Logic and Mathematics

an inquire into mathematics and Platonism

A slight elaboration on a talk given at the Scandinavian Logic Symposium at Roskilde August 20, 2012

I am no logician nor a philosopher, but a mathematician by training and profession. Thus what I am going to present may appear naive. On the other hand in order to have strong opinions it helps not to know so much.

I think that a prevalent attitude towards mathematics and Platonism is nicely summed up by Yuri Manin. He refers to it as intellectually indefensible but psychologically inescapable. As an illustration of what he may refer to, although I very much doubt that this particular example is what he had in mind, I am reminded of the Cambridge Philosopher Moore, who argued for the existence of material objects by simply holding up his hands, one after the other. It is hard to think of a cruder argument, yet it is also hard to think of a more convincing one.

The mathematician is supposed to be naive when he claims that mathematical objects exist independently of people who think of them, yet those mathematicians who take the trouble to stick their necks out usually take an unabashedly Platonist view. Most mathematicians are, however, reluctant to risk making fools of themselves and thus try to avoid the question as being irrelevant, but when pressed they usually, if reluctantly, admit to some Platonist persuasion. I am not going to be afraid of making a fool of myself, thus taking the risk of not only being amusing but also of being offensive. There are, however, mathematicians who are detractors, I would like to mention Reuben Hersh who writes insightfully on what I would like to call mathematical practice and to whom I would like to return later on.

Now, why are mathematicians so convinced of the reality of mathematics? Or more generally, why are people so convinced of the reality of the outside world for that matter? Admittedly there are people who are not, or at least pretend that they are not. Now we have all at some time or another worried about whether we are dreaming or not. But the notion of dreaming, implicitly at least, posits an external reality as a necessary contrast. One may if one wishes interpret Plato as making the claim that the sensual world with which we normally are concerned is but a dream, a mere shadow of the real thing. Anyway, realism is a perfectly acceptable philosophical standpoint and usually does not lead to any raised eye-brows. Why should mathematical realism be different?

For one thing, while the experience of the material physical world is intimately known to most people, this cannot be said of mathematics. True most people encounter numbers and their manipulations, and in classical education the Elements of Euclid and thus elementary axiomatic reasoning were forced upon hapless pupils, and in fact most philosophical reflection on mathematics does not seem to require much beyond this as a prerequisite. Thus people like

Ludwig Wittgenstein feel entitled to expound on the nature of mathematics. But to really engage in mathematics you have to solve hard mathematical problems. In doing so you will inevitably have the experience that you encounter a hard reality that really kicks back at you and which is completely indifferent to your wishes. In standard parlance, you do not make up things as much as discover things. Thus this is a testimony from a few experts, and hence inspirational as it may be, it hardly counts as proof, as little as Moore's demonstration would convince the blind.

I am speaking about proof. Can you really expect to prove such a type of thing? Would not that presuppose hidden assumptions, even less secure and obvious than what it purports to support? This is of course a standard dilemma against which we repeatedly hit our heads until they start to bleed. One obvious way of reasoning about the ontological status of mathematics is to reduce it to logic. Incidentally, such a reduction justifies the title of my talk, a title I actually wrote down before I had any clear image of what my abstract of a yet non-existent talk would be about. Now, logically, this just replaces the problem to logic, and if you think about it, to even shakier grounds, because logic is after all nothing but human reasoning and its validity rests on faith in its soundness. A fashionable scientific explanation of its soundness, which carries with it the same inescapability that Manin referred to, but applied to logic rather than psychology, is an evolutionary one, namely that its soundness is nothing but an expression of its survival value, or more precisely its reproductive advantage. This point of view goes back to American pragmatists such as C.S.Peirce and William James, the latter in his enthusiasm for such type of explanations degenerating into mere silliness. It is of course circular. The evolutionary theory is by itself a logical construct. But of course circularity is inevitable in any kind of meta-physical reflection. C.S.Peirce claimed that the integers are more fundamental than logic itself. That the kind of logic that mathematicians use instinctively (and Peirce was adamant in claiming that mathematics was not a study of forced reasoning but a study by forced reasoning) is really part of ethics (and indeed the project of putting mathematics on a secure footing appears as an ethical one when you first encounter it, which accounts for its felt urgency), while more formal logic is a part of mathematics. I think posterity has proved Peirce right. The efforts of Russell et al to base mathematics on logic seem in retrospect rather monstrous and a definite *cul de sac*, while the approach of Hilbert proved far more fruitful.

