What is Platonism in Mathematics?

A First Attempt

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The Ontology of Platonism

Cogito ergo sum. A few meaningless combinations of characters to those of us who are not fluent in Latin. *I think therefore I am* when translated in your native tongue[1] strikes you as a bolt from heaven, at least to those of us with a philosophical temperament. It does strike at one of the basic features of philosophy, namely its ontology. What exists. It is also phrased in the idealist tradition, what is given is something far more sophisticated than what is merely concluded. A stone does not think, yet it exists nevertheless (or does it? Or maybe, but not deservingly.). It assumes a thinking entity, well individuated as to separate it from the existence it is contemplating, yet being at the same time an object among others of that very contemplation. The idealist tradition is the top-down approach in philosophy, while the materialist tradition is the bottom-up, trying to explain everything from rock bottom. The eternal problem in philosophy is that you cannot separate the two, leading to endless paradoxes, self-references and contradictions. But that is what is given to us, and the more or less futile ambition of philosophy is to transcend it.

Our perception of reality is not direct, it is mediated by a theory of perception to make sense of the multifarious world of the senses. Thus in the process the external and the internal are inextricably intertwined. We often believe naively that direct perception is the ultimate test of truth. But there is of course no such thing as a direct, uninterpreted perception, all perceptions are constructions, and have to fit into our schemes of things. A silly, but not stupid, experiment suggested by Aristotle, is to cross your fingers and rub your nose. The senses tells us that we have two noses, our sense tells us that this is nonsense and explains how we can ever sense such a senseless thing. Theory takes precedence over sensation. The world is not just given, it is constructed[2].

My favourite pre-Socratic philosopher is Parmenides. He taught us that there is but one thing, that differences do not exist. On the face of it is of

[1] The original statement was in fact *Je pense, donc je suis*, rendering the Latin quote to mere 'peacockery'

[2] A slightly more sophisticated phenomenon struck me once when I was a graduate student staying in an apartment in Cambridge, Mass. Through the venetian blinds in the bathroom I saw a plane taking off from Logan Airport. The track of the plane was clearly zig-zagging through the air. But was it? Clearly not, concomitant with the sensation I also realized the reason for the illusion, which I will not insult the intelligence of any bi-ocular individual to explain explicitly.

course nonsense, but it does have a poetic truth to it, and constitutes in fact one of the major influences of western philosophy. Parmenides is a forerunner of Plato, telling us in fact to mistrust the confusing world of the senses and instead look for the unifying principles behind. Plato claimed that the tangible sensual reality with which we are familar through unchecked habit is just the shadowy world cast by a more perfect reality. This conception of Plato, immortalized (and occasionally confused) by the metaphor of cavemen staring at the shadows cast on the wall by unattainable objects illuminated by a fire, has been made much fun of. If taken too literally it degenerates into mere silliness. And maybe Plato took it too literally after all (although I would be careful to dimiss him so easily, irony which was his preferred medium of expression, does not travel well) but of course we are more interested in the Platonic sense of Platonism, than any merely historical manifestation of it that Plato may have happened to have. We are concerned about thought, not its history.

Naively what exists is what we can touch, smell or see. What impinges on us, whether we want it or not. In short what appears to have an objective existence independant of the thinking self, being it ours or that of Descartes. But the ostensibly concrete, turns out to be far more elusive than we would have thought from the beginning, which we need not be reminded of, children as we are of the scientific revolution. By thinking of objects we invariably conceive of abstractions, simplistically thought of as relations (whatever they are) between objects, and once you do that, the process can be inductively (and hence indefinitely) continued, eventually leading to such notions as higher cardinalities, to which we will return later. It is thought feeding on itself in a self-referential process. But is it anything but ephemeral clouds to be dispersed by the rays of the early sun? Clearly there is some kind of hierachy of existence, and Platonism is about taking this hierarchy seriously.

Platonism and Essence

The idea of form or essence lies at the core of the popular interpretation of Platonism. It is most easily explained in the context of emerging Greek geometry, eventually codified by Euclid. Geometry is about the real world, and out of its spatial character, certain concepts are being isolated, like those of points, lines, triangles, angles etc[3]. One may argue persuasively that such choices are in fact arbitrary and reflect accidents of history and

[3]It may be instructive for me to recall that my first systematic introduction to such concepts was of course not at the relatively late age of instruction in geometry, but in carpentry in elementary school. We were told that a line had no thickness, and indeed I convinced myself of that fact by looking at the edge of a planed piece of plank through a magnifying glass and observing no magnification of its thickness. To the mathematically inclined pupil, mathematics is not defined by the scheduled partition of the school day. Needless to say I was a very poor carpenter, but perked up when scaling and such things were occasionally brought up.

human brain architecture (the latter ultimately beng reduced to the former conceived on a geological scale) and that other civilizations or certainly other intelligences, even if sharing the same physical space, would have conceived of it in radically different terms. Leaving those comments aside for the moment, it becomes clear to everyone that the lines we draw in the sand, or chalk on the black-board or plot by computer, are not absolutely straight, nor infinitely thin (in the sense of not magnifying under magnification) or more to the point, as we will see below, not indefinitely extended. But how do we perceive of such imperfections when we cannot manifest the real thing physically? How do we see that a line is not straight without having a firm sense of straightness? And if we have such a one, from where do we have it when we will always be deprived from seeing one actually manifested? This forces a view of Platonic reality which is inappropriate when generalized. It is one thing to think of a perfect line, but a perfect chair! Of all the conceivable chairs will there be one which is perfect, and to which all other chairs want to conform? This is silly. And if Plato seems to claim such things, it is not entirely clear at least to me, that it he is not pulling our legs. From this primitive misconception comes the idea that the Platonic world is a kind of Noahs Ark in which one particular representative is chosen from each entity constituting its essence. Thus there is a Real Lion, a Real Chair, a Real Sunflower fixed for eternity, Real standing for the canonical Ideal. Or to return to mathematics 0 is the empty set, while 1 is the set of the empty set, 2 is the set of 1 and 0 etc, or that the ordered pair (a, b) is simply the set $\{a, \{b\}\}$. In biology such a view of things invariably puts an intolerable straight-jacket and is rightly laughed at nowadays, and even in the mathematical setting, useful as those definitions may be for clarity and manipulation, they nevertheless fail to catch the 'essence' of what it means to be one or two or an ordered pair.

To get a fairer description of Platonism it could be helpful to return to his metaphor. The various projections of a typical 3-dimensional object make up a confusing multitude of shapes. All of those are explained, not by picking the most typical of them, but to look at the original 3-dimensional object itself, which in no way resembles any of its flat manifestations. In fact for creatures confined to Flatland, the Platonic object would be inconceivable, or if conceived at all, only indirectly through some crude reconstruction. Once again I am just giving a metaphor, and like all metaphors it should not be taken too literally (then it just becomes silly). Instead metaphors are there just to suggest the essence (once again a Platonic simile) which they are supposed to manifest sufficiently concretely to our thoughts in order to gain access to them. Or to give a slightly more sophisticated mathematical example, with a somewhat anachronistic touch. The Platonic solids are just the 'Platonic' shadows of their symmetry groups. True with this extended interpretation, the subject of Platonic existence becomes far less controversial and too elusive to be pinned down for easy ridicule. In fact by its very elusiveness it is in danger of becoming too vapid and insipid. The point of this essay is to try to show that this conception of Platonism has nevertheless some meat to it.

Geometry and Deduction

To return to the primitive geometric objects we left a while ago, we marvelled at the mystery that we conceive of such features as we cannot directly perceive with our senses. Now the Axiomatization of Geometry has at its object to fully internalize external space. By making it accessible to our minds we can manipulate them with a clarity and rigor denied the fallible senses. And in all intellectual pursuit, the power of rational argument takes precedence over anything else. On what grounds do we base such an unassailable faith in what seems to us moderns simply to be merely a most elusive electrical activity of some lump of folded wet material?

The ambition of the Greek geometers was to arrive at unassailable truth, using the most powerful tool they could conceive of - rational reasoning. What was the lithmus test of truth? Conviction. However, rational reasoning comes in many forms. Some of it is immediately perceived by the mind in a flash of recognition. 'Indeed this is how it has to be!'. We all as mathematicians recall such experiences, some of us may describe them as the 'highs' we all seek to attain, making up the ultimate motivation for our exalted pursuits. We may describe such convictions as 'obvious', they allow a direct embrace with truth, and as such are more convincing than the testimony of our senses, and of course they are mostly employed in situations where our senses gain no admittance[4]. But most mathematical reasoning employs long chains of arguments. In many ways such reasoning is an externalization and can often not be conducted by the mind alone, but needs props like pen and pencil, and in the modern electronic age those props are susceptible to further enhancement. The most primitive example is of course the calculation, and while much of the technical arguments in a paper is of a calculational kind, it usually cannot be automatically reduced to it literally[5]. How do we trust a long calculation, be it done step by step by ourselves, or by a machine? Trusting as we may be of the exalted insight, painfully aware we are when it comes to the more mundane activities of our minds, knowing from experience that mistakes are legion. We know that there is always room for doubt, that no matter how detailed we make

[4]It ties in with the Platonist image of remembrance, to which we will have many occasions to return

[5]This is why the presentation and reading of a proof is often tedious. Some particular methods are used over and over again in a kind of seemingly Brownian motion of combinatorical steps. The accomplished presentator prefers to wave his hands and to focus on some 'key-ideas'. Ideally this should inspire the members of the audience to fill in the gaps themselves, but most often just resulting in leaving out the devil himself.

the proofs, how finely grained we make the reasoning as to maximize the accuracy at each step, we also add to the number of steps and thus to the complexity of the reasoning and to the risk of mistake[6]. How indeed do we trust that we are correct, and indeed what is correct objectively speaking? Is there a Truth beyond that which resides in rational conviction? Such a question cannot of course be answered in rational terms, it is more a question of religious faith, but as such not entirely taken out of thin air. Platonism means that there is such a Truth, that we may not attain it, but that it exists anyway.

To explain why such a claim is not entirely outrageous but harmonizes with our view of mathematics, let us backtrack again. The deductive approach to truth is the single most important legacy of the Greek civilization, and as such a triumph of mind over matter, seductive to any so temperamentally inclined student thousands of years later. Heady as such an experience is, the basic realization is a humbling one, namely that we have to take some things on faith. Thus the notion of an axiom. The Greeks made a useful distinction between axioms and postulates, which however has been blurred nowadays. Axioms, as understood by Euclid, concern the very forms of reasoning, while postulates concern the basic features of the objects we are about to study. Both kinds are supposed to be self-evident, but there is an important difference between the self-evidence of principles of reasoning, and the self-evidence of objects of empirical progeny, in fact one would be tempted to think of axioms in terms of hard and soft, although with no precise demarcation. While it is easy to change the soft axioms, it is far harder to meddle with the hard ones. This very self-evidence is a guarantee of truth, the bedrock on which the whole foundation rests. Our faith in the Euclidean axioms is based on the fact that they concern things which have meaning and which is anchored not only in a physical reality but also in a more ideal reality. For historical reasons it is important to point out that the so called Parallel Postulate, note that it was a postulate not an axiom, did not conform to the exacting (Platonic?) standards of primitivity that the other axioms did, and thus infamously many generations of geometers tried to prove it. In fact the impossibility of proving it was already manifest by the work of Sacchieri, Lambert and Legendre in the 18th century, although none of them had the courage to draw the ultimate conclusion. One of the favourite occupations of a mathematician is the creation of impossible worlds, some of

[6]It is a common observation that when we do not understand a proof, it is not essentially because of non-obvious steps being left out. We may insert as many intermediate arguments, and in the end we may not achieve more than a verificational understanding without the immediate global understanding of why it must be true. Convictions may come later, when we have developed a different perspective, when the terms and arguments assume a different kind of meaning. (Surely the reader here suspects anticipations of Platonism to be developed below.) Or, as the cynics claim, when we simply have got used to things and learned how to ignore our incomprehension

which can be very elabourate. It is called proof by contradiction. The whole point of creating such a world is to destroy it in the end, and out of its ashes ferret out a single truth namely the falsehood and thus non-existence of a tentative assumption. The world which those 18th century mathematicians created was strange and absurd but not strictly meaning contradictory. And non-Euclidean geometry, this new continent, had seen the light of the day.

