Synopsis for Thursday, October 4

Dimension

We have defined the dimension for a variety we will consider various interpretations

i) number of algebraically independent functions

ii) number of local co-ordinates, i.e. writing it as a manifold

iii) In the projective setting we will show that if $V \subset P^n$ and there is a linear subspace L with $\dim(L) = k$ such that $L \cap V = \emptyset$ then $\dim(V) < n-k$. This will introduce the important geometric concept of a projection in projective space.

A polynomial ring $k[x_1, \dots x_n]$ has of course n independent functions x_i. If we consider $k[x_1, \dots x_n]/(F(x_1, \dots x_n)$ those functions will have a dependancy given by $F(x_1, \dots x_n) = 0$ while there will be no dependence between $x_2, \dots x_n$.

Now choose a point a such that $\mathrm{grad}(F)_a \neq 0$ and we can assume that $\partial F/\partial x_1 \neq 0$, then we can express x_1 as a function of $x_2, \dots x_n$ and the latter become local co-ordinates for a patch around a

Projections

We have a map $P^n \to P^{n-1}$ given by $(x_0, x_1, \dots x_n) \mapsto (x_1, x_2 \dots x_n)$. This is defined except at the point $(1, 0, 0 \dots 0)$ which is called the point of projection. We can also define it without using co-ordinates as follows. Choose a point $p \in P(V)$ this corresponds to a 1-dimensional linear subspace of V and hence a map $\pi : V \to V/L$. For any $K \neq L$ we have that $\pi(K)$ is a 1-dimensional linear subspace of V/L hence a map $\pi : P(V) \to P(V/L)$. The fibers of this map will be lines passing through p.

More generally we can choose a linear subspace $\Pi \subset P(V)$ corresponding to a linear subspace $\Gamma \subset V$ and a map $\pi : V \to V/\Gamma$ which whenever $p \notin Gamma$ gives a well-defined point of $P(V/\Gamma)$. Here the fibers will be linear subspaces of one dimension higher than Γ and containing it.

The following fact is elementary but fundamental.

Intersection of linear subspaces If V, W are linear subspaces of P^n of dimensions k, m respectively then if $k + m \geq n$ then $V \cap W \neq \emptyset$.

PROOF: Simply consider linear subspaces in of dimension $k+1$ and $m+1$ in a linear space of dimension $n+1$. They obviously intersect in a linear subspace whose dimension is at least $(k+1)+(m+1)-(n+1) = (k+m-n)+1$ with the condition above this subspace is non-trivial.

In fact the conclusion will hold for any projective sub-varieties they need not be linear.

Now For a variety X choose a subspace L of maximal dimension such that $L \cap X = \emptyset$. Then the image of X under the projection π_L must be the whole projective space, otherwise we could consider the inverse image of $x \notin \pi(X)$ giving a linear subspace properly containing L and disjoint from X We will discuss the dimensions of various varieties and introduce the Grassmannians. The dimension of this image projective space will be the dimension of X which is thus a covering of a projective space, and as we will se below, a finite covering.

Grassmannians

While P^n can be thought of as the lines through the origin of a $n+1$-dimensional vectorspace (i.e. all its 1-dimensional linear subspaces) we will define $G(k,n)$ as the space of all $k+1$-dimensional subspaces of a $n+1$ linear subspace.

The remarkable thing is that $G(k,n)$ is a projective variety, this is shown by the introduction of Plücker co-cordinates and the vectorspaces $\bigwedge^k(V)$ of a linear vector space V.

Given a vectorspace V with basis $e_1, e_2 \dots e_n$ we can consider the vector space $\bigwedge^k V$ with a basis of elements $\mathfrak{e}_I = e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}$ with $i_1 < i_2 \dots < i_k$ of dimension $\binom{n}{k}$. Now given a subspace W of V of dimension k choose a basis $f_1, f_2 \dots f_k$ for it and consider $\mathfrak{f} = f_1 \wedge f_2 \wedge \dots \wedge f_k$ where we write down the elements f_i as linear combinations of the basis elements e_i and use bilinearity as well as the alternating $e_i \wedge e_j = -e_j \wedge e_i$ and $e_i \wedge e_i = 0$. In this way we can express $\mathfrak{f}$ as a linear combination $\mathfrak{f} = \sum_I \mathbf{f}_I \mathfrak{e}_I$ and the coefficients $\mathbf{f}_I$ are called the Plücker coordinates (with respect to the basis of W). But if we choose another basis of W all the Plücker co-ordinates will simply be multiplied by the determinant of the basis change, Hence we have the Plücker embedding

$$Pl : G(k,n) \to P^{\binom{n+1}{k+1}-1}$$

This map is in general not surjective, but the image is cut out by homogenous polynomials

Example 1 In particular we have the quadratic line complex $G(1,3)$ of all lines in P^3 which can be thought of a 4-dimensional quadric in P^5.