Hilbert is seen by mathematical philosophers as a formalist, and it is true that the one thing that the second wedding of mathematics and logic produced was the notion of a formalized language (which actually goes at least as far back as Leibniz as a philosophical project). Even if mathematics could be reduced to an axiomatic system where the objects were deprived of any meaning except that imbued by them instrumentally, the system per se becomes a concrete entity of which you could ask empirical questions, such as consistency. In fact the study of meta-mathematical consistency became a mathematical problem, and in a sense every mathematical problem could be codified as an arithmetical one. Gödel pursued those ideas to their surprising conclusion, and it is symptomatic that Hilbert instead of rejecting this conclusion, going contrary to his hopes, he and his school welcomed it, appreciating the power of mathematics when seeing it exercised. As a result, rather than seeing mathematics as applied logic, logic nowadays can be seen as applied mathematics. There are of course circularities in this conception, but as we have already noted, circularity is inevitable in meta-physics.

At this stage in the discussion it may be helpful to bring in the subject of games in the discussion in order to clarify the ontological issues. I am indebted to David Wells for impressing upon me the importance of the notion, beyond the obvious cynical interpretation of mathematical activity being mere game playing. A game is characterized by rules, and the point of the game is to play it, which in particular means following the rules, a rule, which you for obvious reasons cannot include in the set of rules. The notion of a game seems to be innate among children and in no way encoded in the formal rules, hence it is far from certain that if an alien intelligence would be presented with the games of say chess, that they would understand the notion of playing. In the notion of playing there is the understanding that the game constitutes a toy-world, which in one sense is at the mercy of the players, because they are free to end a play whenever they want and change the rules to start a new one; and which on the other hand exerts a domination on the players, because they are voluntarily forced to submit themselves to its rules, an act which is not fully voluntary as breaking the rules does not mean vanquishing them, only to admit defeat and withdrawing from the game, which somehow, although being a creation of ours leads its own life. Every invention has unintended consequences, some of which may not be to our liking, and those consequences are not, so to speak, willed or created by us. The rules create a hard reality which kicks back at us.

To make the point more explicit, assume that the game is chess, which although most of us cannot play it very well its rules are well-known. The rules are about configurations on a board and gives restrictions on how to go from one configuration to another. In fact we can see it as a kind of syntax. Every game can easily be translated into a sequence of zeroes and ones, and legal games constitute a subset of all possible sequences. Thus the game of chess can in one sense be described by a listing of all possible legal games. However, playing the game of chess is not just a question of producing legal games, that would be to play chess with yourself, but to pit yourself against an opponent and to be motivated t win. Those are very human motivations that you bring to the rules themselves and only then does a real game ensue. Nothing like obstruction stimulates the imagination, and while chess-players engage themselves many secondary qualities of the game emerge, notions such as weak squares, strong pawns, and whatever. Those notions are not implicit in the axioms they arise from the confrontation of the human intellect and the rules. Alien intelligences, if consenting to play the game, would presumably come up with other notions. But whatever notions which will appear they are somehow constrained by the rules. The rules are there in the background and they kick back at you. You do not so much invent strategies as discover them.

Mathematics formalized into a deductive system can likewise be identified with syntactic sequences of zeroes and ones. To make the analogy as close and relevant as possible to the game of chess, we should not just think of end results, such as theorems, but should include the whole deductive derivations from first principles. The correctness of the latter can in principle easily be checked mechanically, while theorems famously cannot in general. (The chess analogue would be to just present configurations and ask whether they can be embedded into legal games, that would be a totally different kind of chessplaying, more like solitary. How difficult it would be, I cannot tell, but it shows that one can ask all kinds of 'mathematical' questions about chess which has little if any relevance to winning a game.) From this point of view mathematics does not seem to differ essentially from chess, the whole thing is just a question of listing legal sequences. For this reason I liken the formalization of mathematics to presenting an imagine pixel by pixel. Great for many purposes, but in no way indicative of the way we experience pictures.