Is non-Euclidean geometry true and if so old Euclidean geometry false and ready to be thrown on the refuse heap of history? This depends on what we mean by true? Physically true? Lobachevsky alluded to astronomical observation. In hyperbolic geometry[7] there is always parallax, even for objects infinitely far away. At the time of Lobachevsky no parallax had as yet been measured, which meant that if the world was indeed hyperbolic, the natural unit[8] would be very large from the human point of view. The story that Gauss tried to measure the sums of angles of a triangle in order to settle matters must be apocryphal, unless he anticipated Riemann (and Einstein) imagining a locally varying curvature of space. (Uniformly curved) hyperbolic space would be way beyond the physical possibility of measuring triangle deficiency for terrestial triangles, as inferred from simple astronomical observation, which Gauss obviously must have been aware of[9].

[7] We are leaving aside the case of spherical geometry which was clearly known to the ancients. This was of course never seen as an alternative to Euclidean geometry for various reasons. First it clearly violated many of the axioms and postulates, like two lines meeting only in one point. Furthermore, and more to the point, the great circles of a sphere are clearly not straight, and all the geometry of a sphere can be reduced to 3-dimensional geometry, a tendency which ironically has been revived with the availability of modern computer power making the former skill of spherical geometry enjoyed by astronomers obsolete. The Greeks did not conceive of the real projective plane, nor did they entertain the idea of 3-dimensional sphere. Incidentally the most direct geometrical object to human perception is the sphere, not as one to which you are external, as the (imperfect and oblate) Earth; but the perfect sphere of vision, usually refered to as the celestial sphere, parametrizing all directions. Obviously mans mobility adds a radial direction to his conception of space, although he is traditionally restrained to move on a surface. While we have no problem of resigning ourselves to the fact that our vision is finite, we cannot conceive of any directions but those given to us; on the other hand we do imagine that the radial lines emanating from our eyes can be extended indefinitely, any other thought invariably leads the naive mind to conclude that we get to a boundary, with the obvious question of what lies beyond.

[8] A notion familar to any mathematician having more than a passing acquaintance with hyperbolic geometry

[9] On a sphere the size of the earth, it would be practically impossible to discover the angular excess of triangles a few square kilometers in area. In hyperbolic space with the same size of the unit, the sun would not be visible at the poles, and the luminosity of stars would vary dramtically as the earth would orbit the sun. Further absurd and easily observed consequences can easily be listed.

Formalism

The discovery of non-Euclidean geometry had some momentous consequnces as to what is mathematics and mathematical truth. While the correctness of Euclidean geometry was based on the fact that it perfectly described space, nay indeed it was space itself, the discovery of the alternative geometry made a separation between physical truth and mathematical truth, thus in fact highlighting the Platonic nature of Euclidean truth. With that was the idea of a mathematical model born. Euclidean and Hyperbolic Geometry being considered as creations of the mind, each with its internal logic, providing models of the real world, not being the real physical world (of which there is but one)[10]. It did away with the notion of axioms (and postulates) having to be self-evident, and relating to something real and tangible, in other words being infused with 'meaning'. Instead they could be chosen at will, under the proviso (but what a proviso!) of being internally consistent. With this was born the idea that mathematics was a formal game. A totally unsentimental vision most succinctly formulated by Hilbert, in which the objects of study had no external meaning, what mattered was the rules under which they could be manipulated. Mathematics became a game, in which seemingly arbitrary rules were imposed, which were for all intents and purposes followed mechanically. Then of course the objects could be given any number of concrete manifestations and thus the mathematical results suitably interpreted. From being an ontological codification of the universe in its mathematical aspects, axiomatization became simply a method of economy in the sense of parsimony[11]. Ultimately mathematics turned out to be not something to understand or something that meant anything at all, it was just to be viewed as a long sequence of tautologies[12]. This formalization of mathematics is often opposed to Platonism. I want to suggest that formalization is in one sense subsumed in Platonism, in fact that its proponents ultimately resort to Platonist thinking, and in another direct psychological sense is indeed incompatible with Platonism. The first part is the easiest and the least interesting, so let us start with that.

The formalist approach to mathematics, often popularized and thus vulgarized, is, as we noted above, nothing but the conception of mathematics as a formal game of rules set out to be applied. In fact what we are doing is just to manipulate signs on paper, or whatever physical manifestation we choose.

[10] Does Hyperbolic geometry pertain to physical reality? Penrose points out a striking context. The configuration space of all velocities make up a hyperbolic space with the unit being observationally small. The aberration of stars, observed already in the 17th century, is nothing but the perceived parallax of the infinitely distant light-rays. A velocity is as physical as a point.

[11] On the other hand this unification is also very Platonic in spirit, as we will discuss later

[12] Paraphrasing Russell, whose attitude to mathematics was one of haughty ambivalence.

The results are theorems, in principle checkable as results of valid reasonings. To a working mathematician this is a crude travesty of what mathematics is about, but yet one he is more than ready to admit to in any philosophical discussion, and hence one which dominates the philosophical view of mathematics, and provides the starting point for most philosophers (whose mathematical expertise may not go beyond that of classical Euclidean Geometry) on the subject of mathematics. Devoid of any meaning, those various mathematical games, nevertheless exist as entities of study, and the interesting thing is to find out their internal consistencies, because although subject to only a relativistic notion of truth, they still have to conform to intrinsic, and as it turns out, exacting conditions.

The study of formal systems is usually called meta-mathematics to avoid confusion. But this in itself sows confusion, because meta-mathematics is just mathematics, but now turned to the study of formal games. Instead of studying classical mathematical objects of great beauty and significance, we are now studying various strings of symbols meaning nothing by themselves. Paradoxically this seemingly all-encompassing study seems far less interesting and intricate than many of the particular cases it purports to subsume. Now the point is that we may be completly free to set up any such set of rules, but once we have done that, we lose control. The consequences of those rules, in particular their consistencies are not a matter at our discretion[13] they constitute a hard external reality that is liable to kick back at us. Meta-mathematics is mathematics, hence we argue mathematically about those formal systems. How do we know that they exist, and if so in what sense do they exist? Naively, and every sincere thought has to start out naively, we think of this infinite (potential or actual) set of finite strings of symbols, and we imagine we can contemplate them one by one. In particular we use notions like the shortest proof and so on. A statement like a certain diophantine equation lacks a solution may or may not be true. If false there is a counter-example, and a counter-example is of course a proof of the counter statement, even if the length of it may be inconcievably large (and mind you large integers are very large and far more difficult to fathom than the principle of infinity itself, something to which we will have occasion to return). If there are no such counter-examples, and this is something we may never know, unless of course if we are beings which can 'in finite time' check an infinite number of cases, the theorem is true although we can never prove it. Thus we easily conceive of a difference between provable truth and Truth itself, the hallmark of Platonism. All of this is very simple and certainly Platonic in character. We are arguing about an infinite set of whose *a priori* existence we are convinced.

[13]As Popper stated. The integers may be an invention of humans, and certainly the operations of addition and multiplication so are, but not the laws of distribution which are forced upon us.

So how do we 'prove' that Hyperbolic geometry is consistent? The way we do it is through the model. With a formalist approach, the various objects (or signs) have no fixed meaning but can be given various interpretations. The different models of hyperbolic geometry are too well-known to mathematicians to be repeated here, suffice it to say that they reduce the question of consistency to that of Euclidean geometry, which for reasons of familiarity seems quite reassuring (so from a pragmatic point of view we may as well stop here). Of course we can go one step further, by encoding space into arrays of numbers, and ultimately reduce the question to the consistency of the integers.

As is well known, the free reasoning leading to the Paradise discovered by Cantor, also brought about a crisis in mathematics, or rather a philosophical crisis of the formal foundations of mathematics. The working (Platonist?) mathematician could not care less (secure in their own heaven?). Poincaré even remarked sarcastically that the sterile pursuit of logic at least now seemed to have progressed far enough to produce contradictions. The attitude of Hilbert (another Platonist mathematician?) was one of hygiene. Let us solve the problem of logical foundations once and for all and then go on with our business (as usual?)! The optimistic program foundered and turned out to be intractable. In retrospect how could it have turned out otherwise? How can we go about investigating the consistency of the integers, when any meta-mathematical study seems to involve taking them for granted, as in the naive discussion above? Gödel did exactly that, mixing mathematics with its meta-mathematics, and through Cantors diagonal principle, (i.e. the constructive use of self-reference, or as I prefer to term it, the principle of free will), established his celebrated theorem, which has become something of a cult theorem on the mathematical-philosophical fringe, often quoted, more seldom understood, even if it is technically and conceptually far simpler than most celebrated theorems in mathematics.

It would be pointless to state exactly what it says, let alone prove it. But just to get the flavour we may recall the Richard paradox which allegedly served as an inspiration to Gödel. 'Consider the smallest integer not defined in less than thirteen English words'. That integer has just been defined in twelve words! What does it mean to be defined by a string of integers? Most strings of characters are 'meaningless' some, however, form meaningful English sentences (or Finnish or transliterated Russian, Chinese or Mayan for that matter) and may or may not refer to integers. How do we assign a possible integer to a string of characters? Should we in this age restrict ourselves to English or also allow any of the formally recognised languages written on this planet? Or should we also include extinct languages, or even languages not yet evolved? We are clearly digressing and the discussion is going out of hand. Formally of course we may simply write down a list of strings and to some of them associate numbers. The relation between the strings and the numbers is purely formal, there is no 'meaning' attached to those strings, beyond that of being arbitrarily paired with a number. A simple such list would be obtained by counting the number of characters in the sentences. If so there is no paradox, only if we start to impute some 'meaning' to the strings, and out of that somehow the list would emerge, and in doing so becoming a candidate for reference, as it was not 'before'. Clearly it is a case of self-reference, leading to many *ad hoc* solutions, like Russells' theory of types.

Platonism and Popperism

More interesting though is the different attitude that Platonism imposes. This leads to various elaborations of Platonisms, some of which may go out of hand and give to Platonism a bad name. I am certainly not going to defend all of those, but I will nevertheless look upon them sympathetically.

First and foremost not all formal games are equally interesting. The criteria for interesting is clearly one far more subjective than truth, although in mathematics as a human practice, to which we will have occasion to return later, it is of extreme importance, in fact paradoxically more important than truth, at least in the sense of the latter interpreted as being merely a case of being correct. From a formal point of view, the consistent systems would be interesting, the others not, but this is of course a far too simplistic point of view. Some formal systems are just more 'fruitful' than others. And what do we mean by 'fruitful'? And here the actual practice of mathematics enters. To the real working mathematician the subject is one of extreme vividness and beauty. So many diverse things fit together in striking ways. Commonly one problem in one area is only solved through the methods of some entirely different area. Mathematics makes up a web every part harmonizing with every other part. Often the same result can be obtained from two radically different approaches. That this should be so strikes the working mathematician short of the miraculous. But above all the actual practice of mathematics, as opposed to the transcendent insights which are comparatively rare, involves details. Intricate chains of details, in which the devil is to be found[14].

