The Mathematician's Brain

A personal tour through the essentials of mathematics and some of the great minds behind them

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This book promises a lot and delivers but little (of what it promises). This is not to say that it is not a good read, it is entertaining enough[1], only not providing what you expect. Least of all it gives you any information about the brain of a mathematician, which, if you think about it, is a relief. People have suggested that the only way to get a scientific understanding of number is to study the human brain, which supposedly has created them. Frege would turn in his grave.

Now the point of departure is that mathematics is the study through logical deduction of systems of axioms with the point being that it is meaningless to ask whether the axioms are true or not, what is the concern of the mathematician is only what can be derived from them (and damn the consequences). This view of mathematics can be compared to the presentation of a picture pixel by pixel. Undoubtedly an objective presentation, with its undeniable uses, such as copying and manipulation in digital media, yet one that does not give any clue as to what is a picture really? The human mind when seeing a picture understands it in some mysterious sense, as it somehow pops up; but is baffled and left in the lurch when confronted with a pixel codification involving millions of bytes. The meaning of it all remains in the dark. In fact 'seeing' is a classical metaphor for 'understanding' one can be presented by long chains of logical reasoning yet be puzzled by what they mean. In such cases you often talk about 'local understanding'. You can see how the different pieces of the jig-saw puzzle fit together with each other, but the picture itself is opaque to you.

The idea of reducing mathematics to logic and thereby give it unassailable foundations was enthusiastically pursued by people like Frege, Russell and Whitehead, and even Hilbert fell partially under its spell, as he is seen as responsible for the formalist view of mathematics. But Hilbert never was a formalist at heart, he had a definite purpose, namely to show the solidity of mathematics, meaning as being free of contradictions, as well as its power (*wir müssen wissen, wir werden wissen*). That project foundered as Gdel showed, thus giving the death-knell to the idea of mathematics just being a formal game. Contrary to what outsiders may believe, Gödel's theorem has had no real impact on living mathematics, only killing conceptions of it as suggested by 'Principia Matematica'. Now to understand the axiomatic method one can do well by making the distinctions between axioms and postulates. In the treatise by Euclid, whose importance cannot be overestimated, axioms refer to principles of thought, postulates about the facts of physical space. Those axioms and postulates are not arbitrary but are based on intuition. One of the great

[1] It gives some light gossip on mathematicians such as Grothendieck and Turing, but there are actually few of them who make even a come-on appearance. On the other hand the book is commendably compact.

insights of the Greeks was the limitation of the deductive method, ultimately it was based on facts and principles lying beyond the reach of logic, just as the faith in rational thinking cannot be proven, as opposed to tentatively justified, by rational thinking itself. The ultimate point of the deductive approach was not to achieve absolute knowledge, for that the old Greeks were too sophisticated, but to make reasoning transparent and communicable (be it only in a mechanical way) and amenable to critical scrutiny and improvement. One may see this as the essence of democracy, namely not the vulgar notion of representation and vote, but of free discussion, in which the power of an argument has nothing to do with its originator and its way of originating, but is intrinsically determined by its power as argument *per se*, in other words how it fits with other arguments, confirming or contradicting, opening up new arenas or closing others.