Now playing chess differs essentially from doing mathematics, although admittedly some aspects of doing mathematics undeniably have the same flavor as playing a game of chess. Thus there is some overlap between good mathematicians and good chessplayers, but that is not a philosophical issue and hence not worth dwelling on. In chess formalization comes first, and in playing the game of chess you are actually in some sense literally doing a search on the list of possible games. Thus when search procedures are made more efficient by computers, humans have to concede defeat. True, the secondary features of chess-playing may be so well-developed that the chess-player does not see the configuration on the board, he sees something else, just as we as practiced readers are not aware of the letters on the page, only the story they convey. (The phenomenon is usually referred to as 'popping up'). And thus when he plays, he is not aware of making any explicit calculation of possible consequences, he just 'intuitively' feels what he 'has' to do. Thus in chess meaning is imposed on a given formalization, while in mathematics meaning comes first, and formalization afterward. Of course this is not strictly true in individual encounters with mathematics. The pixel-by-pixel presentation of a picture has many advantages when it comes to transportation, say such as copying, and in much mathematical instruction pseudo-formalizations (pseudo in the sense of pseudo-code which is not formal enough to be implemented on a computer) are conveyed, for which the student only gradually acquires a sense of meaning. Mathematicians do not employ any searching when they try to discover things, for that the system, unlike the case of simple chess, is too unwieldy, and thus it is not clear how to program a computer to do mathematics in a systematic way, as opposed to perform specific calculations assigned to it by humans. And even if such machines were constructed, it is not entirely clear in what sense they could beat humans. Further speculative inquiry would lead into the notion of artificial intelligence, although ultimately related to the question of Platonism, it would only distract at this stage. Although mathematics is abundantly richer than chess, the analogies may, as I indicated above, clarify ontological issues, on which I believe everyone competently and seriously reflecting on mathematics agree, regardless of their stand on the issue of Platonism. In a game there is an objective component, the rules, and a subjective component of the imaginative response to the rules. The first are god-given so to speak (regardless of whether they are given by humans or not) and they have autonomous consequences, the second are obviously human. So why does the issue of Platonism engender such strong feelings? On one hand determined anti-platonists condemn platonistic leanings as naive, primitive and outdated: while platonists resent the implication that mathematics is but a social game, which if viewed closely and unsentimentally dissolve into mist.

Now so called logicism and formalism (I must admit that I cannot really see the difference) are often seen as alternatives to Platonism. I am puzzled. I see them simply as attempts to bring Platonism into the realm of a systematic discourse. In the same way I am also puzzled by so called nominalism and intuitionism and in what ways they are really opposed to mathematical realism. Platonism is like Hinduism, it is perfectly able to incorporate within itself all religions, including atheism. For the record Hilbert was no formalist in any temperamental way, formalism for him had just one purpose, namely of a rear-guard action (which it failed), after which he was about to go onto bigger and better things. The point of formalism is to make mathematics mechanical, and thus to make it into an external object. I guess Leibniz understood this well, but technology was not up to par in his time to allow an implementation. Now to a large extent the spirit has become flesh. However, I concede that the issue of formalism allows you to ask a fundamental question, namely is it possible, in principle, to completely formalize mathematics. It is a philosophical question, which may have an affirmative answer, but which cannot be implemented in practice. For one thing mathematicians may implicitly use principles of reasoning that have not yet been identified and tamed. We recognize a piece of sound reasoning when we see it (just as Plato claimed that all acquisition of knowledge is a case of remembering the forgotten). And it is possible that this process will never end. Thus one may argue that Platonism goes beyond formalism in denying that its project cannot be realized. That formalism, not even formally, can capture mathematics. On the other hand a Platonist attitude can also be seen as compatible with complete formalization. The Platonist standpoint would then be that the rules of mathematics, unlike the rules of chess and other human inventions, are discovered and so to speak given by God. To paraphrase the existentialist dictum, with mathematics existence precedes essence, with games it is the other way around, the rules and the formalization is given, then existence follows. Admittedly in mathematics there are many games in which axiomatic rules are set up, but such game become meaningful as mathematical games only if they relate fruitfully with mathematics as a whole.

