Scale

The universal laws of life and death in organisms, cities and companies

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In the fifties there appeared a cultural anthology on mathematical writing edited by James Newman[1]. The English title of this multi-volume enterprise was 'The World of Mathematics' and it was translated into Swedish as 'Sigma'. However, it was in its original English incarnation I encountered it in my mid-teens, and as with most books you encounter when young and malleable, it made a lasting impression. However, it was not a sample of pure mathematics which made the deepest impression on me, but a chapter called 'On Magnitude' lifted out of d'Arcy Thompsons classical 'On Growth and Form' which really caught my imagination. It was on scaling and its implications in biology and mechanics and it did really, as they say, open my eyes. The fascination was to be found in how some very simple mathematical principles could explain and illuminate diverse and complicated phenomena. This is of course nothing but the seduction of Platonism, that behind the bewildering complexity of the world known to us by our senses, there lies a simpler truer world, accessible to us only by our minds and intellects. At the heart of the matter we find Galileo's principles that the strength (be it of muscles or support) scale as surface areas, thus as squares, but masses by volumes, hence as cubes. Similarly for heat loss, which gives a rather accurate estimation of the metabolism of hot-blooded animals[2]. In fact one sees that it then would scale as the power 2/3 with linear dimensions, underestimating it for long and skinny animals, with a relatively large surface area. The typical scaling would then be given by power-laws. Among the illustrative examples of such laws we may consider Keplers Third Law, namely that the orbital times T of plaets scale as $R^{3/2}$ where R is the distance[3]. From this we note, observing that T^2 is also inversely proportional to the mass, that if we scaled the Solar system the orbital times would not change, thus we could not determine the dimensions of the system by simply looking at its qualitative aspects, while in many other applications there are indeed drastic consequences. E.g large animals need to be much stockier with a arger proportion of their

[1] 1907-66, a lawyer and mathematician, whose anthology was fifteen years in the making

[2] One of the group assignments I used to give to my teacher-students was to fill a milk-cartoon with hot water and see how quickly it cooled, by scaling this gave a fairly accurate estimate of the energy need of humans, under the assumption that most of the energy we need is simply to maintain our body temperature, just as almost all the energy the Earth receives from the sun merely keeps it at a clement temperature.

[3] This is easy to derive from the inverse square law in the case of circular motions, where R is the radius. In the general case of elliptical orbits R is replaced by the major axis, and the derivation is slightly more involved and maybe not as transparent. See the appendix.

weight taken up by their skeletons, than small animals[4] The fact that things change with scaling indicates that matter cannot be scale invariant, which in the case of biology points to the presence of building blocks, such as cells, and more generally to atoms, which gives to the physical world natural units that do not exist in the idealized Euclidean world[5], an important observation not made by the author.

Now for readers unfamiliar to scaling in general and d'Arcy Thompson[6] in particular, this must be very exciting and seductive, at least if they are mathematically competent and inclined; and thus view the book as much more spectacular and groundbreaking than it really is (not the the author claims undue credit, he does in fact later on in the text, refers not only to both Galileo and d'Arcy Thompson[7]ut also to Haldane and Huxley[8] and his allometric scaling, but only as an afterthought and thus wittingly or unwittingly misleading the reader about the originality of what is being presented). The real contribution of the author is to extend those classical scaling ideas into the realm of the social world meaning the growth and decay of companies and cities, noting that companies die, most of them quickly, but cities on the other hand never seem to die, but flourish for centuries. The research that goes into this strikes a pure mathematician as rather pedestrian, which to some extent (maybe even to a large extent) is due to a lack of proper appreciation of the nature and challenges of applied mathematics. As the author notes: in biology as opposed to physics the hard work is to formulate the relevant mathematical problems, once that is done, they are fairy simple to solve; while in physics it is the other way around, in particular calling for much more sophisticated mathematics.

Now the ambition of the author is to explain it all to the mathematically illiterate, which is a classical challenge, which sadly is seldom successfully met. The author takes great pain not to be technical, and eschews any semblance of a mathematica formula, but does include a lot of diagrams, maybe forgetting that diagrams may be almost as much of a challenge to a general reader as a simple formula. The mathematics needed is not difficult let alone sophisticated but should be within the grasp of anyone with normal intelligence and in fact ought to be known by any educated person, as far as the notion of beong eduated has any real meaning. In short what the author is trying to convey should have been part of basic school education, and if school has not managed to convey such understanding during the years of an individuals optimal period of learning, why should an hour or so of reading, often at a mature age, be able to make up for years of neglect? You do not

[4] As a child I was puzzled by the fact that when I had my toy cars colliding they did not suffer the dramatic damage that real cars did. The point is of course that kinetic energy grows with the fifth power not the third.