The description may admittedly be somewhat idealistic, some parts seem less connected than others and thus, at least tentatively may be dismissed as bad mathematics. But the actual experience of things reinforcing each other is a very deep and satisfying one, and in fact it is exactly this that gives to mathematics its solidity. Deductive proofs alone are not enough to instil the convictions of truth, without the mutual corroboration of results, mathematics would be a very shaky construction indeed. It is this solidity

[14]This is why a mathematical text can ultimately prove far more satisfying than the reflection upon far vaguer concepts, which the present text illustrates. Hume remarks that it is this precision of mathematics which allows the long chains of reasoning unparallelled in other disciplines. (Hume had in mind geometric arguments as in Euclid.)

that gives to the working mathematician a deep sense that what he or she is dealing with is something 'out there', an objective reality as tangible as the real one 'out there'. Because in what sense is the so called 'real world' so 'real' to us, so palpably tangible? Because we interact with it continuously. Not only passively through our senses, we also manipulate it, we get feedbacks, and everything seems to 'fit together'. My point is that there is no essential difference between the empiricism of the real world and the empiricism of mathematics. It is true that the 'real world' is more accessible to us through our senses (although as noted initially much of our perception of the world is a theoretical construct, a result of mostly unconscious thinking, the mastery of which is both a product of evolution and constant habit and which we would only lose at our peril), while the 'mathematical world' is only accessible through the ardurous process of thought peculiar to humans (or rather a fairly small subsection of it). Thus from a social point of view people who deny the objectivity of the real world, or even deny it altogether, are dismissed as lunatics; while those who profess the objectivity of the mathematical world being on a par with that of the real physical tend to be ridiculed. We are now entering the realm of social consensus and social construct and the emergence of the phenomenon of the post-modern thinker[15].

In this context it could be appropriate to recall Popper and his criteria of falsification which I believe applies as much to the supposedly deductive science of mathematics as to the so called inductive (natural) sciences. Popper teaches us that we cannot prove things, we can only disprove them. I will not enter into a pedantic discussion of why those seemingly symmetric projects are indeed asymmetric, except by pointing out that an infinite number of cases cannot be verified, but a single counterexample can[16]. According to Popper what distinguishes science from mere social constructs is the possibility to produce statements that can be falsified. Any theory produces a host of consequences all amenable to falsification. Science does not progress by claiming unassailable truths, but by weeding out falsehoods and false leads. Its truths are all provisonal and tentative to be continuously

[15] The post-modern thinker is of course very much a pre-modern phenomenon. He was present already among the Ancient Greeks in his incarnation as a sophist. Much of Platos' dialogues are in the nature of attacking the invidiousness of the sophists, who, in the words put to Socrates, are as ready to argue one side of the question as the other, getting drunk on their own cleverness. Then it is quite another thing that Plato often lets his Socrates argue as a sophist himself, playing on words and strained analogies, but this I suspect is simply a manifestation of the irony of Plato, to which I have already referred.

[16] Of course there is a symmetry, but on a higher level. Providing a single counterexample to a statement actually proves that there is no proof of it, that among an infinite number of possible proofs, none applies to it. And this rejection of an infinite list we are able to do, provided we assume that mathematics, as we know of it, is free of contradiction. Which in many ways is a religious conviction.

challenged. This does of course not mean that Popper is a post-modernist[17]. He does not deny truth as an ontological entity, only despairing of its epistemology. The principle of falsification has a few very important consequenc. First you do not convince yourself by adducing evidence, this can be done as much as you wish, in fact there is no theory, no matter how stupid, for which you cannot amass unlimited evidence. The secret is to look at the weak points to try and undercut what you are claiming. And this is indeed the way a mathematician works. You may be convinced of every step in your proof, but yet you want to test it as much as you can with what is known, until you feel more comfortable. Such tests can simply be numerical verifications, and no real mathematician is so stuck in his deductive ways that he does not welcome additional independant confirmation. Secondly by chosing your potential falsification suitably as to common grounds you can settle matters with an opponent. Few people may be convinced by the theoretical arguments for an atomic explosion, including the theoretical physicists themselves, but the actual explosion of one convinces everyone. It is in this that the principle of falsification exhibits objectivity. No one can disqualify someone from judging a theory by his or her ignorance, as there should in principle always be consequences amenable to checking for the most ignorant. And this indeed lies behind the well-known challenge 'Show me'[18]. It is through this process of hypothesis and testing that any organism, in fact any entity, interacts with the environment and fits with it or in the end expires, lying behind the process of evolution to which we will have occasion to return. We do not passively learn from the real world, because why should the real world care to teach us anything unsolicited; we only do so by asking the right questions. This incidentally is the difference between Lamarckian and Darwinian evolution[19]. My point is that it is through this process of testing and mutual confirmation that mathematics engenders in its serious practitioners this sense of objective palpable reality liable, as we noted above, to kick back at you. This process is of course not a substitute for deductive reasoning, in fact it builds on it and employs it, but it is an addition and a corrective.

Intuition

Much is made about intuition. In fact sustained rational inquiry is distasteful to most people and exhausting to us all. Much more romantic to have the shortcut of intuition. Intuition should be no mystery, and if so it

[17]Although some people tend to think so, notably Dawkins in one of his essays.

[18]Missouri, the homestate of Truman, is sometimes referred to as the 'show-me state', maybe because of the attitude of Truman

[19]Although many naive scientists in immature disciplines are not aware of this crucial difference. One does not progress in science by observing and drawing conclusions, the uninstructed observation does not exist.

should have a rational and mundane explanation. This usually takes the form of a subconscious integration of a vast experience. The lucky insight might prefer the prepared mind, while intuition requires it. Intuition is usually something that is associated with a social setting. We 'intuit' the feelings of others without having any rational basis for it. In fact people who are thought to have no reliable intuition about other people are shunned as non-human, and those who try to compensate for this deficiency by some kind of rational calculation (whatever that means) are abhorred. It is temptingly easy to explain this ability as an evolved feature and thus to describe it as a natural state of the human mind, to which systematic disinterested inquiry is somewhat alien, although, to continue unchecked speculation, it constitutes a natural developement of innate curiosity[20]. Thus the serious mathematician does interact with mathematical concepts as if he would be interacting socially. That means he uses intuition, he is motivated by emotions, concepts have deep meaning and he really cares about them. In other words he is able to employ the entire 'social structure' of the brain. In fact, continuing our speculations, we may refer to specifically human brainpowers such as our uncanny ability to recognise faces to bring to the task of mathematical thinking[21].

What does this have to do with Platonism? It does seem dangerously close to advocating an extreme view that mathematics is nothing but a human construct gone haywire! In a sense it is about religion (another human construct?). The idea of God is a very powerful one, in fact far deeper than the merely superstitious belief many of us were subjected to in childhood[22]. Platonism is if not about God (certainly not a personal and revengeful one)

[20]The convergent development of 'intelligence' at least the social kind, in the mammalian world is striking. Social animals such as dogs and horses may not have our analytic powers, but are supposed to be good at reading our 'moods' (although one should be wary of sentimental anthropomorphism with respect to our domestic companions). The story of the horse Clever Hans, which supposedly could count, turned out to be a fraud, as he was 'merely' reading off the expectations of his master. This seems to me to be by itself a feat far superior to the trivial task of counting.

[21]As a trivial illustration of this one may refer to calculating prodigies or magicians of recollections, where it is assumed that their amazing (but pointless) feats are due to their ability to tap into those parts of the brain we exploit in order to recognise people and other subconscious acts of great complexity.

[22]To me God was someone who would punish you if you did not believe in him. How would you dare not to believe in him? H.G.Wells in his popularly addressed History of the World, singles out the Jews as the inventors of Monotheism. Namely the idea of God being a unifying concept, and in no way identifiable with the various 'effigies' constructed to represent him. Such primitive practices were resented and in fact prohibited as idolatry. The iconoclastic tradition of the Muslim world is a radical way of continuing to reject it, while the Christian church, especially the Catholic is far softer on the issue. Needless to remark, the idealization of a deity is very Platonic, in fact predating Platonism. One may remark that religious fundamentalism (which seems only to really pertain to the monotheistic religions) is a way of idolatry, making holy texts serve the same function as idols used to do.

at least about divinity. And intuition as a fanciful interpretation of a direct channel to this divinity. My main point is that a conviction of Platonism is a very powerful incentive to do mathematics. Without it, the activity ceases to be meaningful and degenerates to a game, which may be fine to people of a crass and cynical mind. Now you cannot rationally argue against religion, nor can you argue that religion is good for you and thus you should believe in it. Belief is not in a deep sense an act of volition. Mathematicians who are ardent Platonists tend to view mathematics in exalted ways. I tend to be very sympathetic to them, it would be very hard for me to care about mathematics would I not be convinced of its external nature and its objectivity. Not to be a Platonist is to be something of a formal solipsist. Your heart is not in it, which may be fine if you have no heart.

Mathematics and the Real world

What is the relationship of mathematics to the real world? This unreasonable effectiveness of mathematics of Wigner? It is fashionable to speak about mathematics as a language, a human stratagem to order the inputs from an elusive reality and make sense of them. In fact we have already initially spoken about observation never being naked but always being embedded in our own construction. So rather than think of mathematics in terms of models, of maps to more or less accurately lie on the real world, one may simply think of mathematics as part of it. Thus the Platonic view of mathematics singles out some mathematics as more more relevant and fruitful (I hate to use the term 'useful' with its suggestions of applicability, especially commercial such). Mathematics is something that grows organically (meaning in particular that it is 'connected'), suggesting its own developments, not something we arbitrarily posit.

Touching upon applications we cannot evade a brief discussion of the role of models. Conventional wisdom sees mathematics as a stratagem of ordering and organizing the world, of providing simplified models which we can manipulate and thus use to gain insight. Thus mathematical models have no intrinsic value in themselves they are just more or less useful. Ultimately mathematics then loses its priviliged position and becomes something dispensible[23]. How typical is it not when a model fails to blame mathematics. But the issue is a bit more complicated. The fact that very disparate phenomena can be illuminated by a single simple model[24] is intellectually very satisfying,

[23] How many people would not be delighted if it would turn out that mathematics will after all turn out to be an osbolete tool, an unfortunate digression in the history of mankind, to be discarded along with the slide-rule and similar artefacts, when new methods of studying the real world have become available. Needless to say such a view of mathematics goes against Platonism. Or otherwise put, Platonism is a bulwark against such an ultimate degradation.

