Thetafunctions and Linebundles on Elliptic Curves

Let $E = \mathbb{C}/\Lambda$ be an elliptic curve (Λa lattice$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\frac{\omega_1}{\omega_2}$ not real).

We would now like to describe all the linebundles on E. It is then natural to consider the universal covering and pullbacks

$$\begin{array}{ccc} \pi^8 L & \longrightarrow & L \\ {\scriptstyle \pi^* p}\big\downarrow & & \big\uparrow{\scriptstyle p} \\ \mathbb{C} & \longrightarrow & \mathbb{C}/\Lambda \cong E \end{array}$$

Where $\pi^* L$ is a linebundle on $\mathbb{C}$ (defined by $\pi^* p^{-1}(U) = p^{-1}(\pi U)$ for any small open set U in $\mathbb{C}$, small in the sense that π is an isomorphism on U onto its image $\pi(U)$)

We will then make use of the following unproven fact

Fact:*Any linebundle on $\mathbb{C}$ is trivial*

(Topologically you may think of $\mathbb{C}$ as a contractible space, more seriously though $\mathbb{C}$ is the simplest example of a Stein space)

We can now describe the twisting of L via an action of Λ on $\pi^* L = \mathbb{C} \times \mathbb{C}((z, \zeta))$

The action is given by (for each $\lambda \in \Lambda$)

$$(z, \zeta) \mapsto (z + \lambda, e_\lambda(z)\zeta) \qquad \text{(action)}$$

defining

$\phi : \Lambda \to A(2, \mathbb{C})$ the group of affine transformations

where e_λ is a nowhere vanishing entire function, and the collection (indexed by λ) satisfies

$$e_{\lambda+\check{\lambda}}(z) = e_{\check{\lambda}}(z + \lambda)e_\lambda(z) \qquad (*)$$

(technically this is known as a cocycle condition in group cohomology)

To see this just expand $\phi_{\lambda+\check{\lambda}} = \phi_\lambda(\phi_{\check{\lambda}})$

Now the $e_\lambda(z)$ are not uniquely determined, they maybe modified by changing the fibre coorinate ζ with an arbitrary non-vanishing entire function $f(z)$

An equivalent system of transition functions would be given by

$$\check{e}_\lambda(z) = f(z + \lambda)e_\lambda(z)f(z)^{-1}$$

technically $\check{e}, e$ differ by a co-boundary

We would now like to put $e_\lambda(z)$ into normal form. Observe that we can write $e_\lambda(z) = e^{2\pi i Q_\lambda(z)}(f(z) = e^{2\pi i \gamma(z)})$ and that (*) and (**) translate into

$$Q_{\lambda+\lambda'}(z) = Q_{\lambda'}(z + \lambda) + Q_\lambda(z) \quad (\mathbb{Z}) \qquad (*')$$

$$(**') \qquad Q_{\lambda'}(z) = Q_\lambda(z) + \gamma(z+\lambda) - \gamma(z)(\mathbb{Z})$$

The natural questions are whether there exists systems of entire functions that satisfy $*'$ anf if so what the simplest (but yet interesting) kinds are.

For the first question it is enough to observe that given a basis ω_1, ω_2 one can specify $Q_{\omega_1}, Q_{\omega_2}$ arbitrarily defining $e_\lambda(z)$ by (*) for an arbitrary element $\lambda = n_1\omega_1 + n_2\omega_2$.

The second is now naturally resolved by letting $Q_{\omega_i}(z) = a_i z + b_i$ to be linear functions. (As we will see, letting them being constants would be too restrictive)

We can then write

$$Q_{n_1\omega_1+n_2\omega_2}(z) = (n_1a_1+n_2a_2)z+(n_1b_1+n_2b_2)+\frac{1}{2}(n_1(n_1-1)a_1\omega_1+n_2(n_2-1)a_2\omega_2)+n_1n_2a_1\omega_2$$

Remark i) For 'suitable" quadratic forms we can write

$$Q_\lambda(z) =< \lambda, a > z+ < \lambda, b > + < \lambda, \lambda >'$$

(thus linear in z and "quadratic" in λ)