[5] Thus in the Newtonian world matter has no finer structure, thus making it more geometry than physics

[6] The true significance of Thompson's work was to exlain the physical constraints on evolution withot which an acount if it would be sadly incomplete. Many things can evolve, but the laws of physics cannot be flaunted. In fact external constraints lmits evolution to certain furrows and thus explains much of the convergence it exhibits. While the vision of Darwin explains the great confusin variety, uderlying constrains can be thought of as Platonic forms.

[7] b

[8] Grandson of the great Huxley, and brother of the novelist

learn mathematics by simply being fed information, you need to internalize it and obtain understanding, that elusive concept which lies at the heart of mathematics, and which the author raised and educated as a theoretical physicist, well understands. One can compare it to finding your way by GPS or a classical map. The GPS gives you clear commands what to do and you just follow them mechanically as would you be a machine, while in a map you figure out yourself what to do and thereby 'understanding' your choices. The first case is very unstable, if you miss one command, you are literally lost, while the second allows continuous modifications. It not only allows internalization it demands it, and as a consequence your imagination is engaged, while it is totally absent in the former case. Sadly, as technology gets more and more sophisticated as well as 'user-friendly' there is less and less demand on the imagination, and by implication the intelligence as well, because it is hard to separate them. When it comes to mathematics most people seem to prefer the linear way of GPS orders rather than creating their own two (multi?) dimensional maps.

There are certain things every person should know, among them some elementary but basic mathematics. If you do not, you are lost in many settings, such as the one presented by the author, and your ignorance will not be remedied by a crash course of a few pages. If I had not known what an exponential function was I would hardly have been enlightened by the explanation of the author[9]. An elementary explanation does not need to be rigorous, this is not the problem, but it need to engage the imagination of the reader, and there is the real challenge. Among the things people should know are arithmetic and geemetric series, constant differences versus constant quotients. Nowadays they are replaced by lnear and exponential, sexy buzzwords most people do not understand. They were good for Malthus and should be good for the general reader as well. Intimately connect to exponential are logarithms and without a reasonable firm understanding of them you are not properly an adult[10] The interplay between the arithmetic and geometric in the above sense is a fundamental theme in much of basic mathematics as well as its applications. Weber's law that the strength of sensory impression is subjectively proceeding arithmetcally, but really proceeding geometrically, the notion of magnitudes of stars being a prime example[11]. Now power laws come ut very nicely when graphed on log-log paper (i.e. both scales being logarthmic) because then the graphs will be straight lines (easy for anyone to see and understand) with slopes corresponding to the exponents.

The metabolic scaling has actually exponent 3/4 rather than 2/3 as you would get if you naively thought that it was mainly a question of staving off heat loss. But how do you get the right exponent? The explanation given by the author is completly incomprehensible to me, maybe a mathematical explanation would have been more accessible? This points to a common pitfall, of what the French calls vulgarization, making things

[9] There is one exception. The author imagines bacteria growing exponentially in a petri dish, say with a doubling time of a minute say, and if the dish is full after half an hour, when was it half full? Illustrating the sudden emergence in exponential growth

[10] It is hard not to sympathise with Snow, without such understanding science will for ever be beyond you.

[11] The reason being that there is no canonical unit to measure them all, so you need to compare the sensations two by two, i.e. relatively to each other. William James is a bit skeptical about it in his famous book on Psychology, but I suspect he did not understand the principle.