[24] Examples of such are of course legion, exponential growth and the harmonic oscillator are standard examples known to everyone.

and also very Platonic in character, showing that differences in manifestations are but accidental. And indeed that a 'higher' reality underlies it all. On the other hand what are models used for? Prediction and simulations are obvious objectives. A standard strategy to produce mathematical models is to set up differential equations, ordinary as well as partial. So deeply established is this that a large part of physics is simply described by that 'language'. Differential equations lend themselves naturally to simulation and prediction, and thus an essential part of mathematical applications concern strategies of numerical approximation, a subfield greatly enhanced by modern electronic capabilities[25]. But there is more to differential equations than their predictive power. There are also qualitative questions to be asked. And here a problem arises. Models for prediction and simulation can always be tinkered with in response to empirical feed-back. What results in the end is complicated and opaque, and hardly appealing to the mathematical mind. Early examples thereof is the system of Ptolemy with its intricate structure of epi-cycles. But for a model to reveal something deeper it has to be more canonical. I would like to contrast the case of the Navier-Stokes equation and Maxwells' equations. Both are based on first physical principles, but while the former has turned out to be rather intractable, the latter, by its very structure, pointed to relativity theory and the finite speed of light. Thus the former is in the nature of a model, while the latter? Yuri Manin has suggested that theories are aristocratic forms of models. They certainly have a very Platonic element to them.

The modelling of physics through differential equations has been very successful. In fact so successful that one has sometimes been tempted to identify them. Dirac famously claimed that mathematical elegance of an equation was more important than empirical confirmation. In the end he was always vindicated. In string theory, where the option of empirical confirmation does not exist, mathematical beauty remains the only tool and corrective[26]. Thus a physical intuition is often very useful in solving math-

[25] Numerical analysis, which strike many pure mathematicians as 'boring' not to say 'dirty', provides fresh points of view to classical analysis, as well as providing a host of pure mathematical questions, many of the results being expressed asymptotically and with no immediate application to down-to earth calculation

[26] String theory ties up with cosmology in which pure mathematics comes into the most direct contact with the physical world. Is the universe infinite? Olbers Paradox addressed it. One should keep in mind that before the Universe was believed to be exceptionally large it was understood to be infinite. Why did not all the stars converge to one point, Newton speculated, and came up with that there could not be any distinguished point, all essentially being equally probable. Thus this catastrophe does not happen due to some 'indecision'. Another more mathematically acceptable explanation is symmetry of forces. The Olbers paradox can be resolved by assuming that the densities of stars decrease as we increase our balls, and this, as the Swedish astronomer Charliers pointed out, through some inverse Cantorset construction, does not mean that we have a priviliged position visavi it. An amusing thought-experiment in connection with the Olbers paradox is to insert into the middle of a perfect cube walled by perfect mirrors a tiny lightsource. It

ematical problems, but no intuition in biology or economics seems to serve the same function. Mathematical models in social sciences are very controversial, their greatest sophistication to be found in economics. It is probably quite safe to claim that no mathematical model in the social sciences has yielded transcendent insights even remotely approaching those in physics[27]. Although the ambition to apply mathematical models to social sciences still remains strong or at least relentless[28], the optimism is not as great as it was during the heady days of the Enlightment. It is true that classical economics has provided some solid insights such as supply and demand, and the conflict between the arithmetic and the geometric[29] pointed out by Malthus, and that they as a consequence inspired Darwin in his thoughts on evolution. In fact much of the mathematical study of biological evolution is based on 'economic thinking', optimization and game-theory in particular. As to the micro-aspects of biology, as far as I understand the elucidation of the structure of the DNA-molecule was essentially a mathematical one, be it that various 'dirty' tricks may have been employed. The determination of the spatial structure of complicated molecules, such as proteins and hormones, so important in order to predict their chemical and biological functioning, is something accessible from first principles and thus amenable to calculation. It might provide job-opportunities for future math-graduates, but is it mathematics? Complicated, using a mixture of simulation, guesses and empiricism (model cases to be elaborated) but with no transcendental insights, which after all is the hallmark of true mathematics.

The Practice of Mathematics

Why is not chess mathematics? In both cases we have a formal system of rules, or so we do if we take the purely formalist point of view. Both mathematics and chess involve complicated chains of reasonings, and in both the real experts do not just bow down to long humble calculations, but use intuition and general strategies. Many people are quite good at both chess and mathematics, but in general there is not much overlap[30]. Chess ultimately appears to be a dead-end, a sterile occupation like all games. Mathematics

would create an Olberian universe, unless the speed of light would be finite. In addition to the extended infinity, we also have the issue of the infinitely small. Does our mathematical conception of the real numbers have a physical counterpart? Can black holes be arbritarily small? In particular did the Big Bang proceed from an arbritarily small lump? Will there be an infinite hierarchy of physical laws as conditions become more and more extreme, thus making it impossible to have an encompassing understanding?

[27]There is a Prize in Economics retroactively associated with Nobel. It is basically given to mathematical models. The mathematics involved appears rather pedestrian compared to the mathematical feats of the great mathematicians.

[28]The various implementations of statistics tend to caricature mathematics in which many ignorant practitioneers view them as so many recipes to be cooked.

[29]Or in modern 'sexier' terminology, linear versus exponential.

[30]I myself am terrible. Tempting as this may be in order to claim a divisive difference,

is not a game. Could it be that parts of it may be formalized for clarity, but that the very formalization fails to catch its essence. This is another take on the Platonic character of mathematics. It is about things that matter, that mean something beyond themselves. The good mathematical concepts are not mute as the strings in the Richard paradox. They speak to us. Chess moves do not speak to us. This does not mean that chess does not have a lot of associations to the chess-master, his imagination is obviously engaged, and in many ways the expertise of a chess-master and an expert mathematician may be the same. Formally the same. Rich webs of associations out of which intuition may grow. Yet, and here Platonism sticks out its neck. In mathematics there is more to the individual expertise than the expert brings and generates himself. Mathematics is more than a human construct. Then of course the moves of chess may be intrinsically meaningless, but if chess is considered a formal game, all kinds of questions can be asked about it, some of quite a mathematical flavour. With a few additional rules, like the impossibility of repeating positions indefinitely, the number of possible games becomes finite. To compute that number is a mathematical task, not one specifically suitafble to a chessplayer. In fact his special expertise may have no bearing whatsoever on the problem.

The practice of mathematics is different from mathematics. This appears to be an inevitable conclusion to draw from Platonism. The practice of mathematics involves not only the total production of all mathematics ever published, which at the present count amounts to some 50 million pages[31]. This is not much. There are twenty times as many Chinese individuals as there are printed pages of mathematics. Compared to other disciplines mathematics is a small one. How many people are not working full time on say kidneys, or hypertension, and how many pages have not been produced (and how little do we really know). To the practice of mathematics should also be added all the scrap papers that have been used up in preparations, all the discussions, all the talks, all the thoughts. Still it probably does not add up to very much. Still one may claim that what is published mathematics is enough of a codification to allow future generations to carry on what we have left behind. From a formal point of view the axioms of mathematics (if there was some such thing after all) would be enough to codify mathematics in principle. In other subjects this is not the case. Published history is but a little part of the past (in a sense that published mathematics is but a part of what it pertains to describe). We know so little about the past, and of what we do not now, so little is reconstructible. Not so in mathematics. Thus

admittedly it would be too subjective to the taste of most people. But maybe many mathematicians nevertheless would sympatize with me. I have no imagination to think up chess moves, but would have no problem with doing some clever trigonometric manipulations. But wherein lies the difference really? The latter is also some kind of game. Is it just a matter of habit and motivation?

[31] A figure given at the General Assembly in Shanghai 2002

the mathematical literature, in its entirety, is in principle much more self-contained. Hence the temptation to equate mathematics with its practice. In short to claim that mathematics should be formally identified with what is documented. This is admittedly an extreme view, and my point is that Platonism is diametrically oppsed to it.

More interestingly though is another deeper aspect of mathematical practice, to which we have already alluded above, namely the choice of concepts and definitions. Even if we set down the formal rules, that may define the game, but not necessarily direct it. To any set of formal rules and definitions there are secondary definitions and postulates (the latter usually referred to as the theorems). The choice of that seem in many cases arbitrary, not intrinsically forced, but chosen by accident. And here the notion of Platonism may easily become as risible as when we are talking about the Real Lion and Real Chair. What concepts are in fact canonical and which are in fact accidental and arbitrary? Which concepts are discovered and which are invented[32]. It is sobering to consider how formally different two essentially isomorphic theories independently discovered can be, even when the perpetrators are educated in the same tradition. The difference become even more pronounced when we compare say mathematicians and theoretical physicists causing great difficulties of communication. The actual mathematical practice of physicists can be so different from the standard mathematical that one may wonder whether they talk about the same things. Supposedly the elusive real world, or a common Platonic universe? More seriously though, would we be able to communicate with an extra-terrestial civilization[33]. What concepts would we have in common? I agree that an encounter, short of fatal, would have very many interesting suggestions as to canonicity of many concepts, but in fact there are closer examples available. The study of ancient mathematics is a popular but tedious enterprise. It would gain more interest though if comparisons were made on the basis of what has to be and what is fortuitous. The positional system, essentially the logaritmic packing of information, has been reinvented numerous times among independent human civilizations. In particular the notion of number, as opposed to say theta-functions, seems a universal human encounter. In this context I might point out that the history of mathematics has not so far been adequately treated. It is not enough to point out who did what, but really should be an extended reflection of the practice of mathematics. The practice of mathematics, as

[32] In each individual case the distinction may be fine; but the idea is that discoveries are in the long run inevitable while inventions are accidental. Of course this distinction can be shown to be logically void as we will see below.

[33] Many people naively take for granted that such things exist, given the great number of planets in the universe. But there is a great difference between finding an abode which could support human life, and here there may be hundreds of millions in the visible universe, and finding one in which there actually has been a sufficiently advanced evolution of complicated chemical forms.

opposed to timeless mathematics, is what really possesses a history. Why did certain concepts emerge? In what way did they really change the course of the practice of mathematics? As we know, certain 'simple' conceptual insights can break down barriers of thoughts, that no amount of technical onslaught can dispose of. In a sense this is what mathematics is all about. Often we see retrospectively how certain ideas are in the air, how people are struggling with them, before they manage to properly formulate them, giving further illustration of the fact that while mathematical definitions may formally be arbitrary, in practice they often force themselves upon mathematicians.

And finally the practice of mathematics is undeniably a human one, and thus subjected to the usual weaknesses of human nature. The desire for recognition, not to say fame, the competetive spirit of being more brilliant than your peers. Those certainly provide strong spurs to many mathematicians[34]. Also much of the work mathematicians do is in the form of hackwork, applying techniques from a limited furrow of expertise, in order to get some results, something to show and publish, in order to justify continued support and keep yourself active. The achievements of mathematics are in fact due to the nature of mathematics, and the form of practices it has encouraged, rather than to the mathematicians themselves. Mathematicians who may consider themselves brilliant, compared to the obtuseness most of their fellow citizens display in matters mathematical, will often find themselves cutting mediocre figures when venturing outside their discipline. It is not that mathematicians are brilliant, it is mathematics which is brilliant[35] .

Large Numbers and Transfiniteness

We all have an intuitive idea of numbers and how they can be continued indefinitely. In fact mathematically inclined children tend to be initially attracted to mathematics by the contemplation of large numbers. The point is that it is in fact much harder to really imagine very large numbers than infinity itself. Usually when you want to suggest a very large number, whether to mathematicians or laymen, you talk about astronomical numbers, i.e. numbers that have very many digits. Hundreds, maybe thousand of digits. You can speak about such numbers as numbers of the first order, numbers of second order are numbers whose digits are of the first order. You get the idea even if there is no very precise definition. Those are the numbers of the first class. Numbers of the second class are numbers whose order are of the first class. I am vague, but any mathematician can easily write down recursive

[34] Deplore as one may such extra-mathematical features, they nevertheless provide powerful weapons against boredom and inertia, the bane of much human activity.