Remark ii) The formula above is clearly asymmetric. Thus $n_1n_2a_1\omega_2$ could have been replaced by $n_1n_2a_2\omega_1$ hence

$$\text{(L)} \qquad a_1\omega_2 - a_2\omega_1 \text{ is integral (Legendre)}$$

This puts one (and in fact the only) restriction on the linear forms $a_i z + b_i$ (For a direct interpretation of (L) see below)

We are now going to state (without proof) the basic

Theorem *(Appell-Humpert)Every linebundle on $E(= \mathbb{C}/\Lambda)$ can be represented via () where the $e_\lambda(z)$ are normalized to be*

$$e_\omega(z) = e^{2\pi i(az+b)}(\omega = \omega_i, a.b = a_i, b_i; i = 1, 2)$$

where the Legendre condition $a_1\omega_2 - a_2\omega_1$ is integral is fulfilled.

This will motivate the following definition:

A theta function Θ is an entire function with the following quasi-periodic behaviour with respect to Λ

$$\begin{Bmatrix} \Theta(z+\omega_1) = e^{2\pi i(a_1z+b_1)}\Theta(z) \\ \Theta(z+\omega_2) = e^{2\pi i(a_2z+b_2)}\Theta(z) \end{Bmatrix}$$

We see that although not periodic, the zero sets are; and thus can be descended to E

In fact by the above normalizations of $e_\lambda(z)$ the thetafunctions can be considered as sections of (suitable) linebundles on E

There is now a natural interpretation of the integrality condition

Let N be the # of zeroes of Θ on E (a standard integration on the perimeter of a period parallelogram, chosen as to avoid the zeroes, yields)

$$N = \frac{1}{2\pi i}\int_{\partial\Pi} \frac{\Theta'}{\Theta} = a_1\omega_2 - a_2\omega_1$$

Let us from now on call the integer $a_1\omega_2 - a_2\omega_1$ the degree of the theta function

It is not *a priori* clear that there should exists (non- trivial) theta functions

We observe that a theta function is everywhere non-zero (N=0) iff it is a section of a trivial linebundle on E (Strictly speaking we are abusing language, obviously any section of a trivial linebundle on a compact space is constant, its pullback to the universal cover $\mathbb{C}$ in this case will then be constant up to a non-zero multiplicative function (this corresponds to normalizing the fiber coordinates))

The non-vanishing theta functions can readily be classified

Lemma:Θ *is a non-vanishing theta function iff it is of the form* $\Theta(z) = e^{2\pi i Q(z)}$ *for some quadratic polynomial* Q

(If $Q(z) = Az^2 + Bz + C$ then $a_i = 2A\omega_i, b_i = B\omega_i + A\omega_i^2$ with $a_1\omega_2 - a_2\omega_1 = 0$ as expected)

*Proof:*It is trivial to verify that $e^{2\pi i Q(z)}$ is indeed a theta function (with the quasi periodic constants goven). Conversely let $\Theta(z)$ be a non-vanishing theta function, then $\log\Theta(z)$ satisfies

$$(\circ) \qquad \log\Theta(z+\omega_i) = 2\pi i(a_i z + b_i) + \log\Theta(z)$$

from which follows the estimate

$$\log|\Theta(z)| \leq C_1 + C_2|z|^2$$

($|\log\Theta|$ is bounded on Π (the parallelogram spanned by the two basic periods) to estimate $\log\Theta(z)$ we need to apply ($\circ$) $K_1|z|$ times each time $|\log\Theta(z+\omega) - \log\Theta(z)| \leq K_2|z|$) Thus (by the Cauchy estimates) $\log\Theta(z)$ is a quadratic polynomial.$\diamondsuit$

Thus we have characterized all theta functions with N=0

We will now construct an example with N=1

Weierstraß σ-function

Start with the Weierstraß $\wp$ function $(= \frac{1}{z^2} + a_2 z^2 \ldots)$ Its only pole has residue zero so we can integrate $-\wp$ to $\zeta(z)$ normalized to be odd.

An integration term by term of the partial fraction decomposition of $\wp$ gives

$$\zeta(z) = \frac{1}{z} + \sum_{\lambda\in\Lambda^*}\left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2}\right)$$

Now there is no reason (in fact it is impossible) that ζ should be elliptic, in fact let $\eta_i = \zeta(z+\omega_i) - \zeta(z)$.

