$$

(δ is independent of x_0, see for example [34, pp. 260]*.)*

Let us also define a fundamental series in the theory of discontinuous groups.

DEFINITION A.8 (POINCARÉ SERIES). *The* ***Poincaré series*** *is defined as*

$$h_s(x,y) = \sum_{\gamma_n\in\Gamma} e^{-s\,d(x,\gamma_n y)}.$$

The convergence of this series depends on the parameter s, but is independent of x and y. Let us therefore denote $h_s(0,0)$ simply by h_s.

LEMMA A.9.

$$h_s(x,y) \begin{cases} \textit{converges} & \textit{if } s > \delta. \\ \textit{diverges} & \textit{if } s < \delta. \end{cases}$$

See for example [34, pp. 259–260] for the proof.

The subgroup Γ of $\mathcal{M}$ is said to be of **convergence type** if the Poincaré series converges with $s = \mathrm{d}-1$, the dimension of the boundary.

By using the above definitions we see that the series $\sum_{\gamma_n\in\Gamma}(1-|\gamma_n(0)|)^s$ converges if and only if the Poincaré series does.

2. The non-tangential limit set

Let us study the so called orbit of a Kleinian group Γ which is a set $\Gamma x_0 = \{\gamma x_0 : \gamma \in \Gamma\}$ of points in the unit ball[1], see Section 4 in Chapter III for some orbit pictures. Since Γ is discrete, the points can not cluster inside the unit ball, but since it is infinite (unless Γ is trivial) it has to cluster at the unit sphere ∂B. We call this cluster set Λ — the (total) limit set.

There is a special subset of Λ called the non-tangential limit set Λ_c containing the limit points that are the cluster points on ∂B of orbit points "clustering in a non-tangentially way[2]". See Remark II.5 on page 14 for a technical definition of Λ_c.

If a Kleinian group is of convergence type then the non-tangentially limit set has Lebesgue measure zero, see for example [1, p. 93] or [25] (from where we cite Theorem A on page 13 above).

In 1978 Sullivan presented the following complementary relation.

THEOREM J. *Let Γ be a Kleinian group. If the non-tangential limit set has Lebesgue measure zero, then Γ is of convergence type.*

See for example [1, p. 97] for the proof.

[1] We will usually set the reference point x_0 to 0.

[2] Λ_c is also called the conical limit set.

APPENDIX B

Potential Theory

In the study of limit behavior of subharmonic functions one often meets the concept of thin sets. We will use four different kinds of thinness at the boundary, two local and two global ones. But first we give a short introduction.

1. A smooth introduction

Let us study the class of real valued smooth[1] functions u in the unit disk U in the complex plane. We say that u is harmonic if

$$\Delta u(z) = \frac{\partial^2 u}{\partial x^2}(x+iy) + \frac{\partial^2 u}{\partial y^2}(x+iy) = 0,$$

for all $z = x+iy$ in the unit disk.

Let $f(s)$ be a function on the unit circle $\mathbb{T}$, and let us try to find a harmonic function u that has the radial limit $\lim_{r\to 1} u(rs) = f(s)$, $r \in (0,1)$. If we can find such a u, we have solved the so called *Dirichlet problem.*

Let us now pick an arc I on $\mathbb{T}$ and choose f to be the indicator function for I, i.e.

$$f(s) = \begin{cases} 1 & \text{if } s \in I, \\ 0 & \text{if } s \notin I. \end{cases}$$

Then the solution to the Dirichlet problem $u(z)$, can be interpreted in the following probabilistic way. The value of $u(z)$ is the probability that a Brownian particle[2] starting at z hits the boundary $\mathbb{T}$ for the first time at the arc I. The bigger I seems, seen from the point z, the bigger will $u(z)$ be. Therefore we call $u(\cdot)$ the harmonic measure of I and denote it by $\omega(\cdot, I, U)$.

Let us now solve the Dirichlet problem for a given continuous boundary function f. We can give the solution in the following integral form.

$$u(z) = \frac{1}{2\pi}\int_{\mathbb{T}} P_s(z) f(s)\, ds, \quad \text{where } P_s(z) = \frac{1-|z|^2}{|z-s|^2} \text{ is called the Poisson kernel.}$$

The Poisson kernel is also a harmonic function on its own[3]. It is minimal in the sense that if $h(\cdot)$ is a positive harmonic function such that $h(z) \le P_s(z)$ for all $z \in U$ implies that $h(s) \equiv 0$ or $h(s) = cP_s(z)$ for a constant c.

Let us now make a variant of this. Let h be a positive superharmonic function, i.e. $\Delta h \le 0$, such that $h(z) \ge P_s(z)$ holds on a subset E of the unit disk. How

[1] $u \in C^2(U)$

[2] A dizzy particle with no memory, no preferences and no sense of directions.

