Involution on generalized Weyl algebras preserving the principal grading

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Abstract

We classify all *-representations of generalized Weyl algebras by bounded operators with respect to * preserving the principal grading.

Keywords: generalized Weyl algebra, involution, bounded operator, unitary equivalence, tensor product.

1 Introduction and preliminaries

Fix a positive integer, \( n \), and set \( N_n = \{1,2,\ldots,n\} \). Let \( R \) be a commutative unital complex algebra, \( \sigma_i, i \in N_n \), a set of pairwise commuting automorphisms of \( R \) and \( 0 \neq t_i \in R, i \in N_n \), such that \( \sigma_j(t_i) = t_i, i \neq j \in N_n \). The generalized Weyl algebra \( A = A(R,\{\sigma_i\},\{t_i\}) \) is defined as an \( R \)-algebra, generated over \( R \) by elements \( X_i, Y_i, i \in N_n \), subject to the following relations ([1]):

- \( X_i r = \sigma_i(r)X_i, i \in N_n, r \in R; \)
- \( rY_i = Y_i \sigma_i(r), i \in N_n, r \in R; \)
- \( Y_i X_i = t_i, X_i Y_i = \sigma_i(t_i), i \in N_n; \)
- \( X_i Y_j = Y_j X_i, X_i X_j = X_j X_i, Y_i Y_j = Y_j Y_i, i \neq j \in N_n. \)

We will call these relations the defining relations.

Let \( \mathbb{Z}^n \) be a free abelian group of rank \( n \) with canonical generators \( e_i, i \in N_n \). Then \( A \) has a natural \( \mathbb{Z}^n \)-grading, which we will call principal, by assigning \( \deg(R) = 0, \deg(X_i) = e_i, \deg(Y_i) = -e_i, i \in N_n. \) It is well known that many classical algebras can be realized as generalized Weyl algebras (e.g. see examples in [1]). Representations of generalized Weyl algebras have been studied intensively (see e.g. [1, 2] and references therein), in particular, it has been shown that the corresponding classification problem is usually wild.

A natural problem for generalized Weyl algebras is to study involutions over them and *-representations of the corresponding *-algebras. We remark that by an involution on \( A \) we mean a \( \mathbb{C} \)-skewlinear map, *, from \( A \) to \( A \) satisfying \((ab)^* = b^*a^*, a,b \in A, \) and
$(a^*)^* = a$, $a \in A$. For the involutions satisfying $(X_i)^* = \pm Y_i$ the corresponding problem was solved in [3] for a much wider class of algebras. In the present paper we will be interested in involutions preserving the principal grading on $A$, i.e. $(A_g)^* = A_g$ for any $g \in \mathbb{Z}^n$. We give a complete answer to the classification problem, listing all irreducible $*$-representations up to unitary equivalence.

In Section 2 we introduce the involutions on $A$, which we will be interested in and show that these involutions exhaust all involutions preserving the principal grading provided $R$ is a domain. In Section 3 we deal with the case $n = 1$ and in Section 4 we consider the general situation. The main result is a classification of all $*$-representations of $A$ up to unitary equivalence. Our answer for the general case uses the classification obtained for $n = 1$.

## 2 Involutions

If $*$ is an involution on $A$ preserving the principal grading then $(A_0 = R)^* = A_0 = R$ and hence it induces an involution on the algebra $R$. It is well known ([1]) that the elements $Z_1^{k_1}Z_2^{k_2} \ldots Z_n^{k_n}$, $Z_i = X_i$ or $Z_i = Y_i$, form a basis of $A$ as a free left (or right) $R$-module. Clearly, this basis is graded, hence all graded components of $A$ are free $R$-modules of rank 1. In particular, if $*$ preserves the principal grading, then $(X_i)^* = r_iX_i$ and $(Y_i)^* = s_iY_i$, $i \in N_n$, for some $r_i, s_i \in R$.