A standard residue calculation gives

$$\text{(Legendre)} \qquad 1 = \frac{1}{2\pi i}\int_{\partial\pi}\zeta(z)dz = \eta_1\omega_2 - \eta_2\omega_1$$

Now due to the $\frac{1}{z}$ term ζ cannot be integrated singly-valued. By exponentiating we kill the indetermancy (or what is equivalent by going to the universal covering $\mathbb{C}$ of $\mathbb{C}^*(= \mathbb{C}/2\pi i\mathbb{Z})$ which is the Riemann surface of $\log z$):

Thus we are lead to the differential equation

$$\frac{d\log\sigma(z)}{dz} = \frac{\sigma\prime(z)}{\sigma(z)} = \zeta(z)$$

where we choose the solution $\sigma(z)$ such that $\lim_{z\to 0}\frac{\sigma(z)}{z}=1$.

Thus $\sigma(z)$ has the product expansion

$$\sigma(z)=z\prod_{\lambda\in\Lambda^*}\frac{z-\lambda}{-\lambda}\exp\frac{z}{\lambda}+\frac{z^2}{2\lambda^2}$$

(observe that $\sigma(z)=-\sigma(-z)$ by rearranging factors)

To investigate the periodic behaviour of $\sigma(z)$ we note that as $\zeta(z+\omega_i)-\zeta(z)=\eta_i$ we have

$$\frac{\sigma(z+\omega_i)}{\sigma(z)}=e^{\eta_i z+\beta_i}$$

As $\sigma(z)$ is odd we obtain putting $z=-\frac{\omega_i}{2}$ that

$$-1=e^{-\frac{1}{2}\omega_i\eta_i+\beta_i}\text{ thus }\beta_i=i\pi+\frac{1}{2}\omega_i\eta_i$$

Thus $\sigma(z)$ is a theta function associated to

$$a_i=\frac{1}{2\pi i}\eta_i,\beta_i=\frac{1}{2}+\frac{1}{4\pi i}\omega_i\eta_i$$

(note $a_1\omega_2-a_2\omega_1=1$)

As we have shown that the theory of theta functions is non-trivial, we can consider operations on theta functions to generate new ones.

Proposition: *1) the product of any two theta functions is a theta function.* (If the multipliers are given by (a_i,b_i) and (a_i',b_i') respectively, then the product is associated with (a_i+a_i',b_i+b_i') in particular the degrees are added)

2) The translate of any theta function is a theta function. (If $\theta_\delta(z)=\theta(z-\delta)$ then (a_i,b_i) is transformed to $(a_i,b_i-a_i\delta)$ thus degrees are left invariant)

PROOF :Straightforward $\diamondsuit$

As an application of the above we can now show:

Proposition: *Given (not necessarily distinct) points $P_1\ldots,P_n,Q_1\ldots Q_n$ on $\mathbb{C}$ then there exists an elliptic function with zeroes at $\overline{P}_i$ and poles at $\overline{Q}_i$ ifff $\sum(P_i-Q_i)\in\Lambda$*

If it exists it can be written as

$$\frac{\prod\sigma(z-P_i)}{\prod\sigma(z-Q_i)}$$

Proof: We know from before that $\sum(P_i-Q_i)\in\Lambda$ is a necessary condition by considering the integral

$$\int_{\partial\pi}z\frac{f'}{f}dz$$

Now the product is elliptic (i.e. a meromorphic theta function associated to $(a_i,b_i)=(0,0)$ iff

$$a_1\sum P_i=a_1\sum Q_i(\mod\mathbb{Z})$$
$$a_2\sum P_i=a_2\sum Q_i(\mod\mathbb{Z})$$

thus $(a_1\omega_2 - a_2\omega_1)(\sum P_i - \sum Q_i) \in \Lambda$ as $a_1\omega_2 - a_2\omega_1 = 1$ we are done.$\diamondsuit$

Note: $\sum(P_i - Q_i) \in \Lambda \Leftrightarrow \sum \overline{P}_i = \sum \overline{Q}_i$ (where $\sum$ denotes addition with respect to the canonical grouplaw)