Now to make the distinction between classical deductive systems and the modern ones more clear, one may consider the notion of axioms for a group. When I first encountered this I was puzzled because it seemed an abuse of the notion of axioms, as their existential qualities were stripped off and what remained was only a convention. You should not think of this as axioms but only as a definition of the notion of a group. While classical axioms pertain to some reality, I am thinking of physical space or our ways of thinking, those seem to be merely inventions of the human mind. Do groups exist? I.e. are the axioms consistent? Yes, we can find a plethora of objects satisfying them, which also points to the fact that those axioms have many, what the logicians call models. Should not axioms define a unique reality? However, groups may be a human invention and thus a historical contingency, yet they seem very natural, and there a lot of facts about them that we only discover through hard work. The concept as such existed long before its 'axiomatic' formalization. Disregarding the trivial examples of groups which existed before the group concept was even conceived (given the trivial group, there is little in it that points to the general concept) the theory was implicit in the works of Lagrange and in the early work of Gauss [2], and became more systematically studied as subgroups of permutations by Galois. The notion of an abstract group only slowly got a hold on the imaginations of mathematicians, now of course this is the way most undergraduates encounter groups, and it seems to us a very natural way of quickly introducing the concept. Then, more importantly, the axioms are so few and so simple that it is very hard to imagine that such a rich theory can arise out of them. Thus when mathematicians study 'axiomatic systems' they bring to it concepts that are no ways related to the axioms but are somehow forced on them in their study. In fact the notion of a group may also be said to be forced on mathematicians. The reason for that is of course when the notion of groups interact with other parts of mathematics it acquires meaning through a rich web of associations, and of course this inter-relatedness lies at the heart of mathematics, its fascination, and not to forget, its 'usefulness', which adds to its fascination.

Now the author wants to address the intelligent layman, but unfortunately you cannot assume that such an entity knows any non-trivial mathematics. In other fields of science

[2] In his teenage solution of showing the impossibility of constructing a regular 17-gon by ruler and compass, in which he anticipates the essentials of Galois theory in an abelian setting, and his later work on quadratic forms in which he endows forms up to a suitable equivalence with what we now would recognize as a group structure.

and endeavor, this is not a problem, you can communicate with your intelligent colleagues and be sure that they understand what is going on, but not in mathematics, where technical difficulties put up unenforceable barriers. Ironically what you can with some success communicate without having to dumb down is the foundations of mathematics. In principle it is possible to explain Gödel's proof to a layman, there are subtle arguments involved, but really no technical obstacles[3]. Even simpler than the Gödel proof is Turing's negative solution to the halting problem, which shows conceptual light on Gödel[4]. What is involved? First the notion of an ideal formalization of mathematics and hence the ability to write down a formal proof in symbols. This indeed is the basis for the metaphor of pixels. For people engaged in programming on some reasonable basic level, this is quite natural. Now it is easy to imagine a mechanical process that systematically produces all strings of symbols, the numbers very quickly become exponentially huge, to the frustration of people in AI. For each such string you can check whether 1) it is a syntactically legal string (otherwise it is nonsense) and if so 2) whether it constitutes a correct proof (which a more demanding syntactic condition). In this way you will produce all valid theorems using a 'simple' computer program, simple in the sense that it is so much shorter than the data it will produce, data which is potentially infinite. Given such a cleverly presented list, cleverly in the sense that it will be arranged along longer and longer strings, you can check whether any proposed theorem is a correct one or not, or rather correctly provable or not, simply by searching a finite list. Remember though that finite numbers can be very large. Is this mathematics or philosophy? In a sense it is more philosophy than mathematics. It is the philosophy of meta-mathematics. Meta-mathematics is not formalizable as our piece of mathematics ostensible is, but you can ask clever questions of it. And Gödel was almost able to formalize meta-mathematics, not quite, but sufficiently to show the limitations of formalized mathematical reasoning. Formalizing means that you strip what you are formalizing of any meaning, but then of course this process of stripping is part of meta mathematics and and not formalizable, because it has meaning, it corresponds to something, namely the formalization of something admittedly meaningless. The formalized strings carry no real meaning beyond that how they fit with other strings, but this matter of fitness has meaning. A formal axiomatic system becomes something almost physical in the real world, as it can be encoded in a computer which is a physical entity subjected to the laws of physics. In fairly straightforward ways you should be able to, at least when it comes to basic logic, make the laws of logic into physical necessities (that is the point of a computer, which obviously engaged the imagination of certain individuals long before the advent of electronic computers). Now you can ask whether a given system is consistent or not? Is that a question of mathematics or physics? In standard ways the question of consistency can be reduced to to whether two contradictory statements can

[3] many presentations spend an inordinate amount of time on Gödel numbering, this can be done in many ways, the main thing being the principle, not the specific implementation