[35] This is in a sense similar to the claim that mans intelligence is to be found in language, which to some extent is collectively owned.

formulas to capture the drift[36]. The point is dizzyness if you really want to endow those numbers with any meaning of cardinality. Many people profess to want to live for ever. Deep down this desire is nothing but to avoid the last moment. But the price you would have to pay for this would be tremendous, and apparently beyond the imagination of most people. Just imagine that you live so long, that it takes several pages just to write down the date on the letter you are writing[37]. Do such numbers exist? Human reasoning and imagination are fallible, and for all intents and purposes such numbers are infinities. (We will return to this below.) How can we really claim that we can imagine them one by one, as we did above in our meta-discussion of formal systems? In many ways the dizzyness seems somehow isomorphic to the dizzyness we feel when we consider higher cardinals.

Formally the nature of the integers is supposed to be captured by Peanos axioms whose driving force is the axiom of induction[38]. From the point of view of formality the integers are only entities obeying those rules and have no transcendent meaning. Yet when we reason about them, as we would have to do would we venture to ask whether the system is consistent, we would have to invoke a sense of integers with meaning, not mute entities as in the Richard paradox. From the view of the practice of mathematics, there would be no problems if the smallest proof of contradiction would contain a number of characters of say the twentieth class. Humans are fallible, and their practice of mathematics likewise. So speaks a true mathematical Platonist.

But what about the higher cardinals in Cantors theory of infinities? What meaning do they have? The inescapablility of the countable infinity we have already discussed. The continuum of the reals provides the foundations of modern analysis. (Lebesgue theory with its countable additivity would of course be impossible would the reals be countable[39].) But what about higher cardinals? Do they really appear in serious mathematics, or are they just sterile generalizations? And if they mean something in a Pla-

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[36] One simple example may be something formally innocous of type $F(n+1,m)=F(n,F(n,m))$, $F(1,m+1)=F(1,m)^{F(1,m)}$, $F(1,1)=2$. What is $F(5,2)$ say? The meaning of that number might boggle your mind.

[37] Needless to remark, this calls for obliteration. Thus the ancient simile of the cyclic phenomenon that is endless yet finite

[38] Is this self-evident as any truly Platonic axiom should be? Some people claim based on empirical experience with dense students that it is not. But how can we tell whether the difficulty is really intrinsic or merely an artefact of the difficulties imposed by handling formal rules? Ask any employed person what would happen if the day after a free day would be a free day

[39] Of course there are countable models of the reals, e.g. by only considering finitely definable reals. But although there would be a 1-1 correpondence with the integers this would not be finitely definable. Thus clearly everything goes through. As to what constitute the 'real' reals I do not really see any deeper contradiction. So much for the claims of constructive analysis?

tonic sense what do they really mean? And if so, maybe not of interest to us humans, only to putative intelligences who can deal with the infinite number of integers as we can deal with the digits of our left hands? All those axioms of inaccessible cardinals what do they really mean? By a formal sleight of hand do we create huge sets? You can of course see it as a mere formal game, which may be the point of those logicians, but then of course those entities have no meaning. They are just mute.

Worse still are the transfinite principles such as the axiom of choice and the continuum hypothesis, which we may or may not accept, the ensuing system being as consistent anyhow. From a formalist point of view, this is not really a problem, those axioms do not really mean anything in themselves, they are just operational in what essentially remains a finite game. And the logicians, whose original position was philosophical in the sense of clarifying the logical foundation of the world and valid ways of reasoning, in particular hoping, as we noted above, to prove the consistency and soundness of mathematics, nowadays have stepped down from their metaphysical heights, and become metamathematicians, turning their discipline into, as we have already remarked, just a tiny (and to most mathematicians not a particularly interesting and relevant) subsection of mathematics. To most mathematicians the continuum remains an almost physically palpable reality, and the fact that infinite sequences really do not mean anything to those involved in actual machine computations, is just an epistemological feature, having no relevance to the ontological fact of what is. In fact they are convinced that only through an understanding of the transfinite nature of the real numbers can you really appreciate computational and constructive aspects, which thus present no threat, but instead offer amusing, as well as often instructive, interpretations in finite contexts.

But how should we really address the confusing nature of set theory, the formal logical basis of modern mathematics? Some of those assumptions, like the axiom of choice, appear so natural and incontestable as the basic postulates of Euclidean geometry. Do we have a plurality, contesting alternate logical universes, with however such marginal influence on main-stream mathematical life, from which we can pick and chose? Gödel proposes two attitudes, in fact only one of them I can confirm in the literature, and the second, and to the Platonist most interesting, may be a Freudian slip of mine. The first[40] is simply that we should chose for the axioms of set-theory (and hence mathematics?) those that have the most fruitful consequences. This is a pragmatic attitude, and in one sense also a formal one, at least on the surface, as it appears to encourage the chosing of the most interesting game in town. This is of course an attitude that pre-dates Gödel, and is and was shared by the majority of mathematicians. Hilbert famously proclaimed that we should not let ourselves be expelled from the Paradise Cantor had

[40]As explained in his article in the Am. Math. Monthly in 1947

pointed our way to. Paradise equal to a Platonic Reality? This is certainly a congenial point of view to a Platonist, and just as the case of the ultimate deity, its existence takes precedence over any other worthy attributes, be they of benevolence or justice[41]. The second, maybe apocryphal statement of Gödel is that there are natural axioms for settheory, the problem is that we have not found them yet. But when we (eventually?) will they will strike us as exactly what we have always been looking for. In short we will recognise them in the true Platonic spirit that what we learn is simply what we have once known but forgotten. Gödel may or may not have meant this, but whatever, the cat is now out of the bag, and its historical precedence is of marginal importance.

Language

Mathematics as a human construct, nothing more nothing less, thus comparable to a host of other human concerns, with which it will share some features and differ profoundly in respect of others. To make this association meaningful, there must be some essential similarity beyond the common, and maybe merely accidental origin of merely being human[42]. It could be amusing as well as instructive to discuss a few other human endeavours. Language being perhaps the most basic and obvious.

Language is used both metaphorically and literally. In the literal sense it refers to human languages spoken by human beings, and there is little controversy of what is meant, just as with the biological definition of life as being DNA-based and connected to all other life-forms by the evolutionary tree. In the metaphorical sense it could of course mean anything, as metaphors tend to do. Just as the case of extra-terrestrial life is a thorny philosophical issue (exemplified by the first Viking lander on Mars in 1976) the notion of 'language' in the abstract is elusive. Linguists tell us that human languages are surprisingly similar once one disregards the superficial differences due to accidental historical developements, and also that they do not differ essentially from each other when it comes to powers of expression. From this it is tempting to draw the conclusion that language has not progressed historically (as opposed to having merely changed), that it reflects innate human intelligence, and hence in a sense sets limits on it. Naively language is about mapping reality symbolically as by dictionary lists and rules of syntax, ultimately it is about metaphors and self-reference. Nothing is fixed in language we are told, there are no such things as inflexible rules, ultimately the meaning of words and correct grammatical usage is a matter of practice. To claim that there is a Platonic template to which each individual language

[41] The ultimate heresy, as noted earlier, being denial

[42] This is reminiscent of the naive, and maybe not so stupid idea, that sets should be defined by some common unified princple, - 'property', in the words of Frege. Mathematically most sets are uninteresting as individuals, only relevant so to speak sociologically.

strives, and that our uses of our mother tongues are but imperfect approximations of the ideal, would of course be preposterous, making a travesty of what is meant by Platonism, and to my mind be an example of that kind of straw man mathematical Platonism is often made out to be. Nevertheless language is an inescapable fact, and conventional as it may appear to be, it nevertheless exhibits intrinsic laws which are as hard to formulate as to break. To put it romantically there is a kind of music to speech, not just in the obvious vocal sense, but in the sense of rythm. (The non-native speaker reveals himself not so much by breaking codified rules, because those he can learn, but by not 'quite getting it', his mistakes being hard to pin down.) Texts are vetted by ear and corrected by an undeniable authority of what is good and what is bad, leaving aside the grey zone of what is a matter of taste. Thus locally it makes perfect sense to talk (within obvious limits) about correct language.

Some linguists intriguingly suggest that language is hard-wired into our brains, thus instead of merely being a convention, transmitted from generation to generation, it is a product of evolution, just like the heart and the kidney, and thus as much part of our initial set-up like our organs, and ultimately a part of our external world. The standard evolutionary explanation has to do with social interaction, to which we have already referred above, but unlike many of the so exquisitely 'designed' features of evolution, language, like the brain in which it is embedded, seems to sport so many fortuitous features, that cannot be explained by survival pressures. This is of course not an argument against evolution, nor intended as such. Evolution to which we will return in the end is indeed a blind force making up 'unintended' combinations.

Is mathematics a kind of language? The question should not be taken too literally, not even in the sense of Galileo, who wrote that the language of nature was mathematics. It is not about mathematics being a medium of description and manipulation of an elusive external reality, the language in which mathematics is conducted and presented is ordinary human language, formulas and calculations notwithstanding[43]. It is about mathematics being given to us. There are of course similarities. Language is its own meta-language, just as we claimed, meta-mathematics is just part of mathematics, and not a particularly important and interesting part thereof. It is often claimed that language has its limits, that we think in language, and that we do encounter things which are beyond language. One should not think of this limitation, to use geometric metaphors, as the boundary of a large ball, beyond which language ceases to make sense; but rather that language is without boundary, but closed in itself, thus never presenting this abrupt

[43]Some people even take it so literally as to suggest that mathematics should be taught as any other foreign language, and that the difficulties some pupils experience could be rectified by translating the 'Greek' of mathematics to the vernacular!

cut-off, as the first geometric image would suggest, but the closedness in itself suggested by the second. There is, however, a very significant difference, and that is that while language does not progress, the intelligence and the verbal expressiveness of our Cro-Magno forefathers, were in no way inferior to that of modern man, one would rather be inclined to argue to the contrary[44], mathematics does by accumulations and through the absorption of new insights. This leads to the question of whether mathematics is a cultural activity on a par with music, literature and art, not to mention science. To be compared to artists is something many mathematicians find quite flattering (and hence true).

Culture

While we may look upon language as a kind of organ, culture is definitely not. Culture is not innate, although supposedly the outcome of innate forces and impulses. Modern anatomical man has probably always employed spoken language, while the emergence of culture and civilization, including in particular the invention of the script[45], is a much later phenomenon. With culture and civilization, history of mankind emerged. Before that one could as well talk about the history of man as of the history of the Elephant. In principle man may have been able to roam this earth for hundreds of thousands of years, living like wolves, ekeing out a precarious existence, only to go extinct, before creating culture. Such a life does have its appeal, at least in retrospect, it would have provided the ultimate in sustainable living[46]. Culture is accumulative, an extension to a social scale of mans innate constructions. While each individual makes sense of the external world in order to survive as an organism, culture makes this sense into a collective one, allowing us to write such things as mankind now understands, without refering to a single individual.

[44] The life of 'primitive' man was demanding, not only physically. What counted for survival, was not so much brute strength as ingenuity, handiness and resourcefulness. As civilization has progressed, the demands on man have softened, making him more and more into a passive consumer of goods rather than a producer. The intellectual satisfaction that the Ice-age man may have derived from his battle for survival, nowadays finds its outlets in scientific and cultural activity, at least for a lucky minority.

