Indiscrete Thoughts

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This is the first book ever I have read on the computer screen. I must admit that as expected I did not like that. How much nicer would it not have been to read it as a regular book. Yet as to the ultimate engagement with the contents of it, did it make any real difference? After all is not the habit of reading books as physical books rather old-fashioned and more of sentimental value than anything else? In fact the less reading has to do with something material the better (although even with a laptop the actual device is much more cumbersome than a book, and I shudder at the thought of having to read a book on a cell phone), as after all it is the contents that matters.

Now this digression on book reading is rather apposite after all, the author himself devotes some time to the example of reading a book, and how the contents of the book is something quite apart from the physical arrangements of letters on its printed pages. The contents of the book are indeed what matters, but that does not on the other hand exist, as Rota provocatively puts it, if by exist we mean something that exists in space and time. And here we touches upon his engagement in the phenomenology of Husserl and the concept of *Fundierung*. And what is that? Do not ask me, the term is like all fundamental terms undefinable, otherwise they would not be fundamental. If something does not exist in space and time, it has to cast a shadow of existence on something material in order to manifest itself, but of course even if it does not cast a shadow, it still exists, or rather not exists in a higher spiritual world. With this I suspect he wants to point at the essence of phenomenology, which seems to have a lot in common with Cartesian dualism, if however, more thought out and careful delineated than in the classical Cartesian version. And that which does not exist materially, but is in a sense the most important, is what Popper would call the third world (going one further on classical dualism) or more classically still, Plato's world of forms (which according to Popper just is the first historically documented discovery of world three).

Now Rotas book, whose preface by Reuben Hersh makes it looks as if it will be a treat to read (and yes I am not disappointed) is a mixture of many things, blending into each other, as well as not being discreet. There is frank gossip on mathematicians, some of whom take exception and find it too be a bit too frank (kill-joys in other words), and as with all gossip it brings forth invaluable and often delightful aspects of a time which has passed. There are also discussions of the nature of mathematics by a working mathematician, which both gives it a freshness and also make it very sympathetic, phenomenology or not. Inevitably there are harsh judgments of the contemporary development of philosophy, which many philosophers found offensive (how could they not?) and which the editor of the book tries to brush off as mere irony. Of course everything could always be given an ironic twist, especially in retrospect, but that is a bit disingenuous not say cowardly. What he writes about philosophy rings true to me. At the end there is a long list of obiter dicta, some of them amusing but shallow such as a textbook on algebraic geometry should be written by an Italian and corrected by a German, others a bit tedious and as soon forgotten as read. The point is that what may appear hot and brilliant in conversation has a short half-life and appears dull and cold on the printed page (in Littlewoods memoirs there are many example of wit at the high table, which when read out of the human context and specific situation appears forgettably bland). The problem is also that an obiter dicta should be enjoyed as a piece of candy in isolation, but when consumed on the binge makes you a bit sick to your stomach. And finally as an afterthought some book reviews have been thrown in, a few lengthier ones, such as the one on Halmos of some undeniable interest, but most of them short and without much substance.

The first part named Persons and Places can be described as nostalgia, but this is not meant, at least not in my opinion as something necessarily bad. Rota was born in 1932 and in 1950 he is a Princeton undergraduate and already encountering some legendary professors some of whom he makes the subject of some critical personality sketches. For this he has received criticism, unfairly in my opinion, as contrary to our conceit, it is our virtues and strength which are universal not our vices and weaknesses, which are indeed specific. If you want to be remembered as a unique individual, you cannot hide your unflattering aspects, because it is those that single you out from others. You are known by your shadows.

First in the firing line you encounter Alonzo Church the celebrated logician, whose name you often first encounter through the so called Church-thesis. What is that? Such a quaint expression. Hardly his thesis. Is it meant as a metaphysical statement that can only be taken on trust? He comes across as the very autistic pedant you expect the quintessential logician to be. While there is a wonderful world of mathematics out there you find those fellows stuck timidly at the very threshold. On the other side, as mathematicians we all hold a certain fascination with logic, because as Pierce points out, it strikes at the very morality of mathematics. To that we will have occasion to return. Rota reminds us that some of the classical logicians, such as Cantor, Zermelo, Gdel, Peano and Post were strange fellows indeed who spent some time finding sanctuaries in lunatic asylums. Church did not, but must be classified as very strange, even by the standards of mathematicians according to the author. The lasting impression you are left with is that of a befuddled man lecturing to two or three bewildered students. Is it fair? The guy had a huge number of students, there must be other sides to him, but what Rota presents is what he encountered, anything else would have been dishonest.