**Theorem 2.1.** Let $*$ be an involution on $R$ and $r_i, s_i \in R$, $i \in N_n$. If the following conditions are satisfied:

1. $\sigma_i^{-1}(r^*) = \sigma_i(r)^*$ for all $i \in N_n$, $r \in R$;
2. $r_i\sigma_i(r_i^*) = 1$ and $s_i\sigma_i^{-1}(s_i^*) = 1$, $i \in N_n$.
3. $r_i\sigma_i(s_i)\sigma_i(t_i) = t_i^*$, $i \in N_n$;
4. $r_j\sigma_j(r_i) = r_i\sigma_i(r_j)$, $i, j \in N_n$;
5. $s_j\sigma_j^{-1}(s_i) = s_i\sigma_i^{-1}(s_j)$, $i, j \in N_n$;
6. $s_j\sigma_j^{-1}(r_i) = r_i\sigma_i^{-1}(s_j)$, $i \neq j \in N_n$.

then setting $(X_i)^* = r_iX_i$, $(Y_i)^* = s_iY_i$, $i \in N_n$, defines an involution on $A$. Moreover, if $R$ is a domain and $*$ is an involution on $A$ then the restriction of $*$ to $R$ satisfies all conditions which are listed above.

**Proof.** First we note that $r_i\sigma_i(r_i^*) = 1$ and $s_i\sigma_i^{-1}(s_i^*) = 1$, $i \in N_n$, is equivalent to $(a^*)^* = a$ for all $a \in A$.

Assume that $*$ is an involution on $R$ satisfying the conditions of the theorem. We have to show that the defining relations are stable under the $*$ defined by $(X_i)^* = r_iX_i$, $(Y_i)^* = s_iY_i$, $i \in N_n$. We will check the relation $X_i r = \sigma_i(r)X_i$. Applying $*$ we get $r^* r_i X_i = r_i X_i \sigma_i(r)^*$.
which can be rewritten as $r^*r_iX_i = r_i\sigma_i(\sigma_i(r^*))X_i$. The last is true because of the first condition of the theorem. All other relations can be checked by analogous arguments.

Conversely, we assume that $R$ is a domain. Then, if $*$ is an involution on $A$ such that $(X_i)^* = r_iX_i$, $(Y_i)^* = s_iY_i$, $i \in \mathbb{N}_n$, the arguments above imply $r^*r_iX_i = r_i\sigma_i(\sigma_i(r^*))X_i$. As $A$ is a free $R$ module and $R$ does not have zero divisors, this equality is equivalent to $r^* = \sigma_i(\sigma_i(r^*))$, which is the first condition of the theorem. Analogous arguments applied to other relations will give us the rest. This completes the proof.

We have to remark that such involutions really exist. For example, taking $s_i = r_i = 1$, $i \in \mathbb{N}_n$, we reduce the list of conditions of our theorem to the following: $\sigma_i^{-1}(r^*) = \sigma_i(r^*)$, $r \in R$, and $\sigma_i(t_i) = t_i^*$, $i \in \mathbb{N}_n$. One can easily construct a plenty of examples of such involutions. Below we present some classical examples as well as some new ones.

**Example 1.** Let $n = 1$, $R = \mathbb{C}[H,T]$, $t_1 = T$, $\sigma_1(H) = H - 1$, $\sigma_1(T) = T + H$, $H^* = H$, $T^* = T + H$. Then all the conditions of Theorem 2.1 are satisfied and $X^* = X$, $Y^* = Y$ defines an involution on $A$. In this case $A \simeq U(\mathfrak{sl}(2,\mathbb{C}))$ and $(A, \cdot)$ corresponds to the real form $\mathfrak{sl}(2,\mathbb{R})$.

**Example 2.** Let $n = 1$, $q \in \mathbb{R}$ (resp. $q \in \mathbb{C}$, $|q| = 1$), $R = \mathbb{C}[T,k,k^{-1}]$, $t_1 = T$, $\sigma_1(k) = q^{-1}k$, $\sigma_1(T) = T + \frac{k^2 - k^{-2}}{q - q^{-1}}$, $k^* = k^{-1}$ (resp. $k^* = k$), $T^* = T + \frac{k^2 - k^{-2}}{q - q^{-1}}$. Then setting $X^* = X$, $Y^* = Y$ defines an involution on $A$. In this case $A \simeq U_q(\mathfrak{sl}(2,\mathbb{C}))$ and $(A, \cdot)$ corresponds to the real form $\mathfrak{sl}_q(2,\mathbb{R})$.