[45] It is noteworthy that children are in as little need of instruction to learn to talk as they are in learning to walk. Although many children learn to read and write by themselves, nevertheless most children need to be explicitly instructed, and usually at a rather mature age (six or seven as opposed to one and two) to slowly acquire this cultural trait.

[46] Among paleontologists it is generally assumed, that the great decline in the Ice-Age megafauna is due to the excessive skills of human hunters, only in disease-ridden Africa, always hostile to humanity, although providing its cradle, were the human pressures sufficiently curtailed to allow it to survive into modern times. Thus one should take with a grain of sand the assumption of sustainable life-styles. On the other hand had man been always dependent upon hunting for survival, the classical cycles of fox and rabbits well known in ecology, no doubt would have manifestated itself.

Cultural activities grow organically. Although not controllable by single individuals, they nevertheless are subject to conventions. Thus it is inappropriate to speak about Platonism, after all man is the ultimate arbiter, in the words of Protagoras, one of the main opponents of Plato. Mathematics is undeniably a cultural activity to the extent that we identify it with its practice. In many, maybe most cultural activities it is impossible to separate the practice from its ostensible subject. Or is it? Examples are Art, Music and Literature. Those are activities of a long tradition and rich in cross-references. Crucial to them are what is meant by a good painting, a good piece of music, and a good novel. The extreme view is that those judgements lie entirely in the eye of the beholder, a more reasoned one is that the power of judgement can be educated, but nevertheless who is the final arbiter? To be honest a similiar situation holds for mathematics. What is true and correct may have an objective answer, at least if we refer to an ultimate Platonic reality; but what is interesting and beautiful, in short what amounts to good mathematics is up to an informed judgement. One ostensible object of visual art was to achieve verisimilitude. This was a non-trivial problem, and was not in principle solved until the late Renaissance, when on the other hand realistic painting reached such a level of perfection, that the problem of mimesis had to be abandonded for more interesting things[47]. Music has never been a question of mimicking the sounds of everyday life, but has always been a rather abstract art. True, music has not sprung out of a vacuum, but is intimately connected to language and speech. Not vocally, as noted above, but somehow, in my opinion, stemming from a common source. Music is supposed to be the most immediate of the arts, probably because it is not representational. Literature on the other hand is mediated through language, and in general not about language[48]. We have touched upon the

[47]The advent of photography at the beginning of the 19th century solved the problem in a mechanical way, and amounted to the death of painting according to some painters

[48]Much of prose and especially poetry is about refering back to itself. In this it is musical and so resistant to translation. This is usually referred to as style, although style can be found at very many different levels. Literature as an artform is rather abstract, unlike painting it cannot try to mimic reality 'pixel by pixel'. It has to suggest and evoke. The point of say a novel is to give a linear narrative, often along lines that the reader already is familiar with. It is sometimes claimed that there are but a few different plots, the most familiar of which is the one provided by life, hence the enduring fascination with biography. Mathematics on the other hand cannot be appreciated linearly, unless of course one happens already to be deeply familiar with the particular subject. This is why one usually can only expect to get disjointed insights from a lecture. On the other hand one would find a lecture without any narrative structure intolerable, even if from the information point of view it should be enough for the lecturer to present an incoherent list of different suggestions. Thus one does not read a book of mathematics like a novel, in particular not from cover to cover, but one tends to go back and forth, each new rereading revealing new aspects, in spite of the fact that the formulation stays always the same. This follows from the universal quandary, that in order to understand A you need to understand B, but in order to understand B you need to understand A. Thus in practice this is solved by doing a long chain ABAB.. at each recurrence instructed by 'monodromy'.

original almost scientific ambition of painting as a means of appropriating visual reality. A deeper and more general object of art is religion. This is something which is nowadays considered obsolete, but religion has inspired some of the finest arts. It is tempting to set up a similar relation between the practice of mathematics and the Platonic reality of mathematics.

As a cultural activity religion may be the most profound of all human enterprises. Not only did it inspire the arts, but also for a large part of its history science as well. Mathematics may have been the language of Nature, but Nature was the manifestation of God. Religion as a practice is mostly about superstition, but religion as a transcendent idea is far harder to dismiss frivolously. Just as we cannot reason from first principles but need to assume certain things - literally take them on faith: such as our trust and conviction in rational reasoning, the external existence of a real world, and in particular the existence of other minds, which we can never directly experience (then they would not be 'other') being examples of things beyond rational inquiry and thus things taken on faith. Faith does not preclude doubt, in fact often doubt is a confirmer of faith. Needless to say, when talking about such exalted things, the risk of being silly is overwhelming. Let me just include this paragraph to claim that Platonism in Mathematics is a matter of religion. Not in the superstitious sense, but in the transcendent.

Law and Justice

Rational reasoning and adherence to rules are also manifest in other realms of life, far closer to the quotidian concerns of most citizens. I am of course speaking about law. While the assumptions in mathematics are few, those in law are manifold. From a Platonic point of view the basic assumptions in mathematics are divine, while laws, even if ostensibly given by God (as the case of the ten commandments) are the products not only of individuals but committees, and their interpretations are not seldom obscure and their contradictions legion. But codified laws are necessary for civilizations, providing, no matter how shaky, an objective basis for society and an arbiter of conflicts of interest. Human affairs are messy, as are the attempts to regulate them. Intimately connected to laws are courts, whose object it is to determine guilt or innocence of individuals (and instititutions) and in the process having to address the issue of interpretations of the laws. Just as in the case of chess one may suspect that the legal mind may be close to the mathematical mind. After all it is a question of rational reasoning, sifting evidence, producing chains of deductions. In fact to produce proofs. The history of mathematics does indeed provide a few examples of outstanding mathematicians who have pursued law as a career. Fermat and Cayley stand out. But in fact in general there seems to be very little overlap between the discipline of law and mathematics, in fact enthusiasm for one subject tends to produce disgust for the other. In the case of Fermat and Cayley there is no doubt where their true loves lay. The practice of law was a necessary chore to them.

In a law-court the truth has to be established. The truth which is aimed at is one of historical truth - what really happened in the sense of Ranke. And besides the ambitions go even deeper, it is not enough to ascertain what happened on a factual objective level, one also wants to infer what the motivations and thoughts were of the perpetrators. There is after all a significant difference between what is purely accidental, and what is intentional[49]. To determine what goes on in the mind of a person is a very subtle thing, and although in a social setting we claim to be able to do it all the time, from a scientific point of view it should be far more problematic. As to the determination of factual history, there is in fact a systematic and objective approach. It is known as forensic science and does in principle not differ from other forensic sciences like geology and paleontolgy, the object of which is to tease as much out of the past as possible from the tenuous traces it leaves in the present. This raises a very interesting philosophical point whether there are such things as true forgetting, or whether every event, however insignificant, leaves a trace, and thus that the past in mathematical jargong gets embedded into the future and can in principle be fully reconstructed[50] But law is not concerned about science, the disinterested pursuit of truth for its own sake, but to come to a definite verdict in a limited amount of time. Thus the products of the forensic investigations do not provide the last word, they are only part of something larger and often disparaged as mere technical evidence. It is not in the laboratory or in the secluded chamber the battle for truth is fought. It is public and in the nature of a theatrical performance, when two sides are presenting evidences for and against. Although directly criminal court proceedings belong to a minority of what courts have to ponder, they make for generally appreciated drama. As we have already noted in the case of Poppers' falsification approach, evidence can always be amassed indefinitely. The real issue is to argue against a hypothesis. Given the impact of human passion and the inevitable conflict of interests, it is of paramount importance to split the two issues of guilt and

[49] Compare the difference between manslaughter and murder (with malice aforethought).

[50] It is far harder for us psychologically to accept that a single cause can have multiple events, than for a single event to have multiple causes. To deny the former implies determinism, and in the present paradigm of quantum physics, determinism in this sense is ruled out, not preventing it from being one of the most succesful predictive theories ever thought out; to accept the latter means accepting forgetting and obliteration. This leads to the quandary whether something that happened in the past but leaving no trace 'really happened'. From a formal pragmatic point of view the answer is no, or rather we can or cannot assume, it does not make a difference. From a Platonic point of view the answer is an emphatic, not to say indignant yes! Finally if everything can be reconstructed, does that not mean the possibility of resurrection? That of the twin horrors that awaits us at death - cessation and obliteration, we can at least discount the latter, the most awesome of the two (the first essentially only being frustrating).

innocence between two independent parties. One who argues passionately for one side, the other against it. In fact it does become a game in the real meaning of a contest between two sides.

But is it really Truth, Platonic truth which is the issue? Obviously not, the pressures to come to a decision one way or another makes the coming to a decision more important than whether the decision happens to be right or wrong. Thus a court is not concerned with Platonic truth, this being an unattainable ideal, but with instrumental truth. The court aims not to decide the Truth only the truth dependent upon the evidence of what has actually been presented in the court and to be without any reasonable amount of doubt. To make the process simple enough to be handled, there are formal criteria for what is allowed to be presented as evidence. Thus technical evidence, the result of painstaking scientific investigation can be thrown out. The roles of the two opponents are parochial, meaning that their duties are not to get at the truth per se, only to present as much as possible in the favour of their party. They are not requested to work against their own interersts, real or vicarious, as would the case be if Truth was the ultimate object. The conclusion is expected to emerge from the clashing of confronting views. Who is the ultimate judge? This depends on the legal tradition. In some traditions it is the judge alone, in others an expert committee, and in the Anglo-Saxon tradition, the most well-known the world over, it is a jury of peers. No matter what, the court is playing for an audience, and the audience has to be persuaded. Thus it is often more important how something is said than what is said. Hence the growing importance of the very skill of speech, usually referred to as rhetorics.

This is of course a very human activity. It is not confined to courts, but spills over into assemblies, and provides the backbone of democratic society, in which the actual nature of a decision is less important than the process through which it has been reached. Namely a process of participation and consensus. Now many people have argued persuasively, Popper among others, that the democratic process is in fact our best bet to arrive at good decisions. Thus the nature of a true democracy has less to do with the spectacle of public elections than with functioning institutions, in particular an independent legal system and the right for free expression[51]. The right to vote should of course never be taken too lightly, but the interpretation of the abstraction known as the general will of the electorate is something rather subtle as well as easily manipulated.

Thus the processes of legal courts and by extension general assemblies may be likened to the crude travesties of lines we draw in the sand. Plato was of course well aware of that, and he held such things in contempt, especially those masters of rethorics, who plied their trades in the market place,

[51]The latter, in spite of all the lipservice, is not fully appreciated among most people when the expression turns out to go contrary to their own opinions.

and did, as we have already noted, just as well argue one side of the question as the other, drunk with their cleverness. As a consequence Plato has retrospectively and anachronistically been branded as an anti-democrat and a fascist. And thus among the feebleminded, Platonism by itself has been tainted, and anything thus connected to it spurned.

Is Mathematics like a legal court, concerned more with attainable truth than unattainable Truth. More concerned with its instrumentality than its fact? Maybe for mathematics as Mathematics is practised. It is after all undeniably a human activity and as such subject to all the foibles and fallacies of the human mind. Many mathematicians may indeed act as lawyers, compiling mere briefs, mainly concerned about the formalities of truth than Truth itself.