[4] It is not quite as strong as Gödel's proof and hence more accessible. Implicit in the latter is the notion of a precise conception of what is meant by formalization. Likewise the Turing machine is an attempt to define what is a computation, and is now seen as the definite notion as it was shown to be equivalent to Church's approach. It is this notion of computability that shows the uncomputability of the Halting problem. The punchline in both approaches hinges on Cantor's diagonal trick of putative self-reference.

be made. Now where does mathematics enter? In the notion of infinity which does not have any physical sense, but seems to be a fruit (or a figment?) of the human imagination. If there exists a contradiction this will be discovered in a finite number of steps. Thus the existence of a contradiction is a physical fact, it says something about the real world, because the axiomatic system is a part of the real world, and if you do not agree, let us just implement it on a physical machine. A simpler example is whether a given Diophantine equation has a solution. You just check by systematically inserting numbers and see if it works out. If it does, you will eventually discover it after a finite number of steps [5]. But what about the statement that it does not have a solution? This method of trial and error will never succeed because you have to perform an infinite number of steps to reach it and this is impossible. Is this not some kind of intuitive axiom pertaining to something real and something we need to assume and accept being beyond our powers of proving on more basic premises. Infinity emerges in our imagination, and if the world is large enough it must contain infinity as it contains the human imagination and that contains infinity. Yet in a physical palpable way infinity has no meaning. Or has it?

As a mathematician you are contemptuous of philosophers who dwell on the paradox of Achilles. This is child's stuff. However, it took me some time to really appreciate this paradox, because what it suggests is to exhibit infinity in a finite setting. As Achilles overtakes the turtle he experiences an infinite number of events. He does indeed count to infinity (and beyond). He cannot do it aloud, nor even think about each event, because it hinges on the assumption of the idealized mathematical point with no extension. Do such exist physically, does the real line (and hence Cantorian uncountability) exist physically? When you as a teenager learns about the real line and various infinite processes connected to it, the real lines gets an almost physical palpability[6]. As a mathematician you get used to infinities, they are no longer metaphysical concepts, you handle them every day. In fact the infinities connected to the real line and related concepts are very different from the infinities related to the integers. We are taught the integers and out of those we derive the real line, which we hence think of much more complicated than the integers, when in 'reality' it is the other way around. As mathematicians we are somewhat thwarted by the educational system, the Greeks were not, they made a clear distinction between integers and quantities the latter eventually to develop into the modern concept of the continuum. As I understand it you can axiomatize the real numbers and avoiding the Gödelian complications connected with including the integers. This is a bit subtle, but professional logicians must become subtle.

Now the kind of reasoning applied above is non-technical and can be understood by a child in its early teens, not really more subtle than the Cantorian diagonal argument. But the kind of mathematical reasoning that is necessarily formalized in order to be amenable to a kind of meta-mathematical treatment is a parody of how mathematicians reason. They do not mechanically manipulate symbols (although for some of ones colleagues one has ones doubts) to them mathematical concepts are ripe with meaning otherwise they

[5] Astronomical numbers are very small, and soon one gets into considerations where the amount of time needed goes well beyond the physical life-expectancy of the known (finite) universe.

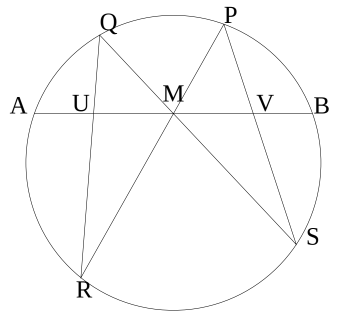
[6] Is your life an open set or not? In other words does there exist a last moment of life, or a first moment of death? Those are teenage reflections.