Lefschetz comes out as much more of a charismatic character than Church, although very rude. But maybe much of his charm was to be found in his rudeness. Aloof to some extent, a bit sloppy in his thinking, just like the Italian geometers he admired. I recall Mackey saying back in the days of the Harvard Math Department's lounge being located at 2 Divinity, that Lefschtez always made mistakes which far lesser mathematicians could fix. How often has it to be said mathematics is ultimately about ideas not deductive chains of rigorous linking. He may be described as a purely intuitive mathematician who never gave a correct proof, nor made a wrong guess either. He had a tough background, starting out as an engineer, and losing his hand in an accident, and then getting a Ph.D. at Clark University, of all places, doing time as an instructor in Nebraska and Kansas, working his ass off in a series of papers that laid the modern foundation for the blend of algebraic geometry and topology[1]. At Princeton he relaxed after his dog years in the Midwest. Did not much more research but wrote a book on topology, not surprisingly with no correct proofs (it was said, Rota reports, that his graduate students were not around to correct him). As a lecturer on courses he was incoherent to the point that only the most faithful stayed on. Rota himself tried to attend a lecture of him. Could not make head or tail of it. Yet during his tenure as the chairman of the math department he made it into the world center it was for some time. He had an uncanny ability to size up a mathematician, at least in fields he knew nothing about, however, when it came to his own, he tended to slip.

Emil Artin comes up for criticism for warping students, they all mimicked his ticks. He is also faulted for his dearth of concrete examples when lecturing, as well his failure to provide proper motivation. His proofs were always the most polished, and hence failed to be enlightening. Pure concepts he tried to communicate directly. A pedagogical shock treatment, according to the author, disastrous to all but the very best and the very worst[2]. Yet in conversation he was very different, then examples and motivations abounded. And Rota also holds him responsible for along with Weil and Chevalley to expunge any traces of determinants and resultants. Rota regrets that the presentation of Modern Algebra, as conceived by Noether and carried out by her students Artin and van der Waerden, is still the norm. Why not more about tensor algebras and less on Galois theory, Rota wonders.

The most sympathetic portrait may be that of Feller, with whom he struck up a friendship of sorts. He lauds his book on probability theory, which he judges to be one of the great masterpieces of mathematics of all time, still read by probabilists in secret refusing to admit to their consulting of it. His lectures were loud and entertaining we learn, but his proofs mainly consisting of hand-waving were often deficient (as with Lefschetz) but nevertheless convincing. And most importantly the main ideas were never wrong. He did not like to be reminded of mistakes, then he got worked up and started shouting, often asking the objector to leave[3]. As a result we have the notion of proof by intimidation. His lectures were rambling, but Rota admits he learned more from them from any others at Princeton[4]. I would also say that the pedagogical value of clear lectures is grossly overrated.

Mathematics in the fifties was an idyllic field, we learn. Students of mathematics

[1] I recall John Morgan quoting him in a lecture, as having said that he wanted to shoot the harpoon of algebraic topology into the flesh of algebraic geometry.

[2] One may interpret this as the best students were able to see through them, while the worst did not care about understanding and motivation, to them mathematics were just something to memorize, in which case the terser the better. But according to Rota, the best appreciated elegant expositions, the worst were infatuated merely by his style.

[3] In fact the was terrified of criticism and as a result his book on probability was rewritten several times, even at the stage of galley proofs, and entire sections of beautiful mathematics were cut out for fear of criticism.

[4] As an example of Fellers style and side remarks one may quote his observation that $\int_0^{2\pi} \cos^2 x dx$ is clearly equal to $\int_0^{2\pi} \sin^2 x dx$ and because the squares add up to 1 it makes it easy to compute either integral.

often were asked whether they intended to become actuaries[5]. There were few centers of excellence (as any university now aspires to become) and it was not that hard to swing a fellowship. Rota himself had received one at Chicago and was about to mail his acceptance letter when he ran into Al Tucker, who had taken it upon him to decide what each math major should go. Milnor remained at Princeton, Hyman Bass were sent to Chicago, Mike Artin to Harvard and so Rota went to Yale as a graduate student. It was an idyllic place in many ways. Hard work and professionalism were looked down upon, according to Rota, professors were ill-paid and teaching assistants even less, so it was in the interest of the administration to have them defer their degrees as long as possible in order to save money. Eventually a thesis was produced, but as Rota writes, strictly an in-house affair. Publication was unheard of[6]. Yet a life of leisure and freedom of pressure could be quite beneficial and instructive and encourage creativity, that pompous word as Rota writes, provided there is congenial company. And the professors provided it, be they poorly paid, they at least enjoyed prestige. It was only in the forties that Yale became a serious math department, starting with Nelson Dunford, which meant that functional analysis would be the core subject, and everyone had to take it. His style of teaching was Socratic, meaning that he gave out problems and had the students solve them. Maybe this lessened his heavy teaching load. Out of this the ultimately three volume book on linear operators would evolve, once he got Jack Schwarz as a Ph.D. student, and it was going to be Jack Schwarz who was to be the attraction for Rota. Functional analysis was a very topical subject in the fifties and sixties and Rota also got engaged in the work by checking exercises, some of them which were very hard. One he could not get at all, neither could Dunford, nor as it turned out could Schwartz. Eventually Dunford assigned the problem to a student who solved it developing an elegant theory as he went along. The student was Robert Langlands.

