Problems for Potential theory (final version modulus misprints)

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(1) Suppose u is subharmonic on an open and bounded $\Omega \subset \mathbb{R}^{\mathrm{N}}$. With $a \in \Omega$, let R be the Euclidean distance between a and $\partial\Omega$. Show that the function $\mathcal{M}(r) := \int_{\partial B} u(a + rs)\, d\sigma(s)$ is increasing. Conclude that the sub-mean-value inequality is valid for all $r \in (0, \mathbb{R})$. (From [2].)

(2) Show that $|u|^p$ is subharmonic on Ω whenever u i harmonic and $p \in [1, \infty]$. (From [2].)

(3) Find sufficient conditions on h so that

$$m_h(E) = \inf_{\mathcal{O} \supset E} m_h(\mathcal{O}),\ \mathcal{O} \text{ open}, \tag{1}$$

holds for each set E in $\mathbb{R}^{\mathrm{N}}$. What can we say when $h(r) = r^{N-\alpha}, N \geq 2$, or when $h(r) = (\log^+ \frac{1}{r})^{-1}, N = 2$? This is Exercise 2.1 in [1].

(4) Is equation (1) true with m_h replaced by m'_h?

(5) Is $\lim m'_h(E_n) = m'_h(E)$ as $E_n \nearrow E$? (Exercise 2.2 in [1].)

(6) Study SRW on $\mathbb{Z}^2$. Define a norm on $\mathbb{Z}^2$ so that using that norm $d(\cdot, \cdot)$ to define balls as

$$B(x, r) = \{z \in \mathbb{Z}^2, d(x, z) < r\},$$

the mean value property is valid for such balls for any harmonic function. What is d, and plot approximately how the the number of points in $B(0, r)$ increase as r increase.

(7) Let T be a triangle with each side S_i having the same length. Suppose x is a point inside T. Let

$$\tau = \inf_{t \geq 0} [B(t) \in \partial T],$$

where $B(t)$ is a standard planar Brownian motion. Let

$$H_i(x) = \mathbb{P}_x[B(\tau) \in S_i].$$

Find an y such such that $H_1(y) = H_2(y) = H_3(y)$. Give that common value. Furthermore, calculate $H_1(x)$ for an arbitrary x using:
  (a) analytic methods
  (b) numerical methods

Compare and comment.

(8) Prove that there are cases when Problem (B) of Section 5.2 in [1] does not have a unique solution. Discuss the case when the compact set F is the unit circle and $K(r) = \log^+ \frac{2}{r}$ in the plane. ($\log^+$ is $\max(log(\cdot), 0)$.)

As possible distributions of mass, we can consider a measure uniformly distributed on the circle and with unit mass, and $c\delta_0$, where δ_0 is the Dirac measure at the origin and c is a constant.

(9) (You don't need to do this!) Let us again study SRW on $\mathbb{Z}^2$ let us add edges between points that are nearest neighbors in the sense that the process can go from one to the other in one jump. We can now view the resulting graph as an infinite electrical network. Let us now assume that the resistance is one for all such edges.
   (a) Show that the resistance between 0 and ∞ is infinite.
   (b) Show that this is not the case if we look at $\mathbb{Z}^3$ instead of $\mathbb{Z}^2$.

References

[1] H. Aikawa, and M. Essén *Potential Theory: Selected Topics*,Springer Lecture Notes in Mathematics 1633, 1996.
[2] S. Axler, P. Bourdon, and W. Ramey, *Harmonic Function Theory* 2nd edition, Springer, 2001. (See http://www.axler.net/HFT.html)
[3] L. Carleson, *Selected Problems on Exceptional Sets* Van Nostrand, Princeton, 1967.