**Example 3.** Let $n = 1$, $q \in \mathbb{R}$ (resp. $q \in \mathbb{C}$, $|q| = 1$), $R = \mathbb{C}[T,k,k^{-1}]$, $t_1 = T$, $\sigma_1(k) = q^{-2}k$, $\sigma_1(T) = T + \frac{k - k^{-1}}{q - q^{-1}}$, $k^* = k^{-1}$ (resp. $k^* = k$), $T^* = T + \frac{k - k^{-1}}{q - q^{-1}}$. If we put $X^* = q^{-1}kX$, $Y^* = qYk^{-1}$, we get an involution on $A$. In this case $A \simeq U_q^+(\mathfrak{sl}(2,\mathbb{C}))$ and $(A, \cdot)$ corresponds to the real form $\mathfrak{sl}_q^+(2,\mathbb{R})$. We remark that the corresponding quantum algebra is slightly different from the one, considered in the previous example.

**Example 4.** Let $n = 1$, $R = \mathbb{C}[H]$, $\sigma_1(H) = H + 1$, $H^* = -H$, $t_1 = f(H^2 - H)$, where $f$ is a fixed polynomial with real coefficients. Then it follows from Theorem 2.1 that $X^* = X$, $Y^* = Y$ defines an involution on $A$.

As an example of higher rank one can take, say, a tensor product of some rank one examples. In the rest of the paper we will assume that the conditions of Theorem 2.1 are satisfied.

## 3 Rank one case

In this section we will deal with the case $n = 1$ and set $\sigma = \sigma_1$, $t = t_1$, $X = X_1$ and $Y = Y_1$, $r = r_1$, $s = s_1$. Let $\pi : A \to \mathcal{B}(H)$ be a $\cdot$-representation of $A$ by bounded linear operators on a separable Hilbert space, $H$. To simplify the notation we will denote the operators $\pi(a), a \in A$, simply by $a$ and hence all equalities below will be operator equalities in $\mathcal{B}(H)$. 


Lemma 3.1. The elements $X^2$ and $Y^2$ commute with all $a \in A$.

Proof. We will prove the statement for $X^2$, the case of $Y^2$ is analogous. We start with elements from $R$. As $R$ is commutative any $h \in R$ is normal. Write $h = h_1 + ih_2$, where $h_1 = (h + h^*)/2$, $h_2 = (h - h^*)/(2i)$. Then the operators $h_1, h_2$ are self-adjoint. For $i = 1, 2$ we have $Xh_i = \sigma(h_i)X$. Applying the Fuglede-Putnam-Rosenblum Theorem ([4, Theorem 12.16]) we get $Xh_i^* = Xh_i = \sigma(h_i)^*X$. Hence $\sigma(h_i)X = \sigma(h_i)^*X$. Using $\sigma(h_i)^* = \sigma^{-1}(h_i) = \sigma^{-1}(h_i^*)$ we get $(\sigma(h_i) - \sigma^{-1}(h_i))X = 0$, which implies $(\sigma(h) - \sigma^{-1}(h))X = 0$ for any $h \in R$. Substituting $h$ with $\sigma(h)$ we get $(\sigma^2(h) - h)X = 0$ and $(\sigma^2(h) - h)X^2 = 0$. The last is equivalent to $hX^2 = X^2h$.

For $a = X$ the statement is clear. The only case left is $a = Y$. Using the equality $(\sigma^2(h) - h)X = 0$ obtained above, we have $X^2Y = X\sigma(t) = \sigma^2(t)X = tX = YX^2$. □

Lemma 3.2. For any $h \in R$ the elements $h\sigma(h)$ and $h + \sigma(h)$ commute with all $a \in A$.

Proof. Using $(\sigma^2(h) - h)X = 0$ we have that $X\sigma(h)h = \sigma^2(h)\sigma(h)X = h\sigma(h)X$ and analogously $h\sigma(h)Y = Yh\sigma(h)$, $(h+\sigma(h))X = X(h+\sigma(h))$ and $(h+\sigma(h))Y = Y(h+\sigma(h))$, completing the proof. □

Lemma 3.3. Assume that $\pi$ is irreducible. Then the spectrum of all elements $h \in R$ is discrete, in particular, there exists a non-zero common (for $R$) eigenvector $v \in H$.

Proof. Assume that $\pi$ is irreducible. Fix $h \in R$. As $h\sigma(h)$ and $h + \sigma(h)$ commute with the action of $A$ and $\pi$ is irreducible these operators act as scalars, $c_1$ and $c_2$ correspondingly, on $H$. Hence $h^2 - c_2h + c_1 = 0$, which implies our statement. □

Lemma 3.4. If $X^2 \neq 0$ then $H$ coincides with the linear span of $v$ and $Xv$. Otherwise, $H$ coincides with the linear span of $v$ and $Yv$. In particular, $H$ is at most two-dimensional.

