Chalmers University of Technology
Exam in **Dynamical systems** (FFR 130) 2001-03-08, 8.45–13.45.
Tool: Calculator.
Telephone support: 0701 - 71 18 71.
Language: Unfortunately only English or Swedish.
The exams will be returned, and discussed, Thursday March 15 at 12.30 in FL74.

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1. What is the difference between a deterministic (or a real) fractal and a natural (or a quasi) fractal? Give examples. (4 p)

2. Let $\{X_t\}_{t=1}^{10} = \{3, 2, 7, 8, 5, 7, 2, 4, 9, 9\}$.
   (a) Plot a pseudo–phase plot with lag 1.
   (b) Compute the value of the *auto-correlation function* with lag 1.
   (c) Plot a *corrologram* with lags from 0 up to 3. (2+2+2 p)

3. Explain what an attractor is. What is the attractor to the logistic equation $x_{t+1} = \frac{3}{2}x_t(1 - x_t)$? (4 p)

4. Suppose you where given a time–series that you where supposed to analyze to come up with a forecast model. Outline a strategy for that. (6 p)

5. Define mathematically the Hurst exponent. Relate it to the Hausdorff dimension. Give also a non-technical description of the Hausdorff dimension aimed to Göran Persson or George W. Bush. (Maybe you can use those lines if you bump into them here in July?) (2+2+2 p)

6. Describe (i.e. give a construction of) a fractal with dimension $\frac{\ln 3}{\ln 5}$. (4 p)

7. Suppose you want to measure the dimension of a "wiggly" line, and you use a compass to measure the length, in cm, of that curve as a function of the compass width. Your measurements give you the following table:

Opening	Steps
16	2.61
8	6.78
4	19.0
2	49.9
1	136.4
1/2	396.0

   Use that table to give an estimate of the fractal dimension of the wiggly line. Display your arguments (and think about how many decimals you should give in the answer). (4 p)

8. (a) Find the *correlation integral*, C_ε, with $\varepsilon = 1, 2, 3$ cm for the point set (which is obviously too tiny for this kind of analyze in reality) in Figure 1.
   (b) Use the data obtained above to give a (very rough) estimate of the *correlation dimension* of the (very–quasi fractal) set. (3+3 p)

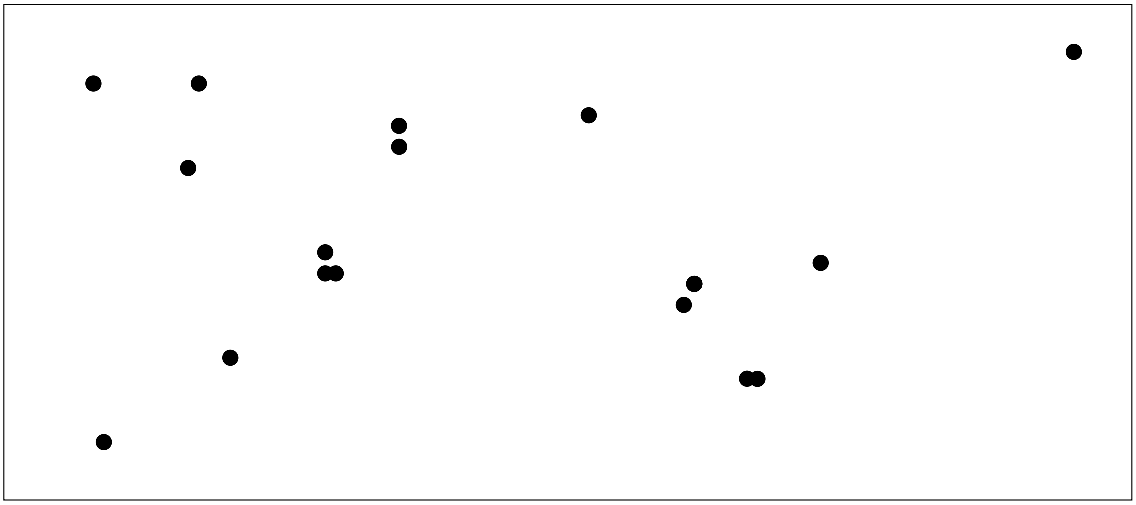

Figure 1

9. (a) Find a suitable compact set K (e.g. K is bounded and limited) in the complex plane and draw the first two resulting images after iterating Ψ twice, i.e draw K and $\Psi^2(K)$, where

$$\Psi = \{\psi_i\}_{i=1}^4, \quad \text{for } \psi_1 = \frac{1}{2}(z+1), \psi_2 = \frac{1}{2}(z-1), \psi_3 = \frac{1}{4}(iz+i) \text{ and } , \psi_4 = -\frac{1}{4}(iz+i).$$

(b) Show that all ψ_i are *contractive similarities*, and describe in a formula how one can obtain a fractal F from those mappings and the above chosen K. (2+2 p)

10. Suppose we have a dynamical system generated by iteration of a single one–dimensional (real valued) map $f(x)$.

(a) Describe how the Lyaponov exponent is related the quotient of δ_n/δ_0 where δ_0 is the input difference and δ_n is the output difference after n iterations.

(b) Starting with the above relation, explain carefully how one can derive a formula for the Lyaponov exponent using the derivatives of the function f. (2+4 p)

11. (a) When is a complex function analytic?

(b) Give an example of a complex function that is *not* analytic.

(c) Give a model of the extended complex plane.

(d) Give an heuristic (i.e. hand–waving) explanation why Julia sets look like self similar fractals.

(e) Let a Kleinian group, Γ be generated by

$$g(z) = \frac{2z - \sqrt{3}}{-\sqrt{3}z + 2}.$$

What is the Hausdorff dimension of its limit set? (1+1+2+2+2 p)

12. Estimate numerically, as good as possible, the Hausdorff dimension of the resulting fractal F from problem 9 above. (Hint: You might find the following scaling property of the Hausdorff measure useful.

$$\Lambda_s(\Psi_i(F)) = \lambda_i^s \Lambda_s(F), \text{ where } \lambda_1 = \lambda_2 = \frac{1}{2}, \lambda_3 = \lambda_4 = \frac{1}{4} \text{ and } s = \mathrm{Dim}_H(F).) \quad (4 \text{ p})$$

Good luck! /TL