equally incomprehensible to the layman and the professional alike. There is some intriguing but non-sensical reference to the body being four-dimensional, and this explaining the denominator being four. But disregarding this rather serious flaw, there is an interesting digression on the vascular system, which lies behind the attempt. The vascular system is a tree that needs to reach every cell in he body. Mathematically it is a fractal, whose closure fills up the whole body. In reality the fractal nature has a necessary cut off point at the size if individual cells. It structure is that of a branching tree, which at each bifurcation splits up into two vessels of the same area adding to the original area of the branching vessel. There is supposedly a physical reason for that we are told, namely to minimize resonance effects in the ocisllating blood flow. There is a similar, but not quite as rigid structure of trees, where at each branching, as already observed by de Vinci the crossections at each banching add up area wise to the original area. There is a striking mechanical explanation for that, namely we can think of the tree as being made up of fibers, each going al the way to the leaves at the very end. Some features should be pinted out, common to both. The size at the end of the trees is the same for all animals and trees, cells (and leaves) being of the same size. What differs is the number of branching levels, the bigger the animal, the more branching levels. Basically each level increases the mass of the animal with a facor of two. Also the original trunk, the main aorta, will predictably increase in size. Another important fact is that when the vessels are small (and narrow) the blood can no longer pulsate. The author likens this to the transmission of electricity by either alternate current or direct current, the former with oscillation being much more efficient. Tus for small animals, most of the volume of blood is flowing in DC-mood, maing up for less efficiency,.

Now this model should not be taken literally, because if so, the number of cells in a body would always be a power of two, and animals would come in discrete sizes, the next size twice as big as its preceding. This shows that in applied mathematics you cannot pursue things literally, the role of mathematics always is to some extent a metaphorical one, not for calcuation but of guidance and illustration, meant to stimulate the imagination not to replace it. As I never tire of saying, when metaphors are taken literally they just become silly. The author speaks about coarse grained pictures that highlights the basic structures. Interesting as the discussion is, it does not explain to me in what sense the human body can be thought as 4-dimensional by adding one dimension to the 3-dimensional vascular system. And even less as how it explains the 3/4 exponent. But anyway it all provokes some elementary mathematical reflections relegated to the appendix.

There is also an explanation why an organism cannot grow beyond a certain size. The idea being that the energy for maintaining the body becomes greater than the ability to supply energy, thus there will be nothing left for growth. Implicit in that reasoning is that an organism has certain constraints suppled by its architecture so to speak, and those cannot be changed uring its life time, i.e. tey are genetically determined. Of course evolution can change the genetic set-up, thus a mammal can be as large as an elephant, but a human cannot grow to such proportions. This is something that the author should have been more explicit about, the argument he presents is incomplete. It is not so much a question of a lack of rigour but obfuscating the main ideas. With some care one should be able to predict the length of growth depending on the size of the animal. And besides the actual biological situation is more complicated, elephans keep on growing for a long time, maybe growh is only checked by death, as is the case of fish.

Now irritating as the rather tedious and repetitive accounts of the underlying mathematics maybe, at least to the mathematically competent, what irritates more are the opinionated presentations of received ideas in the sense of Flaubert. We are treated to the same tired story of the Greeks only speculating and never putting their speculations to empirical tests; while it was not until Galileo came along that thought was confronted with reality. This is if anything an over-simplification. Galileo probably never madethe famed experiment from the Tower of Pisa, like old god stories it is too good to be true, for the good reason that it would not have been conclusive, in fact it was a thought experiment that convinced him of the principle[12]. Likewise, the principle that a body let to itself would follow a straight path with uniform velocity[13] acnnot be easily experimentally verified, but is a fruit of a fecund imagination. But what Galileo pionererd was systematic quantitative investigation, in particular the determination of constant acceleration[14].

Fractals are of course inevitable in the context, after all one of their defining characteristics being scale invariance. The author tells the nice story of Richardson pondering the length of natonal boundaries, which did not make any splash at the time, published as it was in obscure journals embedded in articles trying to elucidate the origins of bellicose conflicts and wars. Mandelbrot may or may not have read them and been influenced. The author fails, however, to note that their mathematics has been encountered and illuminated already over a century ago, and that the contributio of Mandelbort was to see that they were not mathematical pathologies and curiosites but could serve a useful role in description of various forms encountered in nature. It is common to disparage Euclidean geometry when introducing fractals, as if mankind had been blind until recently. It s true that straight lines do not occur visually in nature[15] bt that it is the whole point of Euclidean geometry. It is concerned not with forms and shapes but by the nature of physical space, and the key to this is 'triangulation', triangles not occuring in nature. Without the solid foundation suplied by disparaged Euclidean geomtry, fractal geomery would not have taken off ground.