Anyway it was abstract mathematics. Mathematical abstraction had its heyday between the twenties and the mid seventies, according to Rota, and functional analysis and algebraic geometry led it, especially during the fifties and sixties. Classical analysts at Stanford had a hard time getting students. The more abstract the better, who could doubt that? As to the three volume treatise. The writing started in the late forties and would go on for twenty odd years, the last volume published in 1971. It was meant at first as an introduction to modern spectral theory, pioneered by Hilbert and Hellinger and brought to attention by the work of von Neumann and Stone. One noteworthy feature was the contrast between the uncompromising abstraction of the text, in the words of Rota, and and the variety of concrete examples in the exercises. There are three topics dealt with in the volumes. Those for which the book provides the definitive account. Those which should be well-known but are not and have not yet been properly read. And finally topics which are ahead of their times, yet to be fully appreciated. Among the gems one may find a proof of the Brouwer fixed point theorem which uses no homology theory but is based on some determinental identities, hopelessly old-fashioned in the fifties. You also find much Banach geometry, in spite of the general theory of Banach spaces having been

[5] I was also being asked this question in the mid-sixties when about to enter high-school.

[6] Norbert Wiener a Harvard student writes in his autobiography that a thesis should not be published, it is only practice. This was some forty years earlier. Much can be said for this.

declared dead several times over, as the author gleefully remarks. He finds the work on the classical moment problem treated in the book scandalously neglected. Partial discoveries of the material are still being published by authors who have not done their reading, he remarks with some irritation. Those volumes may now be unfairly neglected, he claims, and points out to the many gems to be found in them. And he goes on to mention other neglected gems to be found by a younger generation.

Schwarz himself he describes as the most influential mathematician having left his mark on so many areas of pure and applied mathematics, physics, economics, you name it, but most of all computer science which he would start a department of at the Courant institute. There is in Rota a hyperbole in praise as well as blame. Maybe even more so, Schwartz gave of course perfect lectures and motivations, such as a Markov chain being a kind of generalized function which put everyone at ease.

After Yale there was MIT. A new bunch of mathematicians. Wiener of course, so tempting to tease, but also Calderon, Nash, Moser, Paul Cohen, along with Garsia and Mattuck. Problem solving, linear ranking of mathematicians, and gossip. It was here that Rota discovered combinatorics engaging him rather than problems in classical analysis managed to do.

Problem solvers versus theorizers. Most mathematicians are a little bit of both, although extreme examples are not difficult to find. The problem solver is attracted by something that seems hopeless. He wants to quench it, not necessarily in the most elegant and instructive manner, just quench it. What matters is that he is the first and that his proof is correct of course. Once that is done he loses interest in that particular problem and looks for something new. As Rota says, budding mathematicians should be problem solvers. The theorizer is not interested in problems, he is more interested in definitions. He does not want to solve problems, he wants to trivialize them, or even better make them irrelevant. While the problem solver is conservative at heart, to him the mathematical landscape consists of isolated facts, the theorizer is a revolutionary who wants to connect and explain them all. Theorems are only interesting so far as they give rise to good definitions. To a problem solver exposition is below notice, after all what can it be other than stating problems not soolving them, while to the theorizer it is actually harder and more important than research. A good exposition does what the theorizer is looking for, connect and explain diverse facts into a whole. Alfred Young the young combinatorial algebraist knew nothing about the theory of group representations a grand theory developed by the Germans with Frobenius in the lead. And this young guy had the effrontery to get results, which the Germans struggled with, and by pure combinatorics to boot. Frobenius studied him reluctantly and wrote it up in his way, going one bit further. Young was silent for about twenty years claiming he was learning German while he was busy giving sermons as a clergyman intent to upping Frobenius which he eventually did. Hermann Weyl discovered him and as a result we now have Young tableaux and Young diagrams. Grassmann on the other hand was a theorizer. He introduced exterior algebra. What was it good for? Rota refers to Pringsheim, a grad old master of infinite series who haughtily took him to task for not having done anything useful, such as finding a new criterion for convergence. Pringshein incidentally was the father-in-law of Thomas Mann. Rota does not mention that, which is a bit curious given the gossipy nature of his account. Maybe he did not know, and if he knew did not care. Anyway back to Grassmann and his exterior algebra. The question to ask was not whether it was useful to old mathematics, the point was that it opened up a new world, as well as also making sense of much formalism in physics which became much simpler in its formulation. And it did in fact revolutionize algebra, although it took some time before that was realized. Elie Cartan took him up on it and we got differential forms, and it was all written up beautifully in Bourbaki, maybe the best part of Bourbaki Rota wonders.