[3] It is the solution to the Dirichlet problem when $f(s) = 2\pi\delta_s$.

strong is this condition? Can there be such a function h and a point z in $U \setminus E$ such that $h(z) < P_s(z)$? The answer depends on how "big" E is close to the basepoint s. Is E minimally thin at s?

2. Definitions of thin sets at the boundary

DEFINITION B.1 (MINIMAL THINNESS). *A set E is **minimally thin** at $\tau \in \partial B^d$ if there is a z_0 in the unit disc such that $\hat{R}_h^E(z_0) < h(z_0)$, where $h = P_\tau$ is the Poisson kernel at τ and $\hat{R}_h^E$ is the regularized function, $\hat{R}_h^E(z) = \liminf_{w \to z} R_h^E(w)$. The reduced function $R_h^E(w)$ is defined as*

$$R_h^E(w) = \inf\{u(w) : u \in SH(B) \text{ and } u \geq h \text{ on } E\},$$

where $SH(B)$ is the class of non-negative superharmonic functions in the unit ball.

In [20] and [4] a metric condition for a set to be minimally thin is stated. Let $\{Q_k\}$ be the Whitney decomposition of B. Let us also use the following notation. We set q_k to be the (Euclidean) distance from the Whitney cube Q_k to the boundary ∂B and $\rho_k(\tau)$ to be the distance from Q_k to the boundary point τ. By cap we denote the logarithmic capacity when d = 2, and the Newtonian capacity when d $\geq$ 3, see for example [30].

DEFINITION B.2. *We put*

$$W(\xi) = W(\xi, E) = \begin{cases} \displaystyle\sum_k \frac{q_k^2}{\rho_k(\xi)^2} \left(\log \frac{4q_k}{\operatorname{cap}(E \cap Q_k)}\right)^{-1} & \text{if } d = 2, \\ \displaystyle\sum_k \frac{q_k^2}{\rho_k(\xi)^d} \operatorname{cap}(E \cap Q_k) & \text{if } d \geq 3. \end{cases}$$

THEOREM K. (Essén [20]) *E is minimally thin at a boundary point ξ if and only if $W(\xi, E)$ converges.*

REMARK B.3. The definition and metric condition are stated in the plane but will later be extended to the space, see Lemma III.22.

DEFINITION B.4 (RAREFIEDNESS). *A subset E in the upper-half space $\mathbb{H}$ is **rarefied** at ∞ if*

$$\sum_1^\infty \lambda'(E^{(n)}) 2^{n(1-d)} < \infty,$$

*where d is the dimension, $E^{(n)}$ the intersection of E with the half–annulus $\{x \in H \cup \partial H : 2^n \leq |x| < 2^{n+1}\}$ and $\lambda'(E)$ is the **Green mass** of E. (See [19] for some of its properties.)*

There is also a Wiener type criterion for rarefiedness. The following theorem is cited from [4, Theorem 3.2], see also [19].

THEOREM L. (Aikawa [4]) *Let $\mathbb{H}$ be the upper half-space and let $X \in \partial\mathbb{H}$. Suppose E is a bounded subset of $\mathbb{H}$. Then E is rarefied at X if and only if E has a decomposition $E' \cup E''$ such that*

$$\begin{cases} \displaystyle\sum_k \frac{q_k}{\rho_k(X)} \left(\log \frac{4q_k}{\operatorname{cap}(E' \cap Q_k)}\right)^{-1} < \infty & \text{for } d = 2, \\ \displaystyle\sum_k \frac{q_k}{\rho_k(X)^{d-1}} \operatorname{cap}(E' \cap Q_k) < \infty & \text{for } d \geq 3, \end{cases}$$

where E'' has a covering $\cup_i B(X_i, r_i)$ with $X_i \in \partial\mathbb{H}, 0 < 2r_i < |X - X_i|$ and

$$\sum_i \left(\frac{r_i}{|X_i - X|}\right)^{d-1} < \infty.$$

In [4], Aikawa introduces two global characterizations of exceptional sets:

DEFINITION B.5. *A set E in B is **thin with respect to capacity** if*

$$\begin{cases} \sum_k q_k \left(\log \dfrac{4q_k}{\operatorname{cap}(E \cap Q_k)}\right)^{-1} < \infty & \textit{for } d = 2, \\ \sum_k q_k \operatorname{cap}(E \cap Q_k) < \infty & \textit{for } d \geq 3. \end{cases} \tag{67}$$

DEFINITION B.6 (THINNESS WITH RESPECT TO MEASURE). *A set E in the upper-half-space $\mathbb{H}$ is **thin with respect to measure** if*

$$H(E \cap D_t) \to 0 \text{ as } t \to 0,$$

where H is a Hausdorff–type outer measure defined below and D_t the strip of height t from the boundary. The measure $H(\cdot)$ is defined as follows,

$$H(A) := \inf\{\sum_i r_i^{d-1} : A \subset \cup_i \bar{B}(X_i, r_i), X_i \in \partial\mathbb{H}\}.$$

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Index