As to logic and mathematics, it is well worth pointing out that mathematicians also rely less on deductive reasoning, than the public (meaning mathematical philosophers) assume. What gives a mathematician his convictions is not so much long deductive chains but how results seamlessly fit in with other results. The mathematician in that sense does not differ from the scientist in the Popperian interpretation (something Popper incidentally never fully appreciated). And the magical interconnectedness of mathematics is at least psychologically a very strong case for its independent ontology. Something that provided the foundations of Freges Platonistic convictions (although he famously dismissed all attempts to base arithmetics on psychology, the least developed of all the sciences).

Now mathematics is undoubtedly a human practice, what else could it possibly be? and as such it is assumed that it is a human invention on par with the arts and the law, including ethics. And undoubtedly much of what meets the eye are due to conventions, starting with the way we represent numbers to more advanced notation and terminology. Many of the objects such as primes to more specialized such K-3 surfaces, seem to owe their existence to personal choice at one time or another, although some of them (such as primes?) seem more natural and inevitable. But as mathematicians are quick to point out, the objects may be inventions, but the facts about them are not at the discretion of their inventors. Any invention, as we have already noted and even people taking an anti-Platonic view may concede, comes with unintended consequences. In fact this is what drives evolution. Mathematics is not art, although many mathematicians imagine fondly that this is what they are doing. If it is art, it is with a twist. There is an old legend about a Chinese painter who wanted to step into his painting. A mathematician can do it. In particular mathematics is not fiction. On whose authority can we rely when it comes to the color of eyes of Sherlock Holmes maternal grandmother? But the properties of a mathematical object are not regulated by its putative inventor. To take a more abstract art form such as music, with which mathematics sometimes is linked. (There is a direct link going back to Pythagoras, but that is of course a literal and hence trivial link.). The mathematician is concerned about his work being true, but what about a composer? Truth means in his case something rather different. He is not interested in rigor, except possibly in a metaphorical sense. As Popper claims, art and science are defined by criteria, while the criteria in science are objective, those in art are subjective. Yet, art works socially, in fact much more successfully than mathematics does. The achievments of a mathematician can only be properly appreciated by other mathematicians who are able to understand the mathematics. But with music, you do not have to be a composer to appreciate it. One may argue that when it comes to the emotional appreciation of music, musical knowledge and ability may be an impediment rather than an asset. The emotions people claim they feel when listening to say Mozart were they in any way present in Mozart himself? Of course there is musical theory, and composers thinking of music, may be engaged in thinking which may be very similar to that of the mathematician. Also would the public be present at a seminar on musical theory they would be as lost and perplexed as at a mathematical. It might then be tempting to equate doing mathematics with engaging in musical theory, but if so what corresponds to the music itself? Applied mathematics? That would be to stretch the analogy too far.

There is mathematical practice and there is mathematics, as the Platonist would claim. The remarkable consensus that characterizes the fruits of mathematical practice, as opposed to the dissension in other human endeavors of the mind, can thus be seen as a manifestation of that unity provided by the sublime object - mathematics. This is, I think, a very fair summary of the way mathematicians think of their activity. If there would only be mathematical practice, few if any mathematicians would be able to go through the effort (if you are interested in money or fame, there are obvious shortcuts to such aims). But if we would say that there is no music but only musical practice, most if not all musicians would just shrug their shoulders. The argument is of course well-known (it is hard to come up with genuinely new arguments in such a well-trodden subject) and has even a name, I forgot what. Mathematical practice also involves more than proving theorems. There is fashion and there is judgment as to what is important and what is beautiful, and there are appointments made and prizes awarded. And there is of course a history. No claims are made that those also go under the heading of Platonism. They clearly belong to the human face of mathematics, without which the pursuit of mathematics likewise would be impossible. The fact that mathematics has such an unreasonable effectiveness in natural science, is of course something that greatly adds to its prestige and fascination. But I think that this is an issue distinct from that of Platonism. Tempting as it is for a mathematician to see the world as mathematics made flesh, one cannot off hand reject the possibility that mathematics in this regard is merely a case of imposed 'gestalt', the way we can conceive of certain amorphous entities in a structured way. However, such convictions, carrying Platonism from reification to deification, will naturally provide a very strong impetus for work.