Definitions have to be precise in the axiomatic spirit in which much mathematics pretends to be done. If we are sloppy with definitions we cannot really be sure that what we are talking about really exists, From a logical point of view definitions are done by fiat, but in reality, there is the problem of finding the right definition, because the objects exist before and are known by their descriptions. Much work was done on real numbers before there was the impeccable definition by Dedekind. But of course the reals existed before. Everyone had a sense of it, the challenge was to pin it down. The closest one comes to rigorous description is to characterize a mathematical objects in many equivalent ways. Grassmann only knew about descriptions. What is a tensor? Something that transforms in a certain way. Vague, yet sufficient for manipulation. It was not until the aforementioned chapter in Bourbaki a clear definition was given.

Bottom lines. This is what mathematicians identify with and always return to sooner or later. Mathematicians with the same bottom lines find each others company congenial, after all sharing similar expertise and above all taste, makes for sympathy. In algebra there are two incompatible bottom lines. There is the main stream of Algebra One, as Rota refers to it. The classical subjects such as algebraic geometry, algebraic number theory, any abstract and overarching subject. Emmy Noether set the tome inspiring her students to develop the by now classical exposition, such as to be found in van der Waerden. Algebra two consisted of combinatorial investigations, concrete formulas, such as excelled by the invariant theorists with Paul Gordan as the prime example. Hilbert dismissed him as a mere Algorithmiker, after all, did not Hilbert outdo him by his own elegant and abstract proof of finitely generatedness of invariants, something Gordon himself had struggled with all his working life.

The longest portrait is of Stanislaw Ulam. Not an entirely flattering one, which of course makes for interesting reading, as it gives at least the appearance of probing through the surface of just appearance getting, as far as it is possible, to the core of truth. As a consequence he was shunned by the widow. I recall Ulam for the first time in 1976 or so. He had just published his autobiography which made a splash. It was recommended by the Wall Street Journal and my first wife gave it to me as a birthday present. I read it of course, how could I do otherwise? it was after all an autobiography of a mathematician (and a birthday present). I remember a few things, one of them being that perhaps too much research was done in algebraic geometry. The picture that Rota paints of somebody he was fairly close to, is of a highly intelligent fellow, quick of mind and with deep insights, holding his own with von Neumann, but not adjusted and not as productive as he might have been, had he been less gifted. He was lonely and he was lazy and abstained from thinking things through. In conversations he was swift jumping form one topic to the another, without the patience to stick to one. Maybe so many ideas popped into his mind that he could not make up his mind where to start, thus eagerly distracted. As Rota remarks any topic only lasted a few minutes in his conversation, once there was a record of a duration of fifteen, and this particular one is the one Rota chooses to report on. It concerned A.I. which was coming into fashion with the philosophers jumping at it, hoping against hope to be taken seriously at last and maybe be given salary raises to boot. Ulam was not too impressed, those problems have been around for a long time ad he mentioned scholastic logicians such as Duns Scotus in the twelfth century up to Wittgenstein. Rota eager to make a contribution refers to the infinite regression about a homonoculus in the brain, who in his turn has a brain with a smaller homonoculus. Where will it end? Ulam patted him on the back and said good boy, but even Descartes thought about that and eventually gave up and invoked divine intervention, ironic for someone who supposedly founded modern rationalism. Rota counters that it is different today the problem is addressed pragmatically involving modern technology and computer experiments. But someone has to do the thinking Ulam retorts, and Descartes is a hard act to follow. Ulam then remarks that what we intelligently perceive are not objects but functions, in other words something on a more abstract level, and functions require a context, and if the context is not there, the meaning is gone. The fallacy of the man in the brain is due to a misleading analogy between vision and photography. They are as different as can be. A Camera registers objects (in fact as I would be tempted to remark a photo is a collection of pixels, also in the pre-digital age, when after all silver compounds on the plate served the same rôles), but humans perceive functional rôles, i.e. meanings. And, he adds, that Rotas friends in the A.I. finally begin to see the importance of contexts, but they are not practicing it. They persist in building machines that emulate cameras. Such an approach is bound to fail as it starts out with a logical misunderstanding. Rota bows to superior authority of insight, has he any choice, but decides nevertheless to play the devils advocate, and wonders what then becomes of objectivity, so definitely formalized by mathematical logic and by the theory of sets to which Ulam devoted his youth. This provokes visible motion and Ulam counters with asking him whether logic really corresponds to what we are really thinking, noting that he suffers from *dèformations proffessionell*. Look at the bridge over there, he points to in exasperation, it surely was built on logical principles, but would you seriously believe it would fall down, would there turn up to be a contradiction in set theory? Should we abandon logic, Rota reacts in amazement, pretended or not. Not all all, Ulam responds, it is fine but limited, we need to enlarge it, to make precise the meaning of analogies, to add 'as' to the 'and','or,'implies' and 'not' in formal logic. Unless this is done, your A.I. friends will get nowhere. But this sounds impossible to me, Rota objects, Not so bad, Ulam returns in a comforting mood. Around the turn of the century no one thought that the common sense notion of simultaneity needed no explanation and than Einstein came along. We only need to do with 'as' as Einstein did with simultaneity. Mathematics will surely rise to the challenge, and he changes topic. And the topic was never returned to in their conversations.