would not make sense nor stimulate the imagination. Mathematicians are as unaware of the axiomatic grid of their theories as painters are unaware of the pixels of their creations. You do not paint Mona Lisa by dividing the canvas into small squares the sizes below that of human optic resolution, and then painstakingly applying minute amounts of paints on each. Mathematics is far more exciting and subtle than any theorizing about it, if many phenomena in the world gain, at least in fascination, by being reduced to abstract mathematical principles, thereby establishing hidden connections to other phenomena, this is not the case of mathematics itself. Yet philosophers tend to think of mathematics in this way, one only needs to be reminded of Russell and Wittgenstein and their conception of mathematics as a sequence of tautologies, as outlined above. This also is the source of the misunderstanding that everything is somehow hidden in the axioms, that deduction does not, unlike induction, reveal any new knowledge. Doing mathematics means inventing new concepts which both enable you to dig deeper into axiomatic structures and to aid your understanding. Such concepts are not to be found in the axioms, as little as various notions of chess are encoded in the rules, but are the creations of human minds wrestling with the facts. When it comes to the issue of Platonism in mathematics, the general consensus is that this is not only a naive but maybe even a stupid idea, suggested by the literal interpretations of Plato's metaphors. The author distances himself from the notion of Platonism, yet his conception of mathematics as something objective and consisting of eternal truths is unabashed mathematical Platonism. It is a mystery that something emerging from the human imagination should have this remarkable stability and create such a consensus unparalleled in any scientific endeavor. There are many fruits of the human imagination, the writing of fiction is one, but you cannot really imagine people working on the same novel without causing logical havoc, much less for future writers to pick up the pieces and develop along intrinsic lines. In mathematics you can pose all kinds of questions and meaningfully answer them, but there is no way you can assign a meaningful answer to what color of eyes Sherlock Holmes maternal grandmother had, it is clearly a matter of whim. In mathematics there are no whims. The old wish of a Chinese painter of being able to step into his painting, is actually fulfilled for the mathematicians, they can literally step into their own creations. Reality is a question of cohesion, of all kinds of senses to reinforce each other. Something you can both touch and see has 'more reality' than something you can only see, and in mathematics things can be proved in many different ways, each reinforcing each other, and that is this that gives to the mathematician the sense of reality, meaning correctness, not so much the result of a long chain of deductive reasoning, where, as we noted, there maybe local understanding but hardly global[7].

The purpose of the book is to persuade the layman reader that mathematics is so much more than the philosophical caricature that constitutes received knowledge, that it is far from a mechanized procedure (although admittedly much of that is also included in the day of a mathematician) but a creative endeavor involving concepts and ideas which are not in any reasonable sense inherent in the axioms. To convince a reader you need to do some real mathematics, and it is here the popularizer of mathematics runs into serious

[7] The author refers to René Thom, claiming that nowhere but in mathematics (and possibly theoretical physics) is logical reasoning employed at such lengths. This ought to be a rather obvious observation, and C.S.Peirce formulates this in more or less the same terms well over a hundred years ago

problems, as noted above. You can of course simply present definitions of polynomials or topologies, but if you do not see them in a context they make very little sense. If you have never manipulated simple algebraic formulas, then what is a polynomial? and the general definition of a topology, what sense can you make of it at all, unless you are already familiar with the topology of the real line, and more generally of metric spaces? The author remains nevertheless undaunted and tries, and in cases his efforts fail, advises the reader to skip if the going gets too tough, but unfortunately the ensuing text will then not be anchored, when lacking the prerequisite mathematical understanding. Different of course if you address somebody reasonably educated in mathematics, then you can still remind them of and then dwell on elementary concepts. For an elementary text it is unusual to encounter the Erlangen program, which the author uses to define different geometries. He gives a very nice example, not particularly mathematically interesting, but intriguing enough to make a professional mathematician pause.



Consider a chord AB in a circle, and let M be its midpoint. Let P,Q,R,S be points on the circle related to each other as indicated by the picture on the left (i.e P, M, R are collinear as are Q, M, S). Then M is also a midpoint for the segment UV. Now the triangles QMR and PMS are similar, but that does not lead us anywhere, as we cannot in Euclid's time-honored tradition find any congruent triangles from which we can derive the equality of the lengths UM and VM. In fact this does not turn out to be a problem in classical Euclidean geometry, but one in the realm of Projective geometry where you use invariance of cross-ratios under various transformations such as inversions in circles. Of course there is always an option of translating everything into some algebraic identity using coordinates, but this would be a bit ugly and hardly conducive to any insights.

