VARIOUS FACES OF OPERATOR SYNTHESIS

Victor Shulman

Department of Mathematics, Vologda State Technical University,
15 Lenin Str., Vologda, 160000, Russia

Lyudmila Turowska

Department of Mathematics, Chalmers University of Technology,
412 96 Göteborg, Sweden

The notion of synthesis for operator algebras and subspace lattices was introduced by W. Arveson [A] in connection with some problems of the Invariant Subspace Theory. In [A] there was also established and made use of the relation of the concept to harmonic analysis. Our aim in this paper is to show that operator synthesis has other sides, related to measure theory, operator theory, approximation theory, tensor products and linear operator equations. Each of them brings to the topic its own techniques and its own circle of problems.

SYNTHETIC SETS

We start with the measure theory. Let $(X, \mu), (Y, \nu)$ denote $\sigma$-finite separable spaces with measures. We use standard measure-theoretic terminology. A subset of Cartesian product $X \times Y$ is said to be a measurable rectangle if it has the form $A \times B$ with measurable $A \subseteq X$, $B \subseteq Y$. A set $E \subseteq X \times Y$ is called \textit{marginally null set} if $E \subseteq (X_1 \times Y) \cup (X \times Y_1)$, where $\mu(X_1) = \nu(Y_1) = 0$. We define $w$-topology on $X \times Y$ (see [EKS]) such that the $w$-open (pseudo-open) sets are, modulo marginally null sets, countable union of measurable rectangles. The complements of $w$-open sets are called $w$-closed (pseudo-closed).

Let $\Gamma(X, Y) = L_2(X, \mu) \hat{\otimes} L_2(Y, \nu)$ be the projective tensor product, i.e. the space of functions on $X \times Y$ which admit a representation

$$\Phi(x, y) = \sum_{n=1}^{\infty} f_n(x) g_n(y)$$

(1)

where $f_n \in L_2(X, \mu), g_n \in L_2(Y, \nu)$ and $\sum_{n=1}^{\infty} ||f_n||_{L_2} \cdot ||g_n||_{L_2} < \infty$. The function $\Phi$ is defined marginally almost everywhere (m.a.e.) in that if $f_n, g_n$ are changed on null sets then $\Phi$ will change on a marginally null set. $L_2(X, \mu) \hat{\otimes} L_2(Y, \nu)$-norm of such a function $\Phi$ is

$$||\Phi||_\Gamma = \inf \sum_{n=1}^{\infty} ||f_n||_{L_2} \cdot ||g_n||_{L_2},$$

where the infimum is taken over all such sequences $f_n, g_n$ for which (1) holds m.a.e. In what follows we identify two functions in $\Gamma(X, Y)$ which coincides m.a.e. By [EKS][Theorem 6.5], any function $F \in \Gamma(X, Y)$ is pseudo-continuous (continuous with respect to the $w$-topology defined above). We say that $F \in \Gamma(X, Y)$ vanishes on a pseudo-closed set $K \subseteq X \times Y$ if
$F \chi_K = 0 \ (\text{m.a.e})$, where $\chi_K$ is the characteristic function of $K$. For arbitrary $K \subseteq X \times Y$ denote by $\Phi(K)$ all functions $\Phi \in \Gamma(X, Y)$ vanishing on $K$. Clearly $\Phi(K)$ is a subspace of $\Gamma(X, Y)$. Moreover, one can show that $\Phi(K)$ is closed. Given an arbitrary subset $F \subseteq \Gamma(X, Y)$, we define the null set of $F$ to be the largest, to within a marginally null sets, pseudo-closed set such that any function $F \in F$ vanishes on it. Such set always exists.

Let $\Phi_0(K)$ be the closure in $\Gamma(X, Y)$ of the set of all functions which vanish on neighbourhoods (pseudo-open sets) of $K$. $\Phi_0(K)$ is a closed subspace of $\Phi(K)$. Clearly, the subspaces $\Phi_0(K)$ and $\Phi(K)$ are invariant with respect to the multiplication by functions $f \in L_\infty(X, \mu)$ and $g \in L_\infty(Y, \nu)$ (we say simply invariant).

**Theorem 1.** If $A \subseteq \Gamma(X, Y)$ is an invariant closed subspace such that null $A = K$ then $\Phi_0(K) \subseteq A \subseteq \Phi(K)$.

This theorem justifies the following definition.

**Definition 1.** We say that a pseudo-closed set $K \subseteq X \times Y$ is a set of synthesis (with respect to measures $\mu$, $\nu$) if

$$\Phi_0(K) = \Phi(K).$$

Since $B(L_2(X, \mu), L_2(Y, \nu))$ is dual to the space $\Gamma(X, Y)$, the notion of set of synthesis can be also given in terms of operators instead of functions.