The comforts of illusions were denied him, Rota pronounces, he was a prophet who realized things in a flash. Everything worthwhile can be stated in fifty words or less, Ulam used to claim. A mathematician may spend a lifetime struggling with a hard problem and when finally it is solved the futility of it all would inevitable dawn on him. Ulam did not have to go through this process to realize the futility, he saw right away what was coming in the end. Maybe this is why he never really exerted himself? He dropped the gems for others to pick up and develop. Measurable cardinals have conquered set theory, his foundations of probability theory is now the bedrock. He invented many new stochastic processes and was the first to describe and note the strange recurrences of dynamical systems. To physics he came relatively late in life but had an uncanny ability to spot the essentials.

But he is not done with Ulam yet. He takes a deep breath and gives a regular mini-biography, starting with Ulam growing up in Lviv as a spoiled boy out of a Jewish background of wealthy bankers. This would change of course with the advent of the Second World War, but still in the thirties the Belle Epoque lived on in Central Europe, a time which in retrospect appears both exotic and idyllic. Mathematics was done in cafes in old Lviv centered around Banach and a host of Polish topologists and logicians. Ulam made his mark in abstract mathematics, he introduced the concept of measurable cardinals, as noted above, and which came to play a dominant rôle in set theory ever since. One wonders what is the real semantics of those high cardinals, do they play any role except in formal logic, testing what can be done and expressed, what sense do they have as cardinalities of objects as apart from phantoms of the speculative human brain? He is also remembered by the Borsuk-Ulam theorem in topology which I vaguely remember from my early days in mathematics. It was Ulam who had the idea and conjectured it, and Borsuk who worked out the details, according to Rota. Ulam was always lazy, he loved having brilliant ideas, but he had no *Sitzfleisch*, the ability to sit down and work out the details. When Germany invaded Poland he left for the States and stayed there ever since. He never really adjusted to the States, he had few publications and were not able to land a good job but were farmed out to places such as Wisconsin and southern Los Angeles. He was lucky that he met von Neumann, they both appreciated each other, and Ulam was probably the only real friend von Neumann ever had. Again according to Rota. He was rescued to Los Alamos where he would stay except for shorter breaks. In fact he went there twice, once during the war time, which was of course the most exciting, with the greatest concentration of great minds ever, at least since the Greeks, Rota adds. Then after the war it was dismantled and he had a miserable life teaching calculus to morons and had a health crisis, a case of emphalities that necessitated brain surgery. After that he never was his old self again, and his laziness, inability to concentrate to deal with details, became even worse. He was more original than von Neumann, Rota claims, but had of course much less influence. He did give out many ideas worked out by others without giving proper credit or acknowledging their debt. Maybe it was so difficult to understand and work out his vague ideas that they did not feel the need to, but assumed that they were fully entitled. He was lazy, but managed to turn his laziness into elegance, he talked far too much but he was worth listening to, he was self-centered but not egotistical. He thought further than anyone else but he wasted most of his professional life. And he died suddenly dropping out of existence suddenly. And so this chapter of the book is closed.

Mathematics leads a double life, Rota claims. One part of it deals with facts, like in any other science, such as that the altitudes of a triangle meet in a point, or that there are only five Platonic solids. The second part deals with proofs. We need proofs, meaning the whole apparatus of formal definitions, generally agreed laws of inference. This is a great innovation, Rota admits, and is indispensable, because mathematical facts are not, due to their abstract nature, amenable to experimental verification. This, however, can be discussed, I claim. Euclidean geometry is to a large extent a model of physical space and is as such repeatedly confirmed by quotidian experience. But axiomatics is just one method among many potential of codifying mathematics. The facts are facts and remain facts, Rota announces confidently, while the ways of verifying facts have changed, and no doubt will change in the future. For one thing, one is eager to add, the standards of rigor has changed.

Philosophy is in many ways similar to mathematics, and as it, it too has tried to lead a double life, but with less success. Philosophy is about looking at the world, ruthlessly and unsentimentally. Forcing us to confront what we rather not. The facts of philosophy penetrate deeper into our existential human predicament than the facts of mathematics. But when it comes to verification, the philosophers have failed to achieve the consensus that mathematicians are blessed with. They are still, according to Rota, hotly arguing about method. Thus the assertions of philosophy are tentative and partial, he asserts. Philosophical reasoning is also much more emotional than mathematical, after all it concerns us deeper. I would like to add, that while we may be rather unemotional about specific mathematical facts, the fact of denying that mathematical facts are indeed facts, may touch us emotionally, after all the truth of such assertion would make our endeavors moot. Thus philosophical discourse has an urgency, a mathematical one does not have. Many mathematicians would disagree with Rota, or rather be puzzled by as to what he means, but I see that the two of us have a similar temperament.