And of course there are the usual digs against Platonism as you expect from the philosophically half-educated. The author actually writes

> *The overall agreement with theory is extremely satisfying. But much more than that, I find the extraordinary unity andinterconnectivity of life that is revealed through this lens to be sprirtually elevating in the pantheistic spirit articuated y the philosopher Baruch Spinoza. As Einstein wrote* We followers of Spinoza see our God in the wonderful order and lawfulness of all that exists and in its soul as it reveals itself in man and animal. *Regardkess of one's belief system, there is something supremely grand and reassuring when one perceives even a tiny piece of the mystifying chaotic world around us conforming to regularities and principles that*

[12] If two (equal) bodies fall, will they fall faster when conected? And if so what counts for a connection?

[13] Galileo also hold open the possibility of cirular movement to account for the movements of celestial bodies. The cean radical formualtion is due to Newton.

[14] the fact that you consider change of velocity per time and not per length turns out to be crucial

[15] The exception being the unbroken horizon by an open sea

transcend th awesome complexity and seeming meaninglessness.

Quite a mouthful. Intended or not it is ironically a celebration of Platonism from an author, who, when asked, very likely would disparage the same and deny any connection to it adhering to received opinions about it. Even if the author would hope for, he certainly would not find the same mathematical beauty in the biological and social world which initially excited him in he physical. To point out that the mathematical models are not perfect may be thought of as pedantic but it also has the consequence that the for all intents and purposes inductive method of mathematics are no longer available. In mathematics the conclusions are as solid as the assumptions and can thus be used at stepping stones for further conclusions enabling long chain of reasonings the results of which are unpredictable from the outset, a trivial example being calculations and thus lead to discoveries.

As noted the author, trained as a theoretial physicist, came rater late in life to what he terms the 'big questions'. His ambition has been to apply mathematics in a more consistent way to fields, such as biology, which traditionaly has shunned them. By the consistent application is meant not only serious quantification but also drawing what C.S.Peirce refers to as the necessary conclusions in his characterization of mathematical activity.

But actually worse than the above-mentioned inadequate explanations, where too much attention is given to the elementary and obvious but where attention is needed there is merely hand-waving, and worse than the reiteration of half-digested, or even undigested opinions as noted above is the abundance of abstract mumbo-jumbo that does not add anything to the text, but only provides a stream of clichés and homilies, such as mindless repetitions on the wonder of the variety of life, going on and on. This is the kind of prose that is mechanically produced and could easily, I gather, be computer generated. As a reader you tend to skip through it impatiently as it does not give any purchase of thought. As an example of it, the following excerpt is typical, but far from the worst as there is at least some content to be distilled

> *Consequently, in parallel with the quest for the Theory of Everything, we need to embark on a similar quest for a grand unified theory of complexity. The challenge of developing a quantitative, analytic, principled, predictive framework for understandig complex adaptive systems is surely one of the grand challenges for twenty-first century science. As a vital corollary to this and of greater urgency is the need to develop a grand unified theory of sustainability in order to come to terms with the extraordinary threats we now face. Like all grand syntheses, these will almost certainly remain incomplete, and very likely unattainable, but they will nevertheless inspire significant, possibly new revolutionary ideas, concepts, and techniques with implications for how we move forward and whether what we have so far achieved can survive.*

A dedicated and competent editor would have cut all that fat out, significantly reducing the bulk of the book. On the other hand its presence should also be viewed as cautionary. It is indeed easy as a writer to fall into the trap thinking to be clever and profound, especially during a the first initial stage of a write-up when the main concern is to get all your thoughts down on paper.

The biological examples are just meant to whet the curiosity of the reader, the real contributions lie in tha author's ambition that in a serious manner bring to the traditionally descriptive and qualitative disciplines of social sciences hard core quantitative methods. Quantification is not just about assigning numbers, the numbers have to be put to work in calculations, by themselves they signify nothing[16]. What is better than the scaling approach which has been so iluminating in biology (and mechanical engineering)? The author is very proud of the extensions to cities and companies, although I as a reader has difficulty sharing his excitement, especially as I do not see any real applications and predictions (applications is of course the one thing you have a right to epect from applied mathematics, otherwise it is just 'pseudo-mathematics'). However, he points to a remarkable fact, namely that companies are short-lived, while cities are long-lived. Why is that? Cities seem to have a remarkable resiliency and they seem to appear spontaneously when the conditions are favorable[17]. The explanations the author gives are predominantly qualitative. In a city a lot of different activities are going on, and the geographically focused extent of a city[18] encourages contacts between different people. The pace of a city is simply faster, people walk at a brisker pace, social contacts are more frequent (if not necessarily deeper), and hence innovations (as say quantified by number of patents) and general economic and cultural activity. Cities in the modern third world grow actually super-exponentially, a trend which is of course not sustanable, only as long as there is a reseroir of rural people to be sucked in. Companies are not diversified and they have to face tourgher and tougher competition and old companies typically cannot keep up and make the necessary adjustments, butbthose are left to new companies being formed. One wuld think that a diversified company would have a better chane, but that does not seem to be the case. The companies which ae surived the longest are actually fairly small companies,which are attached to a fixed niche which has persisted for a long time and shielded it from competition. Just as in the growth of an individual, companies need to maintain a high profit margin to flourish, and when it flags, the end is inevitable. Thus companies should be likened to individuals, while cities to entire populations of species.