To see this. We note that if $\dim V = 3$ then any element of $\bigwedge^2(V)$ is decomposable, i.e. it can be written as $w_1 \wedge w_2$. This is no longer true if $\dim V = 4$ then every element is either decomposable or of form $w = w_1 \wedge w_2 + w_3 \wedge w_4$ where the w_i are linearly independent (if not we would be in the case of $\dim V = 3$). In that case $w \wedge w = w_1 \wedge w_2 \wedge w_3 \wedge w_4 \neq 0$. Hence, the decomposable elements are characterized by $w \wedge w = 0$ which translates into a quadratic relation.

It is amusing the determine the dimension of $G(k,n)$. As a warm up consider the case of $G(1,3)$. Let V_1, V_2 be two planes in P^3. A line L is considered exceptional if it intersects $M = V_1 \cap V_2$. Unexceptional lines intersect V_1 and V_2 in unique points, and conversely given points in $V_1 \setminus M$ and $V_2 \setminus M$ they determine a unique non-exceptional line. Thus we see that they make up a $2+2=4$ dimensional space, in fact isomorphic with $\mathbb{C}^4$.

To describe the exceptional lines, consider a plane P not containing M and intersecting the latter in Q. For every pair of points $p \in M \setminus Q$ and $q \in P \setminus Q$ there is a unique exceptional line, and every such line occurs in this way, with the exception of M itself. Thus the exceptional lines make up a $3-$dimensional space.

More generally, as $k+1$ points determine a $k-$ dimensional linear subspace, and every such meets any $n-k$ dimensional linear subspace, we choose $k+1$ such and consider un-exceptional linear subspaces which meet them all in just one point. From this follows that the dimension of $G(k,n)$ is given by $(k+1)(n-k)$

Degrees and Bezouth's theorem

If $X = V(F)$ is a projective hypersurface, then its degree d is easy to define, it is simply the degree d of the polynomial F. In fact if we restrict F to any line, we get a polynomial f in one variable of degree d, or identically equal to zero. Geometrically it means that any line is either contained in X or meets it in d points counted with appropriate multiplicities, which are automatically given by the factorization of f.

In the general case we consider a maximal disjoint subspace L disjoint from X. L defines a projection of X onto a projective space with the same dimension as X. The fibers are given by intersecting X with subspaces M of one dimension higher than L and containing L. We need the following little lemma

Lemma: *Let X be an infinite projective subvariety, then it intersects each hyperplane non-trivially.*

PROOF: Any projective variety is compact, as it is a closed subset of a projective space which is compact. (By definition we get a map $S^{2n-1} \to \mathbb{C}P^n$ by normalizing the homogenous co-ordinates such that $\sum_i |z_i|^2 = 1$. This in fact gives a very interesting fibration with fibers equal to S^1 ($|\lambda| = 1$). A projective variety disjoint from a hypeplane, is in fact affine, and the co-ordinate functions x_i are holomorphic. As the variety is compact they attain maximal values as to their modulus, hence they are locally constant. The decomposition into irreducible varieties then become unions of a finite number of points.

The lemma has the obvious corollary that all fibers of the projection above is finite. What we need to know is that this finite number of points does not change as we vary the plane. When this is done, we can use this as the definition of a degree, To do so we need another interpretation of it.

Given this projection $p : X \to P^n$ with $n = \dim(X)$ we get an algebraic field extension $K(P^n) \subset K(X)$ by having each rational function 'pulling' back to $K(X)$. The degree of that field-extension is simply the degree of X.