As a consequence philosophers envious of the clarity and consensus of mathematics seek to emulate it and hence we have seen in this century (remember Rota never made it to the twenty-first) a mathematization of philosophy in the quest of making it, in the words of Rota, factual and precise. As a consequence, history of philosophy has been seen as obsolete and something to be discarded. One is here reminded of Collingwood, someone Rota seems unaware of, who rallied against the mathematization of philosophy by the analytic school. This ties in with the vision of philosophy presented by Russell and Popper, namely that it progresses by amputation. As an example, Rota points to logic which is now a part of mathematics, and as such has great applications be it mostly to computer science. In fact I would even claim that mathematical logic is applied mathematics. But it paid a price, now logic can no longer serve as the foundations of mathematics. As Pierce noted, in its latter capacity, logic should really be seen as part of moral philosophy. No one claims, Rota continues, that mathematical logic has anything to do with how we think. Rota classifies it with topology and probability theory. Philosophers are fooled to believe that mathematical logic is in fact concerned with truth, Rota adds with a touch of condescension, but it is not, only concerned with the game of truth. Thus mathematicians look at derision (or at least rise their eyebrows) on philosophers, snobbishly sprinkling their papers with symbols.

Mathematics is undoubtedly the most successful intellectual achievement of mankind, Rota states confidently and why dispute him? Problems get eventually solved and there is such a thing as closures. In philosophy there are really only a few central problems to which we are forced to return to over and over again, as we are never definitely done with anything. In mathematics, and in science in general, we cut off things from further inquiry (think of such matters a flat earth) but not necessarily in philosophy, as Kuhn observed. Philosophy has traditionally not been afraid of failure. Now it has been infected by the business mentality of success.

Rota inveighs against the insidious prejudice of precision and clarity that runs in the twentieth century. The clearest example being Wittgenstein's final statement in Tractatus, only later revoked in his Philosophical Investigations. Everyday reasoning is far from precise, but nevertheless effective[7]. The concepts of philosophy are least of all precise, but that does not mean that they are meaningless. In order to make them precise, we invariably misunderstand them. Rota has been criticized for his criticism of philosophy, unfairly I would claim, and I find the apology of his editor, that what he writes is only tongue in cheek, uncalled for. He is serious. I am in the habit of expressing it less precisely, as philosophy being the poetry of science, something not appreciated by philosophers who are prone to take it as an insult, just as the word 'speculative' nowadays is seen as, when it is meant to be the opposite. However, when he claims that philosophers are entranced by mathematical rigor, which may be true, they should look for their own rigor it is not clear what he means, maybe not even to himself.

When it comes to axiomatics one should not confuse mere *presentation* with *content*. Philosophers labor under the absurd impression that mathematicians use it as a basic tool of discovery. And one may point out even more forcefully than does Rota the folly of trying to axiomatize philosophy. I would say the obvious that the axiomatic method should be an object of philosophical study not the basis for it. Maybe here Rota is writing tongue in cheek, irritating the philosophers needlessly. Anyway to continue the reasoning. In mathematics you may start with a definition, but in philosophy you will end with one. If you could start with a clear one, there would be no need for argument all. But definitions in mathematics are not arbitrary, even if mathematicians love to pretend so. A good theorem is not seldom turned into a definition. And a definition proves it mettle by the theorems it inspires. Mathematical presentation necessarily follows a linear order, just as the exploration of a picture does. But understanding does not. You do not understand a two-dimensional picture, or for that matter the three-dimensional one it suggest, linearly, but in a way that transcends the original exploration, which, in the words of Wittgenstein, is but a ladder to be discarded, once you have reached your destination. In the same way you can amuse yourself by taking a collection of facts, say about the real numbers, and pick any at random and prove the others ,and then reverse the process. It it is a good exercise to familiarize yourself with the notion of the reals. Mathematics is not a string of facts held together, but a warp were everything is somehow connected with everything else. As Rota puts it. *The truth of a mathematical theory is distinct from the correctness of any axiomatic method that may be chosen for the presentation of the theory.*

Axiomatics is an afterthought, Rota claims. It is concerned with syntax not semantics. There are many different axiomatizations of the real numbers, to give one example, thus indicating that the reals existed before their axiomatizations, and presumably would exist without them as well, one is tempted to add. Humans are not machines, and mathemati-

[7] In programming you need of curse to be very precise as the program is mechanical and cannot infer your intentions, as you are repeatedly reminded of

cians maybe even less so, and the axiomatic approach beloved by philosophers and other non-mathematicians is not how mathematics is done. Not that it is superflous and bad, it has its undeniable uses, after all it is, as I am fond of saying, an encoding of mathematics on par with representing a picture oixel by pixel, and as such having its undeniable uses. It allows one to check arguments and systematically look for flaws, but this is often, I would say, a last resort. One could as well characterize the writing of novels as a matter of putting one letter after another. On some level this is undeniably true, but not very enlightening. On the other hand as Emil Artin used to claim, an aximoatic presentation could reveal hidden analogies (such as between algebraic curves and number fields), something which also Dieudonne makes a point of.