The brain appears in the title and some comparisons with computers would not be amiss. The computer has many obvious advantages, such as speed and memory, yet so far it has proved to be inferior in so many other aspects, which more closely seems to be related to the human imagination, its intuitions and specifically its visual and linguistic skills. The book was written before the recent advances in deep learning and similar approaches, which seems to take full advantage of the memory capacities and speed with which they are endowed to set to work on huge data sets. The interesting point is that no one understands how the computer actually can recognize pictures, but the previous structured approach with a lot of intricate programming seems to have become obsolete. With this there is a renewed fear of the potential of AI. That was not the case when the author wrote his book, yet he admits to a certain fear that computers might become better than mathematicians in doing mathematics, because if that happens life would become less interesting and meaningful, but he finds comfort in that there is no indication that this will happen soon.

Mathematicians are a diverse lot of personalities, but what unites them? Do they have any common features? The remarkable thing is that mathematics is both a solitary pursuit, more so than that of the sciences, yet a very collective enterprise, because without the accumulated knowledge and the overarching culture engendered by it, a single mathematician would be lost. This culture is a human thing and has nothing Platonic about it. It is absolutely necessary when it comes to judging other mathematicians, not so much on correctness (but there is also some of that) as to the value and beauty of their work, and ultimately who will become honored in some way. Prizes never really did play much of a role in 20th century mathematics, earlier there were academies around the world who set up cash-prizes for solving explicitly formulated problems, the most famous being perhaps the prize set up by the Swedish King Oscar II and which was awarded to Poincaré in spite of, what would turn out later, an erroneous contribution. The mathematicians argued that prizes were superfluous, it was the purity of the ambition that counted. In fact the early Fields medalists did not get very much attention, it is only in later years the fascination with the medalists has spread outside the mathematical community. The author speculates that maybe it is no longer possible to discuss the work of individual mathematicians, because few are qualified to do so, unlike in the past, so how much easier is it not to refer to some prize such a mathematician has received. One may go a bit further and remark how much on hiring now is determined by various indices and publication records, relieving committees to actually read and ponder what the candidates have really done.

But now there must be some common traits among mathematicians, traits which either have attracted them to the subject, and traits which have been formed by the subject itself. One thing, to really do research in mathematics is very hard and frustrating, and I believe with most mathematicians, actually harder and more frustrating in many ways than in the sciences, on the other hand I suspect that the rewards are keener. One common trait I would attribute to mathematicians is intellectual honesty. A mathematician seldom if ever gives a lecture without real content. In fact a mathematical lecture is hard to follow, and most people in the audience are resigned to being lost most of the time. In the past chalk on black was the common form of a lecture, then later came overheads and then computer presentations. The advantage of the former was that mathematics was presented in real time, with the latter the temptation to get across as much as possible, often motivated solely by the lecturers desire to impress, goes beyond the natural attention span of a human. The inclination to really scrutinize an argument is something the mathematicians shares with the lawyer, but the ultimate motivations are very different. The mathematician want to get hold of the truth, the lawyer wants to obfuscate it[8]. In many walks of life making quick decisions are at a premium, but mathematicians write for eternity and are wary of jumping to conclusions. On the other hand the assertion that mathematics is so hard and demanding that mathematicians are prone to mental collapses, seems a bit harder to substantiate. He refers to Hilbert having bouts of depression, and the serious nervous break-down suffered by Felix Klein, supposedly as a result of competing with Poincareé, yet it is hardly convincing. True mathematics has its share, and maybe more, of eccentrics, and undeniably there have been cases of insane mathematicians, even criminally so[9] but it would be stretching things too far to claim that they are borderline psychotics.

[8] Famous is Cicero's remark that a brief to be read to a jury was only meant to dazzle for the moment, not to survive the night.