The common background to of all this is the Santa Fe institute, devoted to the study of comple adaptive systems, for which the book can be seen as a celebration. If smart people from different disciplines come together wondrous things are bound to happen. A statement of faith as outrageous and necessary as many other statements of faith. But how should a research institute be run actually? One aswer is provided by the late Perutz, a student of the legendary Lawrence Bragg, who has a particularly spectacular track record when it comes to students and achievement. Perutz had a simple advice. Handpick the staff and do away with all tthe trappings of bureuacracy. The tendency in modern science is the opposite as it becomes more and more industrial. You do not learn the quality of a researcher by keeping statistical attention to publications, but by knowing them personally. Old-fashioned human relations lies at the heart, while modern bureacracy wants to automate the process. The result is an overabundance of mediocrity in which real talent runs the risk of drowning. A development made possible by the computer age

[16] The historian of ideas - S-E Liedman - refers to 'pseudo-quantities' when this is not the case.

[17] Such as at confluency of rives or at natural harbors

[18] The phyiscal extent of a city in terms of being defined by how far people are prepared to commute, depends on the availability of transport. Commutig time is typically limited to haf an nour, and when ony walking is anoption, cities are more compact and hence more limited as to growth. With the car, urban sprawl has been made possible.

and its capacity to process huge amount of data. This has suggested the idea that with machines automatically learning from huge data set, science can be automated as well. This is bringing Baconism to its logical conclusion. Bacon preached that truth is manifest (and easy to discover as long as you have an open mind) by observation alone. Now the ease with which to do so is far from clear nowadays, in principle it should be possible, as truth is inherent in the data and only needs to be teased out, a process that may transcend the possibilities of collective human intelligence, let alone individual. In practice it means finding hidden correlations, but the author wisely warns against the pitfalls of fortuitous coincidences. This ties up with the development of AI and the possibility of super-human intelligence, to which he remains skeptical.

So what is the big question? Sustainable living. Superexponential growth is not sustainable, ideas which go back to Malthus who contrasted the arithmetic (linear) growth of food production with the geometric (exponential) growth of population. Malthus has almost always been dismissed as a cynical minister blaming the plight of the poor on their unchecked procreation, and that he did not take into account technological improvement due to the creativity of man. Some people claithat the latter can always keep ahead (an article of faith if any), others like the author are a bit more cautious, if not overly pessimistic as the Doomsday sayers the inheritors of Malthus. He warns that innovation will need to be done at an increasing pace, and after all there may be biological consraints on human cognition. Then of course human cognition could be replaced by super-cognition, such as superintelligence, but then, as many people are quick to point out, history does no longer belong to humans, who will become more and more marginalized. But is it not a sign of sentimentality to decry the demise of our own provincialspecies, the future belongs not to the meek but the strong.