Note:$\prod \sigma(z - P_i)$ is clearly a theta function of degree n

Multipliers

Let M denote a multiplier written in matrix form $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ where

$$a_1\omega_2 - a_2\omega_1 \in \mathbb{Z} \quad \text{(the degree of } M)$$

M determines

1) A line bundle L_M via the theorem of Appell-Humpert

2) A vector space of sections $\Gamma(L_M)$

(Note:Theta functions associated to a given M can be added and multiplied by scalars)

Now M determines a trivial line bundle iff M is of the form

$$\begin{pmatrix} 2A\omega_1 & B\omega_1 + A\omega_1^2 \\ 2A\omega_2 & B\omega_2 + A\omega_2^2 \end{pmatrix} = (Q)$$

(where $Q = Az^2 + Bz + C$)

Or equivalently $M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ where $a_1\omega_2 - a_2\omega_1 = 0$($\deg M = 0$) and $b_1\omega_2 - b_2\omega_1 = \frac{1}{2}\omega_2\omega_1(a_1 - a_2)$

We may call M a trivial multiplier. Note that given e.g the first row (this can be arbitrary) the second row is uniquely determined.

Another characterization is that the associated theta functions are non-vanishing. (so called trivial theta functions $e^{2\pi i Q(z)}$)

Note that any trivial theta function periodic with respect to some ω_i must be associated to $M = 0$ and be constant

We say that two multipliers M, M' are equivalent iff

$$M - M' = (Q) \quad (\text{ we write } M \sim M')$$

Note that two multipliers are equivalent iff the associated linebundles L_M and $L_{M'}$ are equivalent

Furthermore the b entries of M are only defined $\mod \mathbb{Z}$ because of the exponentiation $e^{2\pi i *}$ this equivalence relation is always tacitly assumed and is much finer than the above.

Note: Two theta functions can be added iff they belong to the same multiplier (with b entries defined over $\mod \mathbb{Z}$) M. Thus we need to make the fundamental distinction

$$\Gamma(L_M) = \Gamma(L_{M'}) \text{ if } M = M'$$

$$\Gamma(L_M) \cong \Gamma(L_{M'}) \text{ if } M \sim M'$$

Exploiting the remark that the first row of (Q) can be arbitrary (choosing the right Q) we may put multipliers under some normal form. It will also be convenient to normalize the basis (ω_1, ω_2) to $(1, \tau)$

Proposition: *Any multiplier is equivalent to some multiplier of the form*

$$\begin{pmatrix} 0 & 0 \\ -N & b \end{pmatrix}$$

furthermore

$$\begin{pmatrix} 0 & 0 \\ -N & b \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ -M & b' \end{pmatrix} \textit{ iff } N = M \textit{ and } b = b'(\Lambda)$$

Corollary: *The linebundles up to equivalence are parametrised by* $\mathbb{Z} \times E$

*P*ROOF :Given $M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ consider $(Q) = \begin{pmatrix} a_1 & b_1 \\ a_2' & \_2' \end{pmatrix}$ where

$$a_2' = a_1\tau, b_2' = b_1\tau + \frac{1}{2}a_1\tau(\omega_2 - \omega_1)$$

then $M' = M - (Q)$ is equivalent and $M' = \begin{pmatrix} 0 & 0 \\ -N & b \end{pmatrix}$ $\diamondsuit$

It is instructive to compare this with divisors on elliptic curves. Recall that a divisor D on E is a formal linear combination $\sum n_i P_i$ where P_i are points on E

To any divisor we can associate

1) the degree of $D = \deg(D)$ $(\delta(D) = \sum n_i \in \mathbb{Z}$

2) the sum of $D = \varsigma(D) = \sum n_i P_i \in E$

(Remark in 2) we interpret the formal sum $\sum n_i P_i$ literally! by using the addition in E)

Two divisors D, D' are said to be linearly equivalent iff there is a meromorphic function (i.e elliptic) ϕ such that

$$D - D' = (\phi) \quad (D \sim D')$$

Lemma: *If ϕ is an elliptic function then* $\delta(\phi) = \varsigma(\phi) = 0$

*Proof:*This follows from

$$\delta(\phi) = \int_{\partial\Pi} \frac{\phi'}{\phi}, \varsigma(\phi) = \int_{\partial\Pi} z\frac{\phi'}{\phi}$$

where π is the parallelogram of the basic periods $\diamondsuit$

Corollary $\delta(D), \varsigma(D)$ *only depend on the linear equivalence class of* D