Let $H_1 = L_2(X, \mu)$, $H_2 = L_2(Y, \nu)$ and $B(H_1, H_2)$ be the space of bounded linear operators from $H_1$ to $H_2$. We say that $T \in B(H_1, H_2)$ is supported in $E \subseteq X \times Y$ if

$$P_UTQ_Y = 0 \text{ for each sets } U \subseteq Y, \ V \subseteq X \text{ such that } (U \times V) \cap E = \emptyset,$

and define

$$\mathcal{M}_{\text{max}}(E) = \{T \in B(H_1, H_2) \mid T \text{ is supported in } E\}.$$

For any subset $A \subseteq B(H_1, H_2)$ there exists the minimal, to within a marginally null set, of pseudo-closed sets, in which all the operator $T \in A$ are supported. This set will be called the support of $A$. The support of an operator $T \in B(H_1, H_2)$ will be denoted by $\text{supp } T$. It follows from [A] that every pseudo-closed set $E$ is a support of the family $\mathcal{M}_{\text{max}}(E)$.

Let $\mathcal{P}$ be the space of all projections on the separable Hilbert space $l_2$, equipped with the topology of strong convergence. We say that measurable functions $P : X \rightarrow \mathcal{P}$ and $Q : Y \rightarrow \mathcal{P}$ form an $E$-pair if $P(x)Q(y)$ vanishes on $E$. If, additionally, $P$ and $Q$ take only finitely many values then the pair $(P, Q)$ is said to be a simple $E$-pair. Define

$$\mathcal{M}_{\text{min}}(E) = \{T \in B(H_1, H_2) \mid Q(T \otimes 1)P = 0 \text{ for any } E\text{-pair } (P, Q)\},$$

where $1$ is the identity operator in $B(l_2)$. If $E$ is pseudo-closed one can prove that $\mathcal{M}_{\text{max}}(E)$ and $\mathcal{M}_{\text{min}}(E)$ are the maximal and respectively the minimal weak*-closed $L^\infty(X, \mu) \times L^\infty(Y, \nu)$-submodule of $B(H_1, H_2)$ whose support equals $E$.

**Theorem 2.** Let $E \subseteq X \times Y$ be a pseudo-closed set. Then the following are equivalent:

(i) $E$ is a set of synthesis;

(ii) $\mathcal{M}_{\text{min}}(E) = \mathcal{M}_{\text{max}}(E)$;

(iii) $\langle T, F \rangle = 0$ for any $T \in B(H_1, H_2)$ and any $F \in \Gamma(X, Y)$ such that

$$\text{supp } T \subseteq E \subseteq \text{null } F;$$

(iv) any $E$-pair can be approximated a.e. by simple $E$-pairs.
The equivalence of (i) and (iii) was essentially proved in [A]. As a consequence of the equivalence (i) \(\Leftrightarrow (iv)\) we obtain the following

**Theorem 3.** Let \((X, \mu), (Y, \nu), (X_1, \mu_1)\) and \((Y_1, \nu_1)\) be Borel spaces with measures, \(\varphi : X \Rightarrow X_1, \psi : Y \Rightarrow Y_1\) Borel mappings. Suppose that the measures \(\varphi_* \mu\) and \(\psi_* \nu\) are absolutely continuous with respect to the measures \(\mu_1\) and \(\nu_1\). If a Borel set \(E_1 \subseteq X_1 \times Y_1\) is a set of \(\mu_1 \times \nu_1\)-synthesis then \((\varphi \times \psi)^{-1}(E_1)\) is a set of \(\mu \times \nu\)-synthesis.

Suppose that \(\varphi_i\) and \(\psi_i, i = 1, \ldots, n,\) are Borel functions of the spaces \((X, \mu)\) and \((Y, \nu)\) into an ordered Borel space \((Z, \leq)\). Then the set \(E = \{(x, y) \mid \varphi_i(x) \leq \psi_i(y), i = 1, \ldots, n\}\) is called a set of width \(n\). W.Arveson has proved in [A] that every commutative subspace lattice of finite width is synthetic. Synthesizability of special sets of width two ("nontriangular" sets) was proved in [KT, Sh1].

**Corollary 1.** Any set \(E \subseteq X \times Y\) of finite width is a set of \(\mu \times \nu\)-synthesis.

The same result was proved by I.G.Todorov, [T], using another method.

**LINEAR OPERATOR EQUATIONS**

Let \(H_1 = L_2(X, \mu), H_2 = L_2(Y, \nu)\). Consider the following linear operator equations in \(B(H_1, H_2)\):

\[
\sum_{i=1}^{\infty} B_i T A_i = 0,
\]

where \(A_i, B_i\) are the operators of multiplication by bounded measurable functions \(a_i \in L^\infty(X, \mu), b_i \in L^\infty(Y, \nu),\) and

\[
\sum_{i=1}^{\infty} A_i^* A_i \in B(H_1) \quad \text{and} \quad \sum_{i=1}^{\infty} B_i B_i^* \in B(H_2).
\]

The convergence of the partial sum of \(\sum B_i T A_i\) is in the weak*-topology. To any operator equation (2) we associate the characteristic function

\[
F(x, y) = \sum_{i=1}^{\infty} a_i(x) b_i(y)
\]

and the characteristic set

\[
E = \{(x, y) \in X \times Y \mid F(x, y) = 0\}
\]

of the direct product \(\{X \times Y; \mu \otimes \nu\}\).