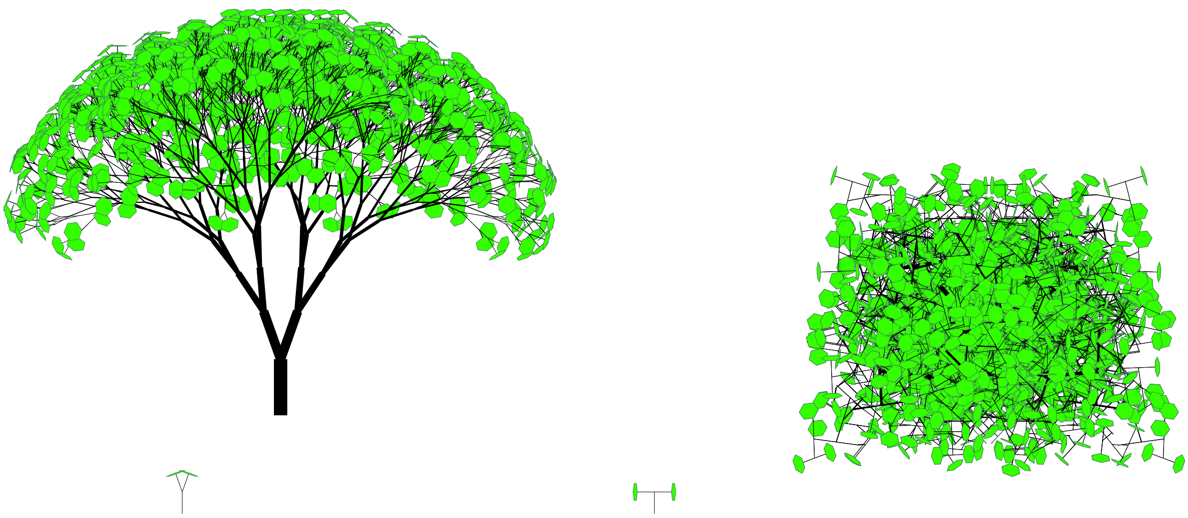
Appendix

1. Kepler's power laws

In the circluar case we have two elementary derivations. In the first approach we observe that a circle with radius R deviates (infitesimally) from a straight line by the amount $\frac{x^2}{2R}$. If it moves with velocity v we have $x = vt$ and if it falls with acceleration a we get $\frac{x^2}{2R} = \frac{at^2}{2}$. Now by Newton acceleration is proportional to R^{-2} (and in fact the constant of proportionality is proportional to the mass (M) of the central attracting body) and of course $T = \frac{2\pi}{v}$. Putting everything together we get that T^2 is indeed proportional to R^3 (and inversely proportional to M). The second approach is to consider circular motion $z = Re^{i\omega t}$ where $\omega = \frac{2\pi}{T}$ and hence the acceleration $\ddot{z} = -\omega^2 z$ taking absolute values $|z| = R$ we get the same formula. Of course in both cases it boils down to the same thing, namely $a = \frac{v^2}{R}$ from which the law is immediate. Note that in a homogenous medium when acceleration is proportional to R, orbital time is constant independent of R.

2. The vascular system

The vascular system is likened to the branching of a tree. However there is a major difference, the vascular system has to be everywhere dense (in some appropriate real-world sense) while the branching of a tree only becomes dense in the canoy which is a surface as the leaves need to catch some sunlight otherwise there is no point. (The root system is often thought of as being the mirror image of the tree itself, but it also needs to be everywhere dense, serving a mirror image of the vasuclar system, drawing nourishment rather than supplying it). Yet the branching principle is the same. What makes the difference? The angle of the branching. If acute we get a tree, if obtuse, say each branch orthogonal to the mother branch. We see the two cases below.



Another similar branching, not mentioned in the book, is given by a river, which is dense in its drainage area.

Now let us get down to some elementary mathematics. Inspired by the second exampe of the branching above we will consider all sums $\sum_{n\in U} \pm r^n$ with $r > 0$ and U a subset of the non-negative integers. If $r \geq 1$ they only make sense for finite number of terms and form an infinite discrete set. From now on we only consider the case $0 < r < 1$ and the closure X of the finite subsums is contained in the closed interval $= [\frac{1}{r-1}, \frac{1}{1-r}]$

We have

Proposition:

In the case of U being an arbitrary subset of the non-negative integers $X = I$ iff $r \geq \frac{1}{3}$, otherwise X is a Cantor-type set; but if U is the whole set of non-negative integers (no term are omitted) then the condition is instead $r \geq \frac{1}{2}$ and if not satsified X is a classical type of Cantor set.

Proof: It is convenient to set $\alpha = r + r^2 + r^3 + \dot{=} \frac{r}{1-r}$ (note that $r = \frac{\alpha}{\alpha+1}$). In terms of α the interval is given by $[-(1+\alpha), (1+\alpha)]$

We have three closed intervals each of length 2α. Unless $\alpha \geq \frac{1}{2}(r = \frac{1}{3})$ they will not overlap and leave two open intervals as indicated in the figure above. Denote them by L, M, R respectively. Numbers in L start with -1, in M by 0 and R with by 1, the complement is inaccessible. To get the second term, we need to repeat the procedure on the appropriate interval and so on. Numbers which survive the whole process will obviously have a representation, so the condition is also sufficient. The case of U being the entire set, it is simpler, then M is removed and we are left with an open interval in the middle, unless of course $1 - \alpha \leq 0$.