Recall that two divisors "belong" to equivalent linebundles iff they are equivalent. The connection being that any meromorphic section of a linebundle defines a divisor (through its zeroes and poles) and that two meromorphic sections of equivalent linebundles define linearly equivalent divisors

Proposition: $\delta\begin{pmatrix} 0 & 0 \\ -N & b \end{pmatrix} = N \quad \varsigma\begin{pmatrix} 0 & 0 \\ -N & b \end{pmatrix} = \frac{N}{2} + b(\Lambda)$

Proof :Should be clear. Note that a theta function belonging to $\begin{pmatrix} 0 & 0 \\ -N & b \end{pmatrix}$ has N zeroes, thus $\delta = N$. Furthermore if we normalize the multipliers for the Weierstraß σ-function we obtain $\begin{pmatrix} 0 & 0 \\ -1 & \frac{1}{2} - \tau \end{pmatrix}$, thus if $\sum n_i P_i$ is a divisor then $\prod \sigma(z - P_i)^{n_i}$ defines it and will have multiplier $\begin{pmatrix} 0 & 0 \\ -N & \frac{N}{2} - N\tau + \sum n_i P_i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -N & \frac{N}{2} - N\tau + \varsigma \end{pmatrix}$ $\diamondsuit$

Note that we have now explicitly verified the assertion of Appell-Humpert.

Theta functions

Theorem (*Riemann-Roch for Elliptic curves)If L is a linebundle of degree N then*

if $N < 0$ then	$\dim \Gamma(L) = 0$
if $N = 0$ and L non-trivial	$\dim \Gamma(L) = 0$
if L is trivial	$\dim \Gamma(L) = 1$
if $N > 0$	$\dim \Gamma(L) = N$

*Proof :*We may assume that L is defined via the multiplier $\begin{pmatrix} 0 & 0 \\ -N & b \end{pmatrix}$ thus the sections $\Gamma(L)$ are periodic functions (invariant under $z \mapsto z + 1$) and maybe expanded in a Fourier series

$$\Theta(z) = \sum a_n e^{2\pi i n z}$$

we obtain

$$\Theta(z + \tau) = \sum a_n e^{2\pi i n \tau} e^{2\pi i n z}$$

$$e^{2\pi i(-Nz+b)}\Theta(z) = \sum a_n e^{2\pi i b} e^{2\pi i(n-N)z} = \sum a_{n?N} e^{2\pi i b} e^{2\pi i n z}$$

giving the functional equation $\Theta(z + \tau) = e^{2\pi i(-Nz+b)}\Theta(z)$ which translates into

$$(*) \qquad a_{n+N} e^{2\pi i b} = a_n e^{2\pi i n \tau}$$

We can now specify $a_0 \dots a_{N-1}$ arbitrarily and define a_n for $n < 0, n > N$ through (*)

When $N > 0$ the Fourier coefficients will be rapidly decreasing and define entire functions, for $N = 0$ we obtain $a_n e^{2\pi i b} = a_n e^{2\pi i n \tau}$ which imply $a_n = 0$ for all n except when $b = k\tau$ for some integer k when $\Lambda e^{2\pi i k z}$ is a solution , but then the corresponding linebundle is trivial.

When $N < 0$ the corresponding Fourier series diverge $\diamondsuit$

Example: *the Riemann theta function $\vartheta(z, \tau)$*

The Riemann theta function is a theta function associated to the lattice $< 1, \tau >$ and with multiplier $\begin{pmatrix} 0 & 0 \\ -1 & -\frac{1}{2}\tau \end{pmatrix}$ and normalized by $a_0 = \frac{1}{2\pi} \int_0^{2\pi} \vartheta(e^{iu}, \tau) du = 1$

Using the inductive formula on the Fourier coefficients we see that

$$\vartheta(z,\tau) = \sum e^{2\pi i(\frac{1}{2}n^2\tau+nz)}$$

Observe that $\vartheta(z,\tau)$ has a single zero at $\frac{1}{2}(1+\tau)$ Indeed as $N=1$ it has a single zero P now on one hand $\varsigma(P) = P$ on the other hand $\varsigma(P) = -\frac{1}{2} - \frac{1}{2}\tau$ thus $P = \frac{1}{2}(1+\tau)(\Lambda)$