Proof. As $X^2$ is central, $X^2 = cI$, where $I$ is the identity operator. First assume $c \neq 0$. Then the linear span $L$ of $v$ and $Xv$ is invariant with respect to $R$ and $X$. We have $Y(av + bXv) = c^{-1}YX^2(av + bXv) = c^{-1}tX(av + bXv) \in L$. This proves the first statement.

If $c = 0$ we have $X^2 = 0$, in particular, $X^*X = rX^2 = 0$ which implies $X = 0$. Now if $Y^2 = 0$ then $Y = 0$ by the same arguments and the statement is clear. Otherwise, the second statement follows by the similar arguments as the first one. □

To state our main classification result in the rank one case we introduce two sets of parameters, $\Gamma_1$ and $\Gamma_2$. Let $\hat{R}$ denote the set of all characters of $R$. Put

$$\Gamma_1 = \{ \chi \in \hat{R} | \chi(h) = \chi(\sigma^2(h)) \text{ for all } h \in R, \chi(h) \neq \chi(h) \text{ for some } h \};$$

$$\Gamma_2 = \{ \chi \in \hat{R} | \chi(h) = \chi(\sigma(h)) \text{ for all } h \in R \}.$$

Theorem 3.1. Any irreducible representation of $A$ is one or two-dimensional and unitarily equivalent to one of the following representations:
1. $\pi_0(\chi)$: $X = Y = 0$, $h = \chi(h)$, where $\chi \in \hat{R}$ such that $\chi(t) = \chi(\sigma(t)) = 0$;

2. $\pi_1(\chi,c,d)$: $X = c$, $Y = d$, $h = \chi(h)$, where $\chi \in \Gamma_2$, $c,d \in \mathbb{C}$ such that $cd = \chi(t)$, $|c| + |d| \neq 0$, $\chi(r)c = \bar{c}$, $\chi(s)d = \bar{d}$;

3. $\pi_2(\chi,d)$:

   $$X = 0; \quad Y = \begin{pmatrix} 0 & \sqrt{d/\chi(s)} \\ \sqrt{d/\chi(s)} & 0 \end{pmatrix}; \quad h = \begin{pmatrix} \chi(h) & 0 \\ 0 & \chi(\sigma(h)) \end{pmatrix},$$

   where $\chi \in \Gamma_1$, $d \in \mathbb{C}$ such that $\chi(t) = \chi(\sigma(t)) = 0$, $\chi(s)d > 0$.

4. $\pi_3(\chi,c)$:

   $$X = \begin{pmatrix} 0 & \sqrt{c/\chi(r)} \\ \sqrt{c/\chi(r)} & 0 \end{pmatrix}; \quad Y = 0; \quad h = \begin{pmatrix} \chi(h) & 0 \\ 0 & \chi(\sigma(h)) \end{pmatrix},$$

   where $\chi \in \Gamma_1$, $c \in \mathbb{C}$ such that $\chi(t) = \chi(\sigma(t)) = 0$, $\chi(r)c > 0$.

5. $\pi_4(\chi,c,d)$:

   $$X = \begin{pmatrix} 0 & \sqrt{c/\chi(r)} \\ \sqrt{c/\chi(r)} & 0 \end{pmatrix}; \quad Y = \begin{pmatrix} 0 & \chi(t)/\sqrt{c/\chi(r)} \\ \chi(\sigma(t))/\sqrt{c/\chi(r)} & 0 \end{pmatrix}; \quad h = \begin{pmatrix} \chi(h) & 0 \\ 0 & \chi(\sigma(h)) \end{pmatrix},$$

   where $\chi \in \Gamma_1$, $c,d \in \mathbb{C}$, such that $\chi(r)c > 0$, $\chi(s)d > 0$, $\chi(t\sigma(t)) = cd$.

The representations $\pi_i(\chi_1, c_1, d_1)$ and $\pi_j(\chi_2, c_2, d_2)$ (the presence of $c_i$ and $d_i$ depends on $i,j$) are unitarily equivalent if and only one of the following holds:

1. $i = j = 1$ and $\chi_1 = \chi_2$;

2. $i = j = 2$, $c_1 = c_2$, $d_1 = d_2$ and $\chi_1 = \chi_2$;

3. $i = j = 3$, $c_1 = c_2$ and $\chi_1 = \chi_2$ or $\chi_1 = \chi_2 \circ \sigma$;

4. $i = j = 4$, $d_1 = d_2$ and $\chi_1 = \chi_2$ or $\chi_1 = \chi_2 \circ \sigma$;

5. $i = j = 5$, $c_1 = c_2$, $d_1 = d_2$ and $\chi_1 = \chi_2$ or $\chi_1 = \chi_2 \circ \sigma$.