Now we come to the gist of Rota's critique of modern philosophy, a criticism related to the concept of amputation to which I have referred to above. By throwing away whatever resists a precise treatment in philosophy, the subject has been impoverished, now dealing mostly with language. This is not a very original nor extreme criticism, but one which is often voiced. I have referred to Collingwood above, but many classical philosophers, such as Russell, have confirmed. In fact Russell deplored the later Wittgenstein as mostly being concerned with social party games. And to relegate the core subjects of philosophy to psychology does not solve the problem. But, as he admits, psychology, neurophysiology, and computer science departments may provide a welcoming home for the expelled problems.

At the heart of it lies the fundamental reductionist misunderstanding that thought is a mechanical process. That the way to solve a mathematical problem is to stare at it, without in any way to become acquainted with its history, what other people have thought about it, in short to get an understanding of what it is all about. This misunderstanding is the basis for the idea that problems can be solved by mental power alone, hence the idea of I.Q. (and which is also quite prevalent in vulgar ideas about A.I.).

What is the solution to the present problems of philosophy? To take the great philosophers of the twentieth century seriously, and he mentions three important philosophers, Husserl, Wittgenstein and far more remarkably Heidegger. And here we find our way to phenomenology in which Rota has found salvation. He starts out by a phenomenological view of mathematics, in particular its notion of truth. There is the truth of facts, as in any other science, and there is a formal truth. In Kantian terms the first are synthetic proofs, the second, analytic. Analytic truths are often identified by tautological, which of course are not the same as trivial. The idea is that the truths are somehow inherent in the axioms, hence the often made claim that deduction does not derive any new knowledge, and Rota would no doubt agree, as mathematician do not discover significant facts by deductions, which should be thought of as glorified computations. Mathematical facts are discovered, then the game is to trivialize them, meaning to make them analytical and thus inevitable. The same things go on in science as well. When a physical fact has been mathematically formulated, it has gone a long way toward finding a analytic inevitability.

Now Rota spends some time discussing the beauty of mathematics and claiming, somewhat shockingly, that truth has no more objectivity than beauty, both are context dependent. Clearly he is here talking about formal truth (or so one hopes). Now he proposes to talk about Enlightenment instead of truth, suspecting what this is really meant when mathematician speak about beauty. You may understand, at least formally, the statement of a theorem in its preciseness; just as you may follow the deductions of a proof (what mathematicians usually refer to as local understanding) yet remain profoundly puzzling. What is it really good for? A reaction one often encounter among students, much to the frustration of instructors. What has not been achieved is enlightenment, which unlike truth and beauty, comes in degrees. How does it fit into a larger picture? And it is enlightenment, as I understand it, which gives the conviction of proof, not formal chains of reasoning. As I would put it: We may take part of an impeccable logical piece of reasoning, but would we trust our lives to it? But if it comes with a sufficient (what is sufficient?) degree of enlightenment, we are more likely to do so, I would add. If there would be no enlightenment in mathematical activity, it would be a strange one indeed, and attract very few, and be most marginal. It is the enlightenment which makes it come to life and make it useful and accord respect, even prestige.

What is the point of mathematics? To discover facts or to prove things? The proof of Fermat's theorem is extremely complicated, but the result is, frankly speaking, uninteresting. One may be tempted to conclude that in number theory proofs are what matters. A result is judged by the effort it takes to prove it. Geometry is different. Here the results are important and have many applications or should we say consequences? The actual proofs are often easy (Rota has plane Euclidean geometry in mind for the latter remark). But of course the distinction is pointless. Proofs and theorems are interchangeable. Just as a theorem may have many proofs, a proof may have many theorems. The actual formulation of Fermat's being one of the least interesting of Wyles proof. The interest of a proof lies in its potential to generate other theorems as well as giving enlightenment why a thing is actually true. A verification may ascertain that a fact is true (in a proper axiomatic setting but it does not explain. Hence the computer verification of the Four Color Theorem fails to satisfy mathematicians. It gives no clue why it should be true.

When it comes to Husserl and phenomenology the going gets tougher. For one thing he introduces some indefinable terminology. łFundierung is one, and *evidence* is another. The first has no English translation, the second is given a new technical meaning. Evidence is hold to be the fundamental notion, truth only a derived one. The point of a proof is not to prove but to communicate evidence. The evidence as experienced by the writer to be reconstructed by the reader. It is not easy to follow Rota here, He ends with the statement that understanding is nothing but synthetic *a priori*, credited it to Kant and Husserl.

Fundierung is different. It is meant to be a new logical concept to change logic and logical reasoning profoundly but it cannot be given a formal definition, not even a description. The one stratagem to follow is to give a number of examples, this being the standard reasoning in philosophy, in fact the way to make its reasoning rigorous.