Associated to ϑ we define four theta-functions.

$$\begin{aligned}
\vartheta_{00} &= \vartheta(z,\tau) \\
\vartheta_{01} &= \vartheta(z+\frac{1}{2},\tau) \\
\vartheta_{10} &= e^{2\pi i(\frac{1}{2}z+\frac{\tau}{8})}\vartheta(z+\frac{1}{2}\tau,\tau) \\
\vartheta_{11} &= e^{2\pi i(\frac{1}{2}(z+\frac{1}{2})+\frac{\tau}{8})}\vartheta(z+\frac{1}{2}(1+\tau),\tau)
\end{aligned}$$

The corresponding multipliers M_{ij} are given by

$$\begin{aligned}
M_{00} &= \begin{pmatrix} 0 & 0 \\ -1 & -\frac{1}{2}\tau \end{pmatrix} \\
M_{01} &= \begin{pmatrix} 0 & 0 \\ -1 & -\frac{1}{2}(1+\tau) \end{pmatrix} \\
M_{10} &= \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & -\frac{1}{2}\tau \end{pmatrix} \\
M_{11} &= \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & -\frac{1}{2}(1+\tau) \end{pmatrix}
\end{aligned}$$

and the zeroes are given by respectively

$$z = \frac{1}{2}(1+\tau), \quad z = \frac{\tau}{2}, \quad z = \frac{1}{2}, \quad z = 0$$

I.e precisely the four 2-division points of the elliptic curve E. As such a point is characterized by $z = -z(\Lambda)$ we obtain from Riemann-Roch

$$\vartheta_{ij}(-z) = a_{ij}\vartheta_{ij}(z)$$

In fact we have.

Lemma $a_{ij} = 1 \quad ij \neq 11$ While $a_{11} = -1$

Thus all are even except ϑ_{11} which is odd.

Remark: This can easily be seen directly from the Fourier expansions by reshuffling terms.

PROOF If $ij \neq 11$ then $\vartheta_{ij} \neq 0$ thus $a_{ij} = 1$. We have

$$\vartheta_{11}(\frac{1}{2},\tau) = e^{2\pi i(\frac{1}{2}+\frac{\tau}{8})}\vartheta(1+\tau,\tau) = -e^{\frac{\tau}{8}}\vartheta(\tau,\tau) = \vartheta_{11}(-\frac{1}{2},\tau)$$

hence $a_{11} = -1$

Observation ϑ_{ij} *are linearly independant entire functions.*

PROOF If $a_{00}\vartheta_{00} + a_{01}\vartheta_{01} + a_{10}\vartheta_{10} + a_{11}\vartheta_{11} = 0$

Then $a_{11} = 0$ as the first three are even and ϑ_{11} is not.

Then $a_{10} = 0$ as the first two are periodic (with respect to $\mathbb{Z}$) and ϑ_{11} is not.

$a_{00} = a01 = 0$ as ϑ_{00} and ϑ_{01} have different multipliers (or more elementarily different zeroes).

The functions ϑ_{ij} are sections of four different linebundles L_{ij} that correspond to the 2-divison points. But their squares are all isomorphic linebundles, and the functions ϑ_{ij}^2 are sections of the same.

PROOF By squaring the different multipliers we obtain the same multiplier

$$\begin{pmatrix} 0 & 0 \\ -2 & -\tau \end{pmatrix}$$

.

Using Riemann-Roch again we may state.

Observation *The four functions* ϑ_{ij}^2 *are pairwise linearly independant and any two make up a basis.*

PROOF Having different zeroes they are pairwise independant. As

$$\deg \begin{pmatrix} 0 & 0 \\ -2 & -\tau \end{pmatrix} = 2$$

by Riemann-Roch, the space spanned by them make up a 2-dimensional vectorspace.