**Proof.** Let $\pi$ be an irreducible $*$-representation of $A$. From Lemmas 3.1-3.4 we know that $\pi$ is at most two-dimensional and $X^2, Y^2$ act as scalars, say $c$ and $d$ respectively, on $H$.

First we assume $c = d = 0$. From the proof of Lemma 3.4 we have $X = Y = 0$ and hence $\pi(A)$ is commutative. Therefore $\pi$ is one-dimensional and is completely determined by $\chi \in \hat{R}$. From $X = Y = 0$ we also have $t = \sigma(t) = 0$ and $\pi$ is equivalent to $\pi_0$. 

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Now let $|c| + |d| \neq 0$ and $v$ be a normed common eigenvector for $R$, given by Lemma 3.3. This means $hv = \chi(h)v$, $h \in R$, for some $\chi \in R$. As $|c| + |d| \neq 0$ either $X^2$ or $Y^2$ is non-zero. Assume $X^2 \neq 0$. Then from $X^2 h = \sigma^2(h)X^2$ we get $h = \sigma^2(h)$ for any $h \in R$ (for $Y^2$ the arguments are similar). Now we consider two cases: $\chi \in \Gamma_1$ or $\chi \in \Gamma_2$. If $\chi \in \Gamma_2$ then $\pi(A)$ is commutative and therefore $\pi$ is one-dimensional. We have $X = c', Y = d', XY = t$ implies $c'd' = \chi(t)$, $X^* = rX$ (resp. $Y^* = sY$) implies $\bar{c} = \chi(r)c'$ (resp. $\bar{d}' = \chi(s)d'$) and $\pi$ is equivalent to $\pi_1$.

If $\chi \in \Gamma_1$ then we consider three cases: $c = 0$, $d = 0$, $c \neq 0$ and $d \neq 0$. If $c = 0$ then $v$ and $Yv$ generate $H$ and they are orthogonal as eigenvectors of a normal operator ($h \in R$ such that $\chi(h) \neq \chi(\sigma(h))$) with different eigenvalues. Writing our representation in the orthonormal basis $v$, $Yv/||Yv||$ (where $s = \sqrt{\chi(s)}$) we get the representation $\pi_2$. Since $X^*X = rX^2$ and $Y^*Y = sY^2$ we have $\chi(r)c \geq 0$ and $\chi(s)d \geq 0$. Analogously, if $d = 0$ one gets that $\pi$ is equivalent to $\pi_3$ by the similar arguments (for the basis $v$, $Xv$). If $c \neq 0$ and $d \neq 0$ then we get $\pi \simeq \pi_4$, using the basis $v$, $Xv$.

The statement about unitary equivalence is obvious. 

\section{Case of arbitrary rank}

We fix an irreducible representation, $\pi$, of $A$ by bounded operators on a separable Hilbert space, $H$, and denote by $a$ the image of $a$ under $\pi$ for all $a \in A$.

\begin{lemma}
The elements $X_i^2$ and $Y_i^2$, $i \in N_n$, commute with all $a \in A$.
\end{lemma}

\begin{proof}
Follows from Lemma 3.1 and the defining relations. 
\end{proof}

Set $G = \prod_{i \in N_n} Z_2^{(i)}$ (n copies of $Z_2$). For $h \in R$ and $g = (\varepsilon_1, \ldots, \varepsilon_n) \in G$ set $g(h) = \sigma_1^{\varepsilon_1} \cdots \sigma_n^{\varepsilon_n}(h)$. Write $G = \{g_1, g_2, \ldots, g_{2^n}\}$. For $h \in R$ and a symmetric polynomial, $f$, in $2^n$ variables put $u(f, h) = f(g_1(h), g_2(h), \ldots, g_{2^n}(h))$.

\begin{lemma}
The elements $u(f, h)$ commute with all $a \in A$. In particular, all $h \in R$ have a discrete spectrum and there exists a non-zero common (for $R$) eigenvector, $v \in H$.
\end{lemma}

\begin{proof}
Clearly, $u(f, h)$ commute with $R$. From the proof of Lemma 3.1 it follows that $(\sigma_2^2(h) - h)X_i = 0$. This implies $X_i u(f, h) = u(f, \sigma_i(h))X_i = u(f, h)X_i$. The arguments for $Y_i$ are similar completing the proof of the first statement.