[9] In the first case the most notable example is of course the schizophrenic John Nash, in the second the French mathematician André Bloch, who killed his brother as well as his aunt and uncle, ostensibly (and ironically) in an eugenic spirit to eliminate mental illness in the family.

Now mathematical papers are never written in a formalized way but use modern natural languages[10], still articles are rather formal, which sometimes make them a bit impenetrable even to professionals (but yet who else would read them?). Mathematics is filled with meaning, but that meaning is not always apparent. Figures do help a lot, even if they are strictly speaking superfluous in a rigorous logical presentation, and side remarks can show a surprisingly effective light on matters. Those seldom appear in published papers, where completeness and brevity are both striven for, but can find a natural place in a lecture, where you cannot expect but a fragment. A standard phrase, used in connection with so called hand-waving, is 'to get the main idea across'. This does not always happen, but when it does it usually make up for the times it did not. The important thing is that an idea can never be properly formulated, but has to be evoked by means fair or foul. A theorem can be given a precise formulation and hence become a component in a jig-saw puzzle to be used in further investigations. An idea belong to the meta-mathematical realm, and can never be a piece in a deductive construction, but instead serve as motivation without which there would be no meaning of the deductive constructions in the first place, nor any effective means of bringing them about. A theorem can have many proofs, but an idea can have many theorems based on it. The capacity to spontaneously grasp ideas is the sign of the budding mathematician, to generate them, the prerogative of the very great. Most people who study mathematics in an educational situation do not grasp ideas, hence its pursuit becomes very strange, just an obstacle thrown in their way. One good way of illustrating it is the attitude towards formulas. For most people they are just ways of computing an answer as in 'what formula should I use?'. A mathematician sees them not as fixed entities, but dynamic ones crying out to be manipulated and transformed. Thus the visual feature of a formula is very important and much of their power lies in the readiness to be visually understood. And finally: In painting, only the actual physical painting has the ultimate value, any copy is inferior. In literature, on the other hand, copies are just as valuable as the original, in fact there is no original (unless you mean the messy manuscript of the classical author, but that has mostly antiquarian interest) but the exact wording is important. Change the words in a poem and it is likely to fall flat. But when it comes to mathematics it is a matter of the ideas, and they have no preferred formulations, as long as they mean the same thing[11]. An idea is worked out in a deeper way that the mere analysis of a poem. In the case of the latter you may wonder why write a poem in the first place, would it not better to go straight to the result of the analysis? To that one may argue that the point of a poem is how it sounds, not what it means. Of course poems can be thought of as ideas when read by other poets and transformed according to their temperaments. Then we are getting closer to the case of mathematics.

And finally again, beauty seems to take an important part in mathematics, but often you have to be a mathematician to appreciate it (exceptions can be made for mathematics

[10] In the past Latin was used for most scientific and mathematical purposes until it was abandoned in the early 19th century. As many people have remarked, this was really an unfortunate development. Latin for mathematical purposes is easy to learn, not that hard to write and even easier to read, while a natural language would require years of study.

[11] Or roughly the same thing? This often does not make sense as ideas notoriously resist being precisely pinned down, but have rather 'cloudy' structures.

that has striking visual manifestations). But, the author reminds us the aesthetic appeal of mathematics has not so much to do with mathematicians being artists. True some mathematicians try to write literature, but even if the results not seldom are not bad, they are never great[12]. Musical ability and mathematics are often said to go together, but then one is talking about performers not composers, of whom there are surprisingly few. And as to painting there are of course a fair share of competent draftsmen among mathematicians, but no real recognized artists, which is noteworthy given the dependence on visual representation in mathematics as opposed to melodical. A picture can support, and sometimes even replace, a mathematical argument, a melody cannot, in spite of all the professed hidden mathematical structures in music.

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[12] Who are the best writers among mathematicians? Many writers have had mathematical talent, such as Stendhal, Solzhenitsyn and maybe also Musil, but not on a professional level, the one exception may be after all Lewis Carroll, a.k.a. Charles Dodgson