In particular we have relations of form

$$\vartheta_{00}^2 = \alpha\vartheta_{01}^2 + \beta\vartheta_{10}^2$$
$$\vartheta_{11}^2 = \alpha'\vartheta_{01}^2 + \beta'\vartheta_{10}^2$$

The coefficients can be determined by plugging in the values of the zeroes of $\vartheta_{01}, \vartheta_{10}$ In particular setting $z = \frac{1}{2}$ and $z = \frac{\tau}{2}$ respectively we get $\alpha = \vartheta_{00}^2(\frac{1}{2})/\vartheta_{01}^2(\frac{1}{2})$ and $\beta = \vartheta_{00}^2(\frac{\tau}{2})/\vartheta_{10}^2(\frac{\tau}{2})$.

In addition to the operations of multiplying and translating theta-functions we have a third operation.

In fact for any given theta-function θ we may form for any integer n the function

$$\theta^{[n]}(z) = \theta(nz)$$

and if θ has multiplier $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ then $\theta^{[n]}$ has multiplier

$$\begin{pmatrix} n^2 a_1 & \frac{1}{2}a_1 n(n-1)\omega_1 + nb_1 \\ n^2 a_2 & \frac{1}{2}a_2 n(n-1)\omega_2 + nb_2 \end{pmatrix}$$

Note in particular that
$$\deg \theta^{[n]} = n^2 \deg \theta$$
On the level of linebundles we have a so called isogeny
$$E \xrightarrow{n} E(x \mapsto nx)$$
The if θ is associated to L then $\theta^{[n]}$ is associated to $n^* L$

Note The cardinality of the fibres are given by $n^2 = \#\{z \in E | nz = 0\} = \#\{\frac{a}{n}\omega_1 + \frac{b}{n}\omega_2; a, b \in \mathbb{Z} \mod n\mathbb{Z}\}$

In particular we want to look at a basis for the theta functions associated to the same linebundle as $\vartheta^{[2]}$. In fact the four theta-functions $\vartheta^{[2]}_{ij}$ are all associated to the same multiplier
$$\begin{pmatrix} 0 & 0 \\ -4 & -2\tau \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix}$$
This has degree four and hence determines a four dimensional space of sections. But we have seen above that those four theta functions are in fact linearly independant (as entire functions) and they span the space of sections to the (unique) linebundle D with $\delta(D) = 4$ and $\varsigma(D) = 0$.

They can now be used to get an explicit mapping
$$z \mapsto (\vartheta_{00}(2z), \vartheta_{01}(2z), \vartheta_{10}(2z), \vartheta_1(2z))$$
into $\mathbb{C}^4 \setminus (0, 0, 0, 0)$. As a map into $\mathbb{C}P^3$ it is periodic hence defining a map
$$\Psi : \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \to \mathbb{C}P^3$$
We want to show that this is an embedding, and to determine the equations that define its image.

A. Ψ separates points.

Let ω be a primitive 2-torsion point, and ϑ an arbitrary section of the above linebundle. Then $\vartheta(z + \omega) = \phi(z)\vartheta(z)$ for some multiplicative factor (which easily can be determined). Thus if $\vartheta(z_1) = \vartheta(z_2)$ then $\vartheta(z_1 + \omega) = \vartheta(z_2 + \omega)$. Assume that Ψ fails to separate z_1, z_2 then one may choose ω a half-period and z_3 such that all five points $z_1, z_2, z_3, z_1 + \omega, z_2 + \omega$ are all distinct. Clearly there will be a non-zero ϑ vanishing at $z_1, z_1 + \omega, z_3$, this will automatically then vanish at z_2 and $z_2 + \omega$ as well, in toto , five distinct points which is absurd.

B. $d\Psi \neq 0$ i.e. Ψ separates infinitely close points.

Argue as in A. by choosing a ϑ vanishing doubly at z_1 and say z_2, it will then also vanish doubly at $z_1 + \omega$.

The image of Ψ will satisfy the two quadratic relations.
$$\vartheta^2_{00}(2z)\vartheta^2_{00}(0) = \vartheta^2_{01}(2z)\vartheta^2_{01}(0) + \vartheta^2_{10}(2z)\vartheta^2_{10}(0)$$
$$\vartheta^2_{11}(2z)\vartheta^2_{00}(0) = \vartheta^2_{01}(2z)\vartheta^2_{10}(0) - \vartheta^2_{10}(2z)\vartheta^2_{01}(0)$$
Thus it will be contained in an intersection of two quadrics. In particular the intersection with a hyperplane will have at most four points. But any intersection by a hyperplane corresponds to a linear combination of the $\vartheta_{ij}(2z)$ i.e. an element

of the linear system $\Gamma(D)$ corresponding to a divisor of degree four. Thus there must be a 1-1 correspondence, and the image is the entire intersection.