From the first statement it follows that all $u(f, h)$ are scalar operators on $H$. Let $f_1, \ldots, f_{2^n}$ be elementary symmetric polynomials. Then each $u(f_i, h)$ is a scalar, say $\gamma_i = \gamma_i(h)$, and we have $h^{2^n} - \gamma_1 h^{2^n-1} + \cdots + \gamma_{2^n} = 0$. This implies the second statement.
\end{proof}

Let $X_i^2$ (resp. $Y_i^2$) act with the scalar $c_i$ (resp. $d_i$) on $H$. Set $N(\pi) = \{i : |c_i| + |d_i| \neq 0\}$ and for each $i \in N(\pi)$ fix $Z_i = X_i$ if $c_i \neq 0$ and $Z_i = Y_i$ otherwise.

\begin{lemma}
The set $B = \{\prod_{i \in N(\pi)} Z_i^{e_i} v\}$, $e_i \in \{0, 1\}$, generates $H$. In particular, we have $\dim(H) \leq 2^{\dim(N(\pi))} \leq 2^n$.
\end{lemma}
Proof. The invariance of the linear span of $B$ under $A$ follows from Lemma 3.4 and the relations $X_iX_j = X_jX_i$, $Y_iY_j = Y_jY_i$ and $X_iY_j = Y_jX_i$, $i \neq j$, in $A$. □

Now we can start the classification. Our representations will be indexed by vectors $u = (\tau_i)_{i \in N}$, where each $\tau_i$ is an irreducible representation of the generalized Weyl algebra $A_i = A_i(R, \sigma_i, t_i)$, $i \in N$, associated with a common character, $\chi \in \hat{R}$. Let $H_i$ denote the Hilbert space of the representation $\tau_i$. Let $N$ be the set of all $i$ such that $\dim(H_i) = 2$. For $i \not\in N$ fix a normed element, $v_i^0 \in H_i$. For $i \in N$ fix an orthonormal basis $v_i^0, v_i^1$ in $H_i$ provided by Theorem 3.1. Set $H(u) = H_1 \otimes H_2 \otimes \cdots \otimes H_n$. Then $\dim(H(u)) = 2^{|N|}$ and \{v_{(\varepsilon_1, \ldots, \varepsilon_n)}^1 \otimes \cdots \otimes v_n^1 \mid \varepsilon_i = 0 \text{ for } i \not\in N \text{ and } \varepsilon_i = 0, 1 \text{ for } i \in N\} \text{ is an orthonormal basis of } H(u).$ Set

\[ h v_{(\varepsilon_1, \ldots, \varepsilon_n)} = \chi(\sigma_1^{\varepsilon_1} \circ \cdots \circ \sigma_n^{\varepsilon_n}(h)) v_{(\varepsilon_1, \ldots, \varepsilon_n)}, \quad h \in H(u); \]

\[ X_i v_{(\varepsilon_1, \ldots, \varepsilon_n)} = v_1^{\varepsilon_1} \otimes \cdots \otimes v_{i-1}^{\varepsilon_{i-1}} \otimes \tau_i(X_i) v_i^{\varepsilon_i} \otimes v_{i+1}^{\varepsilon_{i+1}} \otimes \cdots \otimes v_n^{\varepsilon_n}; \quad i \in N; \]

\[ Y_i v_{(\varepsilon_1, \ldots, \varepsilon_n)} = v_1^{\varepsilon_1} \otimes \cdots \otimes v_{i-1}^{\varepsilon_{i-1}} \otimes \tau_i(Y_i) v_i^{\varepsilon_i} \otimes v_{i+1}^{\varepsilon_{i+1}} \otimes \cdots \otimes v_n^{\varepsilon_n}; \quad i \in N. \]

We will denote this representation by $\pi(u)$.