But also the legal system has a Platonic element, namely that of justice and morality. Those who perform immoral acts should be punished, or at least identified, anything else conflicts with our innate sense of justice, a sense surprisingly strong in spite of its abstract nature. Also we tend to think that morality is something that exists beyond codified laws. Morality is not reduced to a matter of breaking more or less arbitrary rules and regulations, it would exist even if there were no laws at all. Moral laws are in fact on par with geometrical axioms. It is a matter of finding the true ones and formulating them appropriately. Thus the pursuit of justice parallels uncannily the pursuit of Truth. Human legal practice is but a crude approximation of getting at divine decisions. Hence our tendencies to talk about divine or poetic justice, when somebody is punished though having fallen through the interstices of mere legal proceedings. In fact there is almost a notion of a divine justice which is not amenable to articulation by law. Thus Gödels' proof has sometimes been given very fancy interpretations as to the limitations of codified law[52].

To Plato Justice and Morality were higher objects than mere mathematics and hence closer to his temperament and concern. As a moral philosopher Plato has nevertheless more interesting things to say to us and our contemporaries than he can possible say about mathematics. Thus the historical Plato and his modern reincarnation (the more Platonic Plato?) go separate ways here. However relevant Justice and Morality is to us humans, it does not go beyond humanity. In a world without humans, it does not makes sense; but Mathematics would. This is a crucial claim of mathematical Platonism.

[52]It is very easy though to be sympathetic to such views, and in fact legal practice involves many exceptions and special rulings, aimed at getting to the spirit of the law as opposed to its letter.

Evolution

Ultimately all what we have been discussing so far are manifestations of brains. And brains did evolve, so the ultimate story we have to face is that of evolution. In other words the creation of order out of chaos. Not the case of the creation of something out of nothing, this is too difficult a question for humans to ponder.

The principles of Evolution, in terms of variations and natural selection, as formulated by Darwin, are familiar to everyone. When I first encountered them I was surprised that this kind of clever ideas was not confined to mathematics. Or maybe Evolution is mathematics? According to its modern proponents it is mathematics, in particular it is an algorithm that is being unfolded. The incredible richness of the world can be reduced to an almost tautological principle. Is this possible? Many people, if not most, take an instinctive exception to this. Would they formulate their misgivings, it would be in thermodynamical terms. Information is always degraded. The designer is always superior to the objects of his design. Thus the only way out of this dilemma seems to be to posit a superior intelligence. God is the usual name for such a being, although modern men are usually too shy of employing it, and various roundabout ways are devised to get around it. But the principle just enunciated is also a very simple principle, and simple as it appears to be, it nevertheless seems to logically force the existence of God, admittedly given the empirical fact of the rich world. Is this not a kind of boot-strapping by itself[53]?

Once again we come to the question of existence. What is really meant by existence? The naive answer is that existence somehow ties in physically with being connected to time and space. But abstract principles of the type we have just discussed seem to exist to people in very palpable ways leading them to draw momentous conclusions[54]. In one of his stories the Argentinian writer Borges in his the Library of Babel conceives of all possible books. Do they exist? It is easy to give a formal definition of a book as composed of strings of such and such characters of that and that length. It then becomes a mathematical object, a finite one to boot, which we can easily count[55]. Do those books exist? Obviously not as actual physical objects, the known universe is too small for them[56], but as a Platonic object? And how would you find a book in this library? They could of course be

[53]It is reminiscent of the ontological proof of God, in which God is defined as the perfect being, with existence being one of the attributes of perfection.

[54]Of course the existence of a divine intelligence does not necessarily imply a lot of the usual comforting conclusions about the nature of the divinity. It could of course be very hostile, not just totally indifferent, to the fate of man, collectively as well as individually.

[55]It becomes a number of the second order, to use the terminology we have employed above.

[56]If the universe would be hyperbolic, and the books stacked in it, the path to each book would be of a length of first (zero?) order

ordered alphabetically, the entrance consisting of many different doors each corresponding to the initial character of the book. After having made your first choice, you are presented with an identical one for the second character. And so on, through all the characters of the book. Why bother to actually walk through all those doors, why not be content with telling the secretary at the entrance your choices of doors, which she will dutifully enter on her keyboard one by one. When you are through she presents the book to you (beautifully printed and bound). Who said that the Library of Babel does not exist, given any path you take in the library you will eventually end up at the right book. Finding a book is equivalent to writing it!

Now an even more ambitious form of the Library of Babel we have already encountered. By doing away with the finite lengths of books, we can consider a countably infinite library, including in particular all mathematical statements, true, false or nonsensical, as well as proofs thereof, correct, fallacious or simply irrelevant[57]. Borges as the quintessiential librarian, feeding on books as the cow feeds on pasture, is intrigued by all those hidden treasures. Just imagine that among those books would be found true biographies of all the people who have ever lived (as well as 'countless' copies of false, or misleading ones). To Borges every truth, every treasure, in life can be formulated in a book. But all those books are mute, just like the strings in the Richard Paradox. A book which is not read is dead. Only an intelligence, like that of a voracious reader like Borges, can breath life into them. But if we were allowed to live for ever, we would of course be able to plow through all those books, in fact we would have time, not only to do it once, but several times, each according to a new permutation of the books[58]. Would we be any wiser, or would the ingestion of so much nonsense, so much falsehood even out and we would return as wise as when we started out?

But the Babylonic library points at a disturbing feature, namely the role of creativity. Is there really in the end any significant difference between creation and the mere act of plodding through? Whenever a theorem would allow a formal finite proof, the plodder would eventually find it, would he be set on a systematic mission. Of course the procedure is impractical, at least to us humans. For all intents and purposes the number of books in the library is infinite. The length of the book is finite, but its exponential is 'infinite'. The exponential correspond to the potential, the logarithm to the finite and actual. Mathematics as a human activity is finite, although

[57]By abstaining from setting an upper limit of length, we make it of course unrealistic from the point of view of human mathematical practice. Who would ever understand a proof, which takes up a number of pages equal to the number of elementary particles in the known universe? One could of course imagine super-minds, even such that work through an infinite text in finite time by sustainably doubling their reading speed. Such musings can of course be discarded as fanciful but not logically dismissed. Another fear is that we will soon run out of humanly accessible mathematics, compared to other more pressing problems of earthly resources, this is farfetched.

[58]Incidentally making up a number of the third order.

we can in principle imagine a Platonic library, in practice we will only be able to savour some of the growth on the foothills. No human being is ever going to understand a proof that goes on for hundreds of thousands of pages. Mathematics as actual historical practice is but a tiny part of its potential. The Library of Babel (or Gödel say if you want to include proofs with no a *priori* limit of length) does exist, because show me any explicit example of a book that does not exist![59]

So creativity is a mystery. Finding, i.e. writing books, can sometimes be reduced to a strategy of navigation. Simple such strategies, i.e. those that can be codified, can be used and used again, to find the same canon of literature. Such books are never lost, they can be reconstructed. It is often claimed that science is of that kind. Its results are simply hidden in the strategy of navigation (the scientific method), and thus each civilization, terrestial or not, will eventually rediscover them. Is this Platonism? A Platonism of a Canon, of the inescapable truths, compactified in a principle of thought? On the other hand the fruits of literary exploits are different. They supposedly contain accidental features not predictable from a navigational strategy. If once lost, forever lost, because the configuration space of possible books is for all intents and purposes infinite. If Shakespeare had never written his plays, no one else would have done it. They belong to pure accidents, unreplicable. Of course there might have occurred other plays rather similar in philosophy and plot, but the exact choice of words, the exact kind of emotions engendered, would not have been there[60]. Thus what is irreplaceble also becomes precious. Once lost, forever lost. And finally, maybe the most interesting question. How can we recognise the worth of books we discover by the wayside? We have encountered this before in discussing natural axioms for settheory. Recognition is the same thing as remembrance. A very Platonic notion. Then of course, once again, it is not entirely clear whether one may make such a sharp distinction between the fortuitous produce of literary minds and the supposedly more canonical fruits of the mathematical. Many approaches to the solution of problems are indeed quite original, and will as such have profound influence on the further 'organic' development of mathematics. If Gauss had never existed, would mathematics look very different? This is a question to be pondered in the context of a penetrating study of the history of mathematics, or rather a history of its practice.

What does this have to do with Mathematics and Platonism? Ultimately

[59]Borges states in his story somewhat naively, that the library would contain a catalogue of all its books. As far as something actually spelled out this is patently nonsense.

[60]Also in mathematics. It cannot be repeated too often that reasoning is not an inevitable unfolding of hidden principles. The basic assumptions do not form anything like initial conditions. To solve problems in mathematics involves the fortuitous insight, and the very way mathematics then evolves has similarities to evolution, or organic growth to use a similar metaphor.Thus even a particular proof may be an 'accident' and would remain unknown otherwise.

I am trying to sketch a kind of model of a non-Platonist conception of mathematics embedded in a Platonist setting, reminiscent of a hyperbolic model inside Euclidean space, the purpose of which is to suggest that non-Platonist mathematics is inconceivable unless viewed in a larger Platonist universe[61]. One should, however be wary of metaphorical reasoning. It can cut both ways. In fact it is mathematically much more natural to view Euclidean space as embedded in Hyperbolic space as a horisphere, rather than the other way around. The unfalsifiable and hence uncontestable conviction is that thought must ultimately be phrased in Platonic terms. Our convictions ultimately rest on assumptions and principles of reasoning we 'feel in our bones'. Is not the haughty assertion of a post-modernist that there are no universal truths, only social convention, just a social convention and hence false? Maybe a naive rejoinder, yet at the heart of Cantors diagonal principle and Gödels proof.

The reasoning above, with man as ultimate arbiter of meaning and truth, can be extended by refering to the Library of Mendel[62]. Now we are talking about all possible DNA-sequences which make up a huge configuration space in which natural selection provides a navigator. Species are in fact created, not by chance, but through a well-defined process, codified as an algorithm. As noted before. Evolution is just a mathematical process, incorporating among other things, the development of consciousness and intelligence, and in particular that of mathematics as a human endavour. In a way, the nature of this principle encompasses the secret of all higher phenomena of the universe. Yet as a piece of mathematics it seems embarrasingly simple conceptually. As the avowed Platonist Penrose has expressed it[63]:

> The human mind is wonderful thing capable of many things, one small part of which is the development of mathematics. Mathematics is a wonderful and beautiful thing, one small part of which is the development of physically relevant mathematics, useful to describe reality, including that of the physical world and the evolution of life. Life is of course a wonderful thing and all that, one small part of it being the human brain.

And essentially identifying the brain with the mind, (doing anything short of that would be embarrasingly mystical), closes the circuit. What

[61]Brian Davies has a model of a mathematical universe in which the Peano axioms eventually lead to contradictions, by imagining a huge Moebius band, whose non-orientability fails to separate the true from the false. Admittedly it is just meant as a suggestive image, yet it involves the imagination of an un-imaginable vast universe involving huge numbers.

[62]This is a terminology stemming from Dennett in his popularly addressed book on Darwinism. Although I have been intrigued by Borges story since I first read it in the seventies, I cannot of course claim any priority for exploiting it. This being an eminently replicable phenomenon.

[63]What follows is of course a paraphrase, the reader looking for the precise formulation is advised to consult 'The Road to Reality'

to make of it? Our thinking inevitably goes in circles. To reason logically about logic, you need to assume what you are investigating. But this is the way it is. Our mind is given, through which we perceive and conceive reality. The idealistic top-down approach is inseparatly intertwined with the materialistic bottom-up, and from a global point of view, this is inescapable. Only by narrowing our focus and shutting off the rest, are we able to reason coherently and interestingly. The larger picture tends to become vapid, and the cynical reader may be forgiven if he assumes that the illustration of which is actually the ultimate reason for this essay to be written.