By adjunction it is easy to see that any non-singular intersection of two quadrics must have trivial canonical divisor. This will then constitute the natural way of looking at elliptic curves inbedded in $\mathbb{C}P^4$. They will be refered to as elliptic quartics (as opposed to cubics that are plane).

The pair of quadrics defining an elliptic quartic E is not uniquely determined, however they determine a unique pencil $\lambda_0 Q_0 + \lambda_1 Q_1$ of quadrics, any member of which will contain E and any two distinct will cut out E.

Noting that quadrics are represented by symmetric matrices, and that they are singular iff the corresponding determinants vanish, we see that a pencil of quadrics in $\mathbb{C}P^4$ will in general have four singular members. Those are given as cones. By a judicious choice of co-ordinates such a cone may be written as a quadric in just three variables. Given four cones, we may choose a unique system of co-ordinates (up to factors) in such a way that the vertices corresponds to the four points $(1, 0, 0, 0) \dots (0, 0, 0, 1)$. Such a system of co-ordinates may be termed a normalized system, and the basis ϑ_{ij} is just such a system. Omitting any ij there is an element of the pencil of quadric relations involving only the other three, and furthermore in normal form as a sum of three squares.

Each vertex defines a double cover onto a conic, by projection. This will necessarily be ramified at four points, whose cross ratio (up to the action of $\mathfrak{S}_3$) will be constant, defining the j-invariant of E. On the other hand the four singular members of the pencil defines four points on the $\mathbb{C}P^1$ that parametrises the pencil. It is natural to guess that the corresponding crossratios are equivalent to those defined by the elliptic curve by double coverings. To prove this we need to exhibit an explicit geometric correspondence between the quartic E and the double cover of the pencil ramified at the singular members.

There is however a natural such cover. Any non-singular quadric exhibits two rulings by lines. When the quadrics degenerate into a cone, those two distinct rulings coalesce into one, the ruling from the vertiex of the cone.

Each ruling on a quadric containing E defines a $2:1$ cover onto a $\mathbb{C}P^1$ given by say a hyperplane section. This cover can also be considered as an involution and must as such have fixed points, the four ramifications of the covering. It is now easy to classify all such involutions on an elliptic curve. If it preserves zero it must be of the form $z \mapsto -z$ and in general there will be a unique $e \in E$ such that it is given by $z \mapsto e - z$. Thus we have a 1:1 correspondence between involutions with fixed points and points on an elliptic curve.

Now given such an involution ι on E we may consider the line L spanned by $z, \iota(z)$ (we may assume that those two points are distinct). Given a third point p on that line there will be a quadric in the pencil defined by E that passes through p, this quadric Q_p must then contain the entire line. Considering the lines of Q_p skew to L (i.e. of the same ruling), they will define an involution ι' on E which will agree with ι at z thus everywhere. In fact for a ruling the sum of two intersection points will be constant. If we look at all the lines that meet L and E in two additional points, they will constitute the other ruling of Q_p, if ι is given by $e - z$ (where $e = \iota(z) + z$ the other involution will clearly be given by $-e - z$ as four points adds up to zero on a quartic iff they lie in a plane. If Q_p happens to be a cone, there will be only one ruling and e will correspond to a 2-torsion point. Thus we have etablished the desired isomorphism.

The four 2-torsion points will correspond to the singular members, and the four corresponding ramification points will correspond to a set of four 4-torsion points. Such a set will correspond to the zeroes of a ϑ_{ij} and in particular lie on a plane. (The plane through three 4-torsion points obviously hits the quartic in a fourth 4-torsion point). To each such point we may associate the tangent plane at the corresponding cone. This tangentplane will intersect the quartic in just one point, and be so called hyperosculating. (In genral through each point one may find an osculating plane, intersecting it three times). The sixteen 4-torsion points hence play an analogous role to the nine flexes in the planar presentation.