As noted in the introduction one example is reading. Some parts of the act of reading can be described physically as interacting with a text, which also has a physical meaning as ink splatted on paper. Those are part of reading that takes place in space and time. But then there is also the notion of the contents of a text, which is the important thing after all. But what is meant by the contents, can it be placed in space and time? You may remember the contents but not in which language it was written. How do you absorb the contents? Are the contents simply codified into neural connections in the brain? Is the brain 'reading' those neural connections, as it is 'seeing' the image on the retina?

That would be absurd, according to Rota, we need to accept that the most important aspect of a text, namely its contents. do not exist, at least not physically. Fundierung is the fundamental relation between something that does not exist physically and something that does. What does not exist physically, is not called spirit or mind, but function, at least by Rota, taking the lead of Ulam. And what anchors it to reality its facticity. The latter is necessary, but the specific forms it takes, are irrelevant. It does not really matter whether you learn something in Italian or German, provided you command both languages. But if the light is bad and you have trouble reading the letters, or the language is unfamiliar with you, and you have to struggle to understand the words and the way they are put together, the facticty intrudes at the expense of the contents. When you recall the text, what may stand out is not the contents but the struggle you had in reading the particular text. It is the same thing with an idea. It cannot be captured by a precise formulation, as a theorem can, or an argument in a chain of deductive reasoning. But to convey an idea you have to phrase it somehow in the hope that it will be reproduced in another mind. My report on Rotas explanation of Fundierung is not a verbatim one. I have to some extent made it my own by contributing my own interpretations of it, not being content by just quoting Rota. Would I have done so, I would have been a mere passive transmitter of texts. What I am trying to convey is the contents of Rotas exposition and to do so I need to have appropriated those contents to convey them, as one says, in my 'own words'. If I do not do so, I am not honestly trying to communicate. In my own words I go beyond what Rota actually says, and that can be considered bad, but contents can also be seen as ideas, and ideas are not confined but spread by associations to other ideas.

Further examples are given. There is a distinction between 'viewing' and 'seeing'. 'Seeing' is the facticity, which makes the function of 'viewing' possible. 'Seeing' is a purely physiological process taking place in space and time. 'Viewing' is not. The point of 'seeing' is 'viewing', the latter would not be possible without the former. In philosophy you do not argue deductively as in mathematics, you argue by example, as Rota points out. Also in mathematics you argue by example, and in fact this is the most efficient way to learn a subject. You do not understand a definition unless you see examples of it (what Rota refers to as a description), you do not understand a theorem unless you see how it applies in particular examples. In fact this is how you make mathematical discoveries. Precise formal statements serve as facticities. As David Mumford has remarked, it is easier to see a specific representative example and then formulate for yourself what it really means instead of getting a formal statement and try to decipher it.

I would like to digress a little. The modern metaphor of Fundierung is between soft ware and hard ware, although that is a little bit misleading. Even in the computer program, which cannot really be thought of anything radically different from a formal proof, there is much of facticism. Anyway I used to wonder how different hardware figures in the arts. In painting it is the actual painting that counts. It is this which is bought and sold, and if lost, one thinks that something irreplacable is gone and which cannot be preserved by copies. The Art world, when it comes to painting, is actually a commerce in which artefacts are bought and sold, their values seldom related to their artistic, but are purely speculative. When it comes to music the actual physical scores play no rôle, they can easily be multiplied without anything being lost. A musical score is never enjoyed directly but has to be interpreted and performed. It is also important that not a single note is changed in the score, that would amount to a disfiguration. The same thing holds for literature. The actual printed pages can be copied, but formulations are not to be changed, not even a single letter is supposed to be touched, although some minor modernizations seem to be accepted. This is of course even more pronounced when it comes to poetry, compared to prose. Prose can be translated, poetry cannot. If you read something in prose chances are that you have forgotten the actual formulations, sometimes even the language. You do not normally commit prose to memory. It is different with poetrty, and hence a larger part of poetry is ito be found in its form, in its facticide. The sense of poetry makes less sense than its sounds. Contents, although important, is secondary, and more in the nature of a bonus. When it comes to mathematics it is the ideas that matter, faithfully recorded formulations do not. Only a historian of mathematics would be concerned with the exact formulation, for a student and a mathematician, the actual content is all. It does not matter how it is formulated, as long as the content is not effected. The artistic value of a piece of mathematics, as mathematics and not presentation, is independent of the formulation, and resides entirely in the contents.

Now all of this appears fairly clear to me, but this may be because of a certain convergence of temperaments between the author and the reader in this case. In what ways do the notions of 'functions' and their transcendental existence beyond space and time, however suggestively presented, differ from classical notions of mind as different from matter, and Platonic realms, ideas which have been rejected by modern philosophy. Modern philosophy may of course be wrong, after all a consensus is but a social fact, and we are back to the classical outlook, which is so congenial to us. However, the great difficulty, if not necessarily the impossibility, of coming to terms with what Rota calls functions, through mere facticities, that the A.I. project encounters, may very well be connected to phenomenological analysis, without one having to accept the ultimate metaphysical implications of it.

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