But to return to navigation. First there has never been any precise codification of Darwins principle as a mathematical one liable to precise mathematical manipulation. It is and remains a metaphor, and as all metaphors intended to be suggestive and evocative. Secondly as all mathematicians know, even the simplest algorithm can have the most subtle and unpredictable consequences when unfolded. The selection of primes through the sieve of Eratosthenes is a very simple algoritm, but the ultimate consequences thereof are extremly subtle and involve some of the most sophisticated mathematics ever devised, in particular connected with what in our culture is considered as the deepest mathematical conjecture[64]. In short, even if the Darwinian principle is well understood, it can never be used for explicit prediction. Retroactive explanation is of course another matter, quite tempting and as such liable to egregious abuse[65]. One should never forget that Darwins principles were addressing a specific biological context the precise nature of which was never clear to Darwin himself. (I am of course refering to the genetic basis, indicated by Mendel, and only articulated in the 50's, revolutionizing biology by connecting it to its microlevel.) The principles as such are so general and hence applicable to so many different contexts, that they become too vapid[66]. This does not mean, when suitably modified, they can provide insights in many other contexts as well. The Darwinian consensus is that natural selection acts on the phenotype, but the information is carried and transmitted by the genotype (the volumes of the Mendelian library). The way the genotype is articulated into the phenotype, i.e. the reading and meaning of the books of DNA-sequences, is biologically a very subtle and poorly understood phenomenon. The embryological development in particular, constituting the most dramatic phase of this process of organism building. The tragic spectacle of conjoined twins illustrates strikingly how intricate organisms can be built out of blueprints that normally would

[64] I am of course speaking about the Riemann Hypothesis, whose exalted status was not taken for granted in the beginning of the previous century. Hilbert assumed that it would be far more amenable to resolution than many other problems he stated.

[65] So called evolutionary psychology is a case in point. Unchecked speculation doing seriously what Kipling did tongue in cheek.

[66] Dawkins theory of memes, originally intended as an innocent illustration, but eventually taken by its originator seriously, as a result of the all too human weakness of vanity.

result in very different outcomes. Obviously there is no 1-1 correspondence between the genetic information and the phenotype. It has become fashionable to perceive the DNA code as a computer program which is being implemented and set to run on its inexorable path[67]. The DNA-sequences do not provide the entire input, there is also other crucial input coming from the environment[68]. Thus we can do away with the notion of genetic determinism, in spite of the supposedly algorithmic nature of the evolutionary process[69].

Still there must be a certain stability, otherwise the feedback mechanism of natural selection would not work. Of course it is very hard to quantify this. If there would be more phenotypical dispersion, natural selection would work slower, and higher life-styles would not have time to develop within geological times. On the other hand if there was no diversity, the survival of progeny would be less, maybe even wiped out altogether. Flexibility is always an advantage, but of course such advantageous traits would not be handed down to the next generation, but there would at least be a second chance. Much debate has been going on whether natural selection works on the level of the gene, the organism, or even the population[70]. To me, the question depends on the property. A DNA-sequence can in principle be codified, and the identification of the Human Genome is in fact the tedious accomplishment of such a trivial thing, while the properties of a phenotype cannot be so neatly catalogued, features existing on many different levels, and thus directly acted on by selection to a higher or lower degree. Natural selection is essentially one of comparative advantage, hence the speed in which it proceeds varies greatly depending on the particular trait. An appreciation of the various time-scales is fundamental to any systematic inquiry into evolution, without it discussion degenerates to vapid generalizations.

So where is the complexity coming from? On one hand it is a mystery,

[67] What seems to intrigue the biologists is the amount of junk in the code. Most sequences code for nothing, they are just there. Many suggestions have been promoted to explain this. One being of course that the uselessness is only apparent, based on our present ignorance. I have produced a lot of PostScript code in my days. This is not a very structured language, which means that it is very tempting to exploit previous codes, thus in the end producing long codes of which most of it is inactive. The same with genetic codes. Why should nature edit them in order to make them look neat and efficient? What works, works.

[68] The fingerprints of identical twins are not identical, let alone their life histories

[69] An even more radical conception of determinism is expressed by Laplaces thought-experiment of a transcendent intelligence, who when given the positions of all the particles in the universe as well as their velocities, would be able to predict in a flash the future as well as the past. Such a purely mechanistic point of view of the universe, reducing it to Newtons' equations, did not disturb people in general, it just being too abstract; although it did highlight the problem of freewill to the philosophers. Nowadays the quantum paradigms of physics would make such initial assumptions of simultaneous and indefinite precision of position and velocity meaningless.

[70] The most radical solution is of course the 'Selfish gene' of Dawkins, a book who has found millions of readers.

like that of the primes, on the other hand, in a Platonic way all combinations of DNA 'exist' somehow, it is only matter of navigating to them. Then even if DNA may be what defines Life (as we already noted earlier in connection with discussing languages), the process of evolution must also be seen in a wider context. How did DNA evolve? Clearly we are now at the level of the evolution of complicated chemicals. Why is there only DNA-based life? Why could there not be some other life based on a different chemical? Could it be that DNA has edged it out of its niche, on the other hand there is such a diversity of life itself. Or could it be that the real bottle-neck in evolution is at the chemical level? This does have important implications as to the development of extra-terrestrial life. The universe is indeed large, maybe infinite, but the one we know of is very small, the number of particles being just a number of first order, while we may need a number of planets of the second order to have a statistical chance of an independent evolution of life. The configuration spaces are indeed very big, beyond the astronomical.

This of course indicates that we may be 'alone' in the universe. And maybe this is a very good thing. Thus humans and the human brain occupy a very important place in space and time, even if out of the vast number of (Platonic?) possibilities, humanity is but an insignificant bleep. It is this perspective that makes the contemplation of the universe and also the evolution of Life so dismal. Human beings do not seem to have a special place there. Obviously, in spite of actual uniqueness, it was never the purpose of evolution to place us here. In fact evolution seems to have no purpose whatsoever. Everything turning out to be so to speak fortuitous combinations turning out to be stable. You think that you heart beats and the lung expands and contracts in order to sustain you, that the immunity system has been designed to protect you. No such benevolent planning, they just happen to be, because the combination is sustainable. In fact the immune system, which exists in an uneasy symbiosis with you, could as well kill you. And in many cases it turns out in fact to be the proximate cause of death.

Evolution is of course the bottom-up explanation *par excellence.* Materialism at its most subtle. Yet how can we expect it to explain to us the phenomenon of consciousness in general, and the development of mathematics in particular? Until about twenty-five years ago those issues were *persona non grata* in science, beyond its ken. Then with the advent of computer power and simulation, the idea of artificial life and artificial intelligence became a feasible subject of play. But exactly what will we mean by an explanation of consciousness, this quintessentially most subjective of all phenomena? Are their limits, just as there are limits to language, when it comes to scientific explanations? Not in depth but in scope.

Once again what has all this to do with Platonism? Is the intellectual consideration of Evolution a Platonic exercise? On a primitive level it is of course a rejection of it. Species are not made in heaven, the teory of essentialism has no relevance to the messiness of biology. In fact the notion of species is a social construct, what exists is the evolutionary tree and any subdivision into discrete parts is moribund. In fact any two organisms can be 'continuously' deformed into each other, using the steps that constitute parent and off-spring[71]. Evolution supplies in principle a step by step exposition of the development of the human brain starting from unicellular beginnings, not unlike a complicated mathematical proof in which each step is logically obvious but the eventual result opaque to understanding. One usually talks about emerging features, that the whole is greater than the sum of its parts. So just as physics is classically the manifestation of mathematics in time and space, should biology be viewed as another manifestation of mathematics, but now in large configuration spaces? While physics is eminently suitable to the mathematical mind, in fact to the extent that most people tend to confuse them, and in its most speculative incarnation - that of string theory, the classical constraints imposed by empirical experiments have been replaced by those inspired by consideration of mathematical beauty, biology is traditionally not the domain of mathematical intuition.

So once again, what does this have to do with Platonism in mathematics? That Platonism lies behind the messy discipline of biology as well, that there are some simple principles on whose unfolding everything depends. And that Platonism is ultimately about a deeper reality, and its manifestations are everywhere.

Anyway Evolution does in no way explain mathematics as a non-Platonic entity. Evolution itself is subjected to external constraints. The speed of light cannot be evolved into a different one, neither can two and two fail to make four. Some of those external constraints are beautifully illustrated through convergent evolution.

Summary

This has been a long and rambling argument. Like a trip in heavy sea, when the boat has roller-coasted among the waves, while Platonism, like the full Moon by the horizon, has only from time to time been in view. It may be useful to try and recall the salient points. The question of Platonism is one of philosophy, it is hence not to be settled by rational arguments (it is after all like the moon, visible but out of reach to the hand), but by evocations of what it might really be. I have attempted to present different aspects of it, some of which the readers might sympathize with, others which he might find overly speculative, more like the disturbed reflection of the moon, than the moon itself.

First and foremost Platonism has to do with existence. Naive indeed

[71] Continuous deformation is a term borrowed from topology and complex analysis, maybe a more appropriate one would be 'birational equivalence' from the theory of Algebraic Surfaces, to employ mathematical metaphors even more frivolously. Then of course an obvious mathematical codification is the theory of graphs, but a less imaginative and suggestive one.

is the individual who believes that only exists what he can touch with his hands. Once you grant the existence of external real objects, you must also grant the existence of relations between them, and so on inductively. The distinction between the abstract and the concrete, useful as it may be didactically, gets blurred on closer scrutiny. Thus the main claim of Platonism is that mathematical objects exist, not just in the minds of men. That they are independent of man, and in fact constrain and affect the way reality manifests itself. Not through our conception of it. The physical world of space and time exists as a mathematical manifestation, just as the biological world of abstract relations between objects in time and space. Clearly such claims are matter of conviction and faith, and as such religious in nature. However an impersonal religion.

The devil is supposed to be in the details. What really convinces a mathematician about the 'reality' of the objects he deals with, are all the details which fit so well to each other. This interconnected web, which mathematics constitutes, with seemingly disparate parts click together, in a complexity which no single individual can grasp in toto, but only get to 'see' through its grains of sand, presents a coherency that parallels that of the real physical world, which we know through our senses and its systematization and extension provided by science.

Formality and Platonism are contrasted with each other. The difference being that the idea of formality is that something does not 'mean' anything, it only exists instrumentally. Platonism on the other hand is infused with meaning, in particular it means that concepts relate to something, they give off associations and stimulate the imagination. Doing mathematics to the Platonist mathematician is not just a frivolous game. There is in particular a notion of absolute Truth. As such Platonism provides a bulwark against the frivolity and shallowness of so called post-modernist thinking, a phenomenon, which in spite of its name, has been with us since antiquity.

What are human constructs? We give examples such as languages, art, literature, forms of government, including those of courts. Each of those have 'transcendental' aspects, which all however ultimately pertain to the accidental world of humanity. Platonism in mathematics strives to go beyond the human, thus giving a glimpse of a world beyond us.

Finally we discuss biology, the most complex of all description of physical reality. Is biology at its root mathematical? is the real biological world indeed just like a huge mathematical proof, far too complicated for us to comprehend? That it may be mathematical, but at a scale of complexity that prevents those insights of global understanding which constitute the most rewarding human experience of mathematics.

`November 22, 2006`