**Lemma 4.4.** $\pi(u)$ is a s-representation of $A$.

Proof. Recall that $\tau_i$ are representations of $A$, which correspond to the same $\chi \in \hat{R}$. The relations $X_iY_j = Y_jX_i$, $X_iX_j = X_jX_i$ and $Y_iY_j = Y_jY_i$ are obvious. Now consider $X_i h = \sigma_i(h) X_i$ and apply this equality to $v = v_{(\varepsilon_1, \ldots, \varepsilon_n)}$. If $X_i = Y_i = 0$, the relation is obvious. If $\varepsilon_i = 1$ the relation follows directly from the definition of the action of $R$. Otherwise we have $\chi(\sigma_i^2(h)) = \chi(h)$ by Theorem 3.1 and also $X_i h v = \chi(\sigma_1^{\varepsilon_1} \circ \cdots \circ \sigma_n^{\varepsilon_n}(h))X_i v$ and $\sigma_i(h) X_i v = \chi(\sigma_1^{\varepsilon_1} \circ \cdots \circ \sigma_n^{\varepsilon_n}(\sigma_i^2(h)))X_i v$ which is the same. One can use similar arguments to get $h Y_i = Y_i \sigma_i(h)$. Using $\sigma_j(t_i) = t_i$, $i \neq j$, the relations $Y_iX_i = t_i$ and $X_iY_i = \sigma_i(t_i)$ can be easily reduced to the same relations between $\tau_i(X_i)$ and $\tau_i(Y_i)$. This completes the proof. □

Assume that $\pi(u)$ does exist and $\chi$ is the corresponding character of $R$. Set $N = \{i \in N \mid \chi(h) \neq \chi(\sigma_i(h)) \text{ for some } h \in R\}$ where $R = \bigcap \{i \in N \mid X_i \neq 0 \text{ or } Y_i \neq 0\} \subset N$. Denote by $\hat{G} = \prod_{i \in N} \mathbb{Z}_{2^i}$ the corresponding subgroup of $G$ and by $W(u)$ the subgroup of $\hat{G}$ consisting of all $g \in \hat{G}$ such that $\chi(g(h)) = \chi(h)$ for all $h \in R$. Clearly, $|W(u)| = 2^k$ for some $k$.

**Lemma 4.5.** $\pi(u)$ decomposes into $|W(u)|$ pairwise unitarily non-equivalent irreducible representations. In particular, $\pi(u)$ is irreducible if and only if $W(u)$ is trivial.

Proof. For each $w \in W(u)$ we define a monomial, $X(w) \in A$, as follows: let $w = (\varepsilon_i)_{i \in N}$; for each $i \in N$, $\varepsilon_i = 1$, we set $Z_i = X_i$ if $X_i$ is not zero on $H(u)$ otherwise $Z_i = Y_i$; set $X(w) = \prod_{i \in N} Z_i$. We claim that the elements $X(w), w \in W(u)$, commute with the action of $A$ on $H(u)$. Indeed, fix $X(w), w = (\varepsilon_i)_{i \in N} \in W(u)$. That $X(w)$ commutes with $R$ follows from $\chi(w(h)) = \chi(h), w \in W(u)$. Fix $i \in N$. If $X(w)$ contains $X_i$ (resp. $Y_i$) then $X(w)X_i = X_iX(w)$ (resp. $X(w)Y_i = Y_iX(w)$). If $X(w)$ contains neither $X_i$ nor $Y_i$ then clearly $X(w)X_i = X_iX(w)$ and $X(w)Y_i = Y_iX(w)$. Now assume that $X(w)$ contains $X_i$
(for $Y_i$ the arguments are similar). Decompose $w = \sigma_i \circ w'$, where $w'$ does not contain $\sigma_i$. Then $\chi(\sigma_i \circ w'(t_i)) = \chi(t_i)$ and hence $\chi(\sigma_i(t_i)) = \chi(t_i)$. This implies $X_iY_i = Y_iX_i$ and therefore $X(w)Y_i = Y_iX(w)$, proving our claim. In particular, all $X(w)$ are normal operators.

By the definition of $\pi(v)$, each $(X(w))^2$ is a scalar operator with some scalar $\lambda(w) \neq 0$. Hence the spectrum of $X(w)$ is a subset of $\{ \pm \sqrt{\lambda(w)} \}$. As $v_{(0, \ldots, 0)}$ is orthogonal to $X(w)v_{(0, \ldots, 0)}$ we get that spectrum of $X(w)$ coincides with $\{ \pm \sqrt{\lambda(w)} \}$. Let $\{w_1, \ldots, w_k\}$ be a minimal set of generators of $W(v)$. Decomposing $\pi(v)$ with respect to the common spectrum of $X(w_1), \ldots, X(w_k)$ we get $|W(v)|$ non-zero subrepresentations of $\pi(v)$. From the definition of $W(v)$ it follows that each of these subrepresentations has a basis, whose elements are separated by the action of $R$ (i.e. correspond to pairwise distinct characters of $R$). Moreover, as $\pi(v)$ is cyclic, each subrepresentation of it is also cyclic. Altogether this implies that all our subrepresentations are irreducible. The statement that they are pairwise unitarily non-equivalent is obvious.