Hersh does of course not deny the objectivity of mathematics, but points out that its objectivity is only on the level of the individual in the collective, not on the level of the collective in cosmos. Just as we as individuals cannot flaunt the conventions of money, but the collective can. By claiming that mathematical practice is all where there is to mathematics, you put yourself in the position of a classical idealist, such as Berkeley, who claims that perception is all there is, no need to postulate objects that give rise to them. Idealism is a perfectly respectable philosophical stand, regardless of whether or not you like Berkeley postulate a God to make sense of it all. However, the constraints under which an individuals labor in mathematics are somehow not so much externally imposed as internally. In mathematical pursuits there is a great deal of autonomy and tacit assumptions. You do not need to know the formal rules of logic doing mathematics, somehow, as we noted above, you intuit them. To me the rejection of Platonism forces an adoption of some form of Jungian collective unconsciousness. Carl-Gustav Jung is seen as a mystic, and his notion of the collective unconsciousness is of course a mystical entity, although it can be given the semblance of modern scientific respectability, even hinted at by Jung himself. Namely the great similarity of human brains brought about by natural selection. Admittedly we need to account for some element of telepathy and empathy, which do not present any insurmountable problems, in view of current theories of mirror-neurons and such things. Thus we end up with a theory that mathematics is somehow hard-wired in the brain, just as, according to Chomsky language is wired into it. Such a view would of course appeal to people who think of mathematics as a kind of language, and thus like natural language guided by a few principles. I should point out that I in no way impute such a view to Hersh. He has many interesting things to say about mathematical practice and he would probably take an exception to such an idea. Furthermore everyone, who is not pathological, learns language quickly and automatically, and all speakers of a native language are basically as proficient. Language use does not significantly improve beyond childhood learning. While with mathematics there is no such universal phenomenon of automatic acquisition. We also get a problem of how large a part of mathematics is really involved from the beginning. Platonism, does of course not exclude this evolutionary and neurological basis, but the role played by the brain would be more like sensory organs. If we would bring in Popper in the discussion, he would no doubt forcefully argue that some basic structure would be present in the brain, opponent as he is to what he derisively refers to as the bucket theory of mind. Such an approach would nevertheless count against mathematical realism as little as it does against realism per se. Anyway, being a Platonist or not, the question of how the human mind deals with mathematics is a very intriguing one. Mathematical didactics proposes as their ultimate aim to elucidate the structure of mathematical thinking. This, I believe, is a tall order of an ambition, of which very little so far has been achieved.

In summary, to go back to the game metaphor, there is no contradiction in that mathematics is done by humans, and that so much of the activity is clearly subjective, and that there is an objective base. You may be very emotional about mathematics (as the controversy about Platonism testifies to) and the activity itself, but mathematics is not about expressing emotions, in fact it has nothing to do about expressing emotions, as opposed to giving vent to them.

The typical argument against Platonism is the anti-religious one. It simply does not fit into the standard spatio-temporal conception of the world, but does posit a mysterious realm beyond space and time. Of course the argument is circular in a way too obvious to be pointed out, and besides the argument becomes a bit silly when this otherworldly domain is interpreted literally and you ask questions such how could we communicate with such a world. You could as well ask, as did Newton agonizingly, how gravitation acts over distance with no intermediate mediation, or how the angular momentum of the rotating earth leaks and travels across space to the Moon which unfailingly catches it. There is no *a priori* reason to restrict your ontology. Furthermore I think those religious analogies are uncalled for. One is reminded of the British philosopher R.G.Collongwoods conception of history. You cannot bring a bit of the past into the present. The past is the past and can never be resurrected as opposed to reconstructed. The task of a historian is to reconstruct the past into the present using the tools that are available. Thus the past is never fully knowledgeable, it lies in a region beyond our tangible grasp, but that does not mean that it is unreal or just a figment of our imagination, although one may argue that our attempts to reconstruct it are, it certainly exist in a sense, and its very existence is the guiding motivation for our attempts to reconstruct it. In the same way with mathematical platonism. It is beyond our reach, but the ultimate motivation for our mathematical efforts.