If $\alpha = \frac{1}{3}$ then $\frac{1}{2}$ will constitute the intersection $M \cap R$ consequently have two representations $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots$ and $1 - \frac{1}{3} - \frac{1}{3^2} - \frac{1}{3^3} - \ldots$. In fact all numbers of form $\frac{1}{2}t$ with t a triadic number, i.e. corresponding to U is finite, will have ambigious representations, completly analogous to the well-known case of unending 9's in decimal expansions. If $\alpha \geq 1$ (i.e. $r \geq \frac{1}{2}$) then numbers can have infinitely many different representations. Similar results is easily obtained for the second case.

This will have applications to branching trees. First let us start in the plane, and consider stems that decrease with a ratio $r < 1$ and bifurcates with two stem perpendicular to the original. The tree will be contained in a square with vertices at $(\pm(1+\frac{1}{1-r^2}), \pm(1+\frac{1}{1-r^2}))$ and if we project onto the axis we are in the 1-dimensional situation of above, with r replaced by r^2. Thus we get a dense tree, iff $r \geq \frac{1}{\sqrt{2}}$ the border case corresponding to the most parsimonous, where except for a countable number of points, there is a unique path leading to it. This has a straightforward generaization to three dimensions when the crucial scaling factor now becomes $2^{-\frac{1}{3}}$.

The vascular system is a bifurcating tree, 'dense' in the body, such that each branching of a vessel gives rise to two vessels of equal crossectional area and they add up that of the original, thus diameters scale as $2^{-\frac{1}{2}}$ while lengths as noted above scale as $2^{-\frac{1}{3}}$. Thus vessels become skinnier and skinnier as they get narrower and narrower. If the initial stem is referred to as the aorta and given volume V_0 the total volume V of the system will be given by the geometric series

$$V = V_0(1 + 2^{-\frac{1}{6}} + 4^{-\frac{1}{6}} + 8^{-\frac{1}{6}} + \ldots) = \frac{V_0}{1 - 2^{-\frac{1}{6}}} \sim 8V_0$$

This ration is hence independent of the size of the system. The diameter of the human aorta is about 3 cm, its length about 50 cm which is about half a litre, and indeed the volume of blood is about 4 liters.

Furhermore the human body contains about 10^{13} cells, which is about 2^{43} corresponding to 43 branchings, however we do not need a vessel reaching each cell, the vessels are of course made of cells themselves, which gives a lower limit of their diameters. In fact those are given as about $5\mu m$ in diameter, comparable to that of cells. (Note that the human body is about $7 \cdot 10^{-2}m^3$ (a cube with linear dimensions $0.4m$) while that of a sphere with diameter $5\mu m$ has a volume of $6 \cdot 10^{-17}m^3$ thus we are talking about 10^{15} cels, which shows that human cells are on the average larger, more like diameters of $20\mu m$). Starting with $3 \cdot 10^{-2}m$, and as diameters are scaled by a factor of $10^4 \sim 2^{13}$ we are talking about 26 branchings, and thus each final vessel will have to serve 2^{17} cells. Note that the length of the final vessels will be $2^{-26 \cdot \frac{1}{3}} \sim 0.002$ times $0.5m$ i.e. about $1mm$ which a two hundred times the cross-section, thus its walls will lined by about a thousand cells. and crowding around it over a hundred thousand cells to be in diffusive contact. Looking at the end-points we are talking about 60 million, and if evenly distributed one at every cubic millimeter, which tallies with the lengths of the final vessels.

A Blue Whale weighs about 200 tons, thus about 3000 times a human. Its linear dimensions are thus about 15 times that of humans. The number of cells is about 2^{12} that of humans, hence we need 12 more branchings, hence the diameter of its aorta should be 2^6 times that of humans, say about 40 as the exponent 12 was an exaggeration, which translates into a meter and a half. Its length should be scaled by 2^4 thus about 8 meters, which would correspond to about $25'000$ times the blood volume of a human, which seems extreme. In fact the diameter of the aorta is more like $25cm$ which would correspond to only six more branchings and thus a less dense system. Only six branchings would correspond to an aorta the length of two meters, and thus a blood volume only 250 times that of a human, which would not tally with ten tons of blood being pumped through the system. On the other hand if metabolic activity would grow with the surface area (heatloss), it would be in good agrement.

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