Denote by $W$ the group, generated by all $\sigma_j$, $j \in N_n$. $W$ acts on the set of representations of $A_i$ in a natural way.

**Lemma 4.6.**

1. $\pi((\tau(1))_i)$ and $\pi((\tau(2))_i)$ are unitarily equivalent if and only if $\tau(1)_i$ and $\tau(2)_i$ belong to the same orbit of the action of $W$ for all $i \in N_n$.

2. If representations $\pi((\tau(1))_i)$ and $\pi((\tau(2))_i)$ are unitarily non-equivalent then any irreducible component of $\pi((\tau(1))_i)$ is unitarily non-equivalent to any irreducible component of $\pi((\tau(2))_i)$.

**Proof.** Obvious. \hfill \Box

**Theorem 4.1.** Any irreducible $\ast$-representation of $A$ is unitarily equivalent to an irreducible subrepresentation of some $\pi(v)$.

**Proof.** Let $\pi$ be an irreducible representation of $A$ on a Hilbert space $H$. We claim that it is equivalent to an irreducible subrepresentation of $\pi(v)$ for a suitable $v$. From $X_jX_i = X_iX_j$, $X_jY_i = Y_iX_j$, $Y_jX_i = Y_iX_j$, $Y_jY_i = Y_iY_j$ it follows that $\pi|_{A_i}$ decomposes into a direct sum of irreducible representations of $A_i$ which belong to the same orbit of $W$-action. For each $i \in N_n$ fix one representation from the corresponding orbit, say $\tau_i$. As above, set $N = \{i \in N_n \mid \chi(h) \neq \chi(\sigma_i(h))$ for some $h \in R\} \cap \{i \in N_n \mid X_i \neq 0 \text{ or } Y_i \neq 0\}$. Consider $W(\pi)$ and $X(w)$ associated with $\pi$ and $w \in W(\pi)$, as defined in the proof of Lemma 4.5. By the same arguments as there we get that $X(w)$ commute with all $a \in A$ and hence are scalar operators on $H$ with scalars $\pm \sqrt{\lambda(w)}$ (the choice of signs is coordinated by the group structure on $W(\pi)$). Consider the quotient group $G(\pi) = (\prod_{i \in N} \mathbb{Z}_2 / W(\pi))$ and fix some representatives $\Lambda = \{w_1, \ldots, w_{|G(\pi)|}\}$ in the corresponding cosets. For $w \in \Lambda$ define $X(w)$ as above. Let $v$ be a common eigenvector for $R$ in $H$. Then the linear span of $X(w)v$, $w \in \Lambda$, is an orthogonal basis of $H$. Indeed, first we show that this linear span is invariant under $A$. This is clear for $R$. If $\pi(X_i) = 0$, the statement is obvious for such $X_i$. Otherwise, using the fact that $X_i^2 = X(w)$, $w \in W(\pi)$, are scalars on $H$, we have $X_iX(w) = cX(w)$.
for some uniquely defined $c \in C$ and $w' \in \Lambda$. The same argument works for $Y_i$ and we get the necessary invariance. By the construction, all $X(w)v$, $w \in \Lambda$, are common eigenvectors for $R$. Now the orthogonality follows from the fact, that the corresponding common eigenvalues are distinct (definition of $W(\pi)$) and all operators $h \in R$ are normal.

In particular, we get that the irreducible representation with parameters $\tau_i$, $i \in N_n$, and $\pm \sqrt{\lambda(w)}$, $w \in W(\pi)$, is unique up to a unitary equivalence. Therefore it is unitarily equivalent to an irreducible subrepresentation of $\pi((\tau_i)_{i \in N_n})$. 

\textbf{Corollary 4.1.} The dimension of any irreducible $*$-representation of $A$ equals $2^k$ for some $k \in \mathbb{Z}_+$. 

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\section*{References}


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