Formalism may drain mathematics of content, but that does not mean that mathematics is devoid of content, only that the way of representing mathematics formally requires it, just as in mathematical applications the objects of the real world are drained of any of their qualities which has no bearing on the application. Mathematics is about abstraction, and mathematics itself can be abstracted if it is about to become the object of a mathematical analysis, as we have already noted, was the intention of Hilbert. One may wonder whether the study of formal systems, such as mathematics, is as interesting and rich, as the study of the classical objects of mathematics, whether it yields any interesting secrets. One may see Grothendiecks approach to Algebraic Geometry in this vein. By abstracting away the individual features of algebraic objects classically studied, and incidentally in the process being forced to extend them, he was able to go one level of abstraction higher and take a more structural approach, where the individual objects no longer had any compelling interest. In so far he was very successful solving not only internal problems but also classical ones, thus submitting to a more objective standard. The reality of mathematical objects is an issue of Platonism. From an individual point of view. objects with which you work daily assume a remarkable vividness, but this vividness is this to be considered as integral to the mathematics itself, or just in the sense of Locke, a secondary quality? The conceptions of mathematical objects may differ very much among mathematicians, just as our perceptions of the external world differ among humans, and can in no sense be directly compared. (Frege makes a big point of this). Communication between humans is possible only because of a certain isomorphism between individual conceptions, and so in a more literal sense is communication among mathematicians. To return to the issue of idealism. Of a world beyond individual awareness. The vague if suggestive notion of isomorphism making communication between the closed private worlds of individuals possible, is also a way in which to make sense of an external world. In a similar way the Platonic nature of mathematics is what makes sense of mathematical practice. This is just another rewording of Freges argument. As noted it is hard to come up with anything genuinely new in philosophy.

Mathematics is of the mind, and as such it is naturally thought of as a thought, not a thing. Thoughts are effervescent, things are what matter eventually. (This naive dichotomy leaves much to be desired, but it is nevertheless the basis for much suspicion of the exalted ontology of mathematics proposed by the Platonists.) One of the more basic concepts in mathematics is infinity. To the working mathematician it presents no problems, he or she is familiar with it, it is tamed. But large finite numbers are far harder to fathom than infinity itself, as everyone who amuses himself with writing down an innocent recursive function finds out. In what possible sense can infinity be of things? How can it be physically manifested? John Stuart Mill claims that numbers are just based on physical objects such as buttons and pebbles, but it is easy to write down numbers which seem to bear no relation to such innocent assemblages. Is our notion of infinity really a chimera, a fake? Is it not shorthand for a kind of denial? We know how to operate with it instrumentally, but is it really an object? When we do not want the inconvenience of a biggest thing, we imagine an infinitude. Infinity is not a case of over-abundance but deficiency. Magically we think away? (Infinity as generated by the successor operator is tacitly based on the fact that this process of going from one integer to the next is the same, regardless of what integers are considered.)

The problem really becomes embarrassing when we enter Cantor's hierarchy of infinities. In what sense do they really exist? What attention should we really pay to the logicians penchant for postulating ever higher inaccessible cardinalities? Do they only participate in a formal game, the objects which they will into existence, are but ghosts subservient to more or less random instrumental rules. Those big sets are not manifestations of over-abundance but rather fleeting figments of an obsessed imagination. In serious mathematics you need not be concerned with anything beyond that of the continuum, the distinction being crucial to modern measure theory. Yet, I understand that in order to achieve closure, you can never stop, and the real motivation for logicians are to be found in intuitively formed convictions about logical consistencies not in mindless generalization for its own sake. If you embrace Platonism are you forced to embrace higher cardinalities?

Mathematics is sometimes seen as a language. I refute this position, but could it be that set-theory is the language of mathematics. The only genuinely hard mathematical problems concerned with set theory are of cardinalities. As already indicated above, two sets not being equipotent does not necessarily mean that the one is much richer then the other, only that there is a 1-1 correspondence (by itself a set) which is missing. A case of deficiency. And when it comes to things, everything in set-theory has a countable model, 'countable' now having almost a physical tangible interpretation unlike the other cardinalities. or should we think of the hierarchy of higher cardinals as a glimpse of trans-human mathematics. Intellects which can literally contemplate an infinite collection, item by item, can appreciate the 'full' powerset of the integers.