**Theorem 4.** A pseudo-closed set \(E \subseteq X \times Y\) is a set of synthesis if and only if all linear operator equations (2) whose characteristic set equals \(E\) are equivalent.
SOME TENSOR PRODUCTS

Let $(X, \mu)$ and $(Y, \nu)$ be measure spaces with finite measures. Set $R = L_\infty(X, \mu)$, $S = L_\infty(Y, \nu)$ and let $Bil^\sigma(R, S)$ denote the space of all bounded bilinear functions on $R \otimes S$ which are weak-* continuous in each variable with the norm

$$||F|| = \sup\{||F(v, w)|| : ||v||_\infty = ||w||_\infty = 1\}.$$

Here $|| \cdot ||_\infty$ is the essential supremum norm. We let $H_1$, $H_2$ denote the Hilbert spaces $L_2(X, \mu)$ and $L_2(Y, \nu)$ respectively. From [EK] it follows that $Bil^\sigma(R, S)$ is the space of bilinear functionals with representations $(gTx\xi, \eta)$, where $T \in B(H_1, H_2)$ and $\xi \in H_1$, $\eta \in H_2$.

Define $V^\infty(X, Y)$ to be the space of functions on $X \times Y$ which admit a representation $u(x, y) = \sum f_i(x)g_i(y)$, where $f_i \in L_\infty(X, \mu)$, $g_i \in L_\infty(Y, \nu)$, satisfying

$$||\sum |f_i|^2||_\infty||\sum |g_i|^2||_\infty < \infty.$$

Clearly, $V^\infty(X, Y) \subset \Gamma(X, Y)$. One can also show that $V^\infty(X, Y) \subseteq Bil^\sigma(R, S)^*$ (see [EK]). We endow the space $V^\infty(X, Y)$ with $\sigma(V^\infty(X, Y), Bil^\sigma(R, S))$-topology. For arbitrary pseudo-closed set $K \subset X \times Y$ we denote by $\Phi(K)$ the set of all functions $\sum f_i(x)g_i(y) \in V^\infty(X, Y)$, which vanish on $K$, and by $\Phi_0(K)$ the closure of all functions in $V^\infty(X, Y)$ vanishing in neighbourhoods (pseudo-open sets) of $K$.

Analogously, a pseudo-closed set $E \subset X \times Y$ is said to be a set of synthesis with respect to $V^\infty(X, Y)$ if $\Phi_0(E) = \Phi(E)$.

**Proposition 1.** A pseudo-closed set $E \subset X \times Y$ is a set of $\mu \times \nu$-synthesis iff it is a set of synthesis with respect to $V^\infty(X, Y)$.

OPERATOR SYNTHESIS AND SYNTHESIS IN TENSOR ALGEBRAS

Let $X$ and $Y$ be compact spaces. Consider the Varopoulos tensor algebra $V(X, Y) = C(X) \otimes C(Y)$ which consists of all functions $\Phi \in C(X \times Y)$ which admit a representation $\Phi(x, y) = \sum_{i=1}^\infty f_i(x)g_i(y)$, where $f_i \in C(X)$, $g_i \in C(Y)$ and

$$\sum_{i=1}^\infty ||f_i||_{C(X)}||g_i||_{C(Y)} < \infty.$$

$V(X, Y)$ is a Banach algebra with the norm

$$||\Phi(x, y)||_V = \inf_{i=1}^\infty \sum_{i=1}^\infty ||f_i||_{C(X)}||g_i||_{C(Y)},$$

where inf is taken over all representations of $\Phi$ in the form $\sum f_i \otimes g_i$ (see [V]).

Next theorem connects operator synthesis and synthesis with respect to the Varopoulos algebra $V(X, Y)$. Recall that a closed set $E \subset X \times Y$ is a set of spectral synthesis in $V(X, Y)$ if

$$\langle B, F \rangle = 0$$

for any $B \in V(X, Y)^*$, supp$(B) \subseteq E$, and any $\Phi \in V(X, Y)$ which vanishes on $E$. Any element of $V(X, Y)^*$ can be identified with a bounded bilinear form $\langle B, f \otimes g \rangle = B(f, g)$ (bimeasure). Let $M(X)$, $M(Y)$ be the sets of finite Borel measures on $X$ and $Y$ respectively.
Theorem 5. If a closed set \( E \subseteq X \times Y \) is a set of synthesis with respect to any pair of measures \( (\mu, \nu), \mu \in M(X), \nu \in M(Y) \), then \( E \) is synthetic with respect to \( V(X,Y) \).

Theorem 1 and Theorem 5 give now

Corollary 2. Suppose that \( \varphi_i : X \rightarrow Z \) and \( \psi_i : Y \rightarrow Z \), \( i = 1, \ldots, n \), are continuous functions from compact metric spaces \( X \) and \( Y \) to an ordered compact metric space \( Z \). Then the set \( E = \{(x,y) \mid \varphi_i(x) \leq \psi_i(y), i = 1, \ldots, n\} \) is a set of synthesis with respect to the algebra \( V(X,Y) \).

This corollary yields the theorem of Drury on synthesizability of “nontriangular” sets, which are sets of width two (see [D]).

References