Example 2 If we have a hypersurface $F(x_0, x_1, x_2 \dots x_{n+1}) = 0$ we consider the projection from a point $p = (0, 0, 0 \dots 1)$ outside it (Note that the condition is that F does not containing a pure power of x_{n+1} among its monomials) projection onto the plane $x_{n+1} = 0$. The latter has the algebraically independent rational functions $x_1/x_0, x_2/x_0 \dots x_n/x_0$ making up the function field $K(P^n)$ of the hyperplane. Those functions can also be thought of as rational functions on $V(F)$. The latter also have the rational function x_{n+1}/x_0

which is algebraically dependent on the previous, the dependency given by the polynomial F. If we write F in terms of x_{n+1} and coefficients in terms of the other variables, which can be thought of as elements of $K(P^n)$. This gives the defining extension.

There is also another approach. If we project from a subspace L_0 of codimension one in L, the image of X has co-dimension one. The image is also a variety (Note: It is important that we are in the projective setting, it is not true in the affine), and hence has to be given by single polynomial, whose degree is the sought after. (Note: The inverse images of lines, are going to be exactly the planes of complementary dimension meeting finitely)

Bezout in the plane

The most important theorem we will discuss is Bezout's theorem that the degree of an intersection is the product of the degrees. We will in particular consider the case of projective curves in the plane of degree m, n intersecting in mn points with the appropriate multiplicities.

The theorem is trivial if one of the curves is a union of lines and natural proofs of Bezout are given by some kind of deformation argument. So let the plane curves be given by F, G of degree n and m respectively, and let G_0 be the union of m-lines. Then we have the pencil $G_{(s,t)} = sG + tG_0 = 0$ with $0 = (0, 1)$ corresponding to G_0 and $\infty = (1, 0)$ corresponding to G and the rest are so to speak linear interpolations. Now the idea is that the number of intersection $F \cap G_{(s,t)}$ should stay constant. This is obviously false over the reals, as we may have critical cases which disconnect the parameter spaces $\mathbb{R}P^1 = S^1$, while over the complex numbers the paratemeter space $\mathbb{C}P^1$ it is not disconnected.

Recall that if we have a holomorphic function f then $\frac{1}{2\pi} \int_\gamma \frac{f'(z)}{f(z)} dz$ is an integer and gives the number of zeroes of f inside the region enclosed by γ. From this we see that this number stays constant if we perturb f slightly. Locally by the implicit function theorem $V(F)$ gan be given by some local co-ordinate z and G as some holomorphic function in z.

You may recall the Hausdorff metric on compact subsets. We say that $d(X, Y) < \epsilon$ iff $Y \subset X_\epsilon$ and $X \subset Y_\epsilon$ where U_ϵ is the ϵ neighborhood of U i.e. the set $\{x : \exists y \in U \quad d(x, y) < \epsilon\}$. We see from this that the Hausdorft distance between the zeroes of $V(F_t)$ and $V(F)$ goes to zero as F_t tends to F_t. This actually generalizes to several variables. However it is not true if you vary the coefficients of the polynomials of an ideal. We have that xy, yz, zx intersect in three points in P^2 (or in the co-ordinate axi of k^3 if you interpret it affinity) but if you perturb the quadrics ever so little, the intersection is empty!

We can also present another more algebraic approach via resultants of polynomials.

Resultants

Given two polynomials $f(x) = a_o x^n + \dots a_n$ and $g(x) = b_0 x^m + \dots b_m$ when can we tell whether they have a common root (i.e. a common linear factor) without having to find it?

It is easy to see that this is possible iff you can find polynomials $h(x), k(x)$ such that $h(x)f(x) = g(x)k(x)$ and $\deg(h) < \deg(g), \deg(k) < \deg(f)$. Thus we can write $h(x) = c_0 x^{m-1} + \dots c_{m-1}, k(x) = d_0 x^{n-1} + \dots d_{n-1}$

If we want to find such polynomials explicitly we need to solve the following system of linear equations,

$$\begin{array}{rcl} c_o a_0 & = & d_0 b_0 \\ c_0 a_1 + c_1 a_0 & = & d_0 b_1 + d_1 b_0 \\ c_0 a_2 + c_1 a_1 + c_2 a_0 & = & d_0 b_2 + d_1 b_1 + d_2 b_0 \\ & & \\ c_{m-2} a_n + c_{m-1} a_{n-1} & = & d_{n-2} b_m + d_{n-1} b_{m-1} \\ c_{m-1} a_n & = & d_{n-1} b_m \end{array}$$

This is $n+m$ homogeneous linear equations in $n+m$ variables, and in order to have a non-trivial solution the corresponding determinant

$$R(f,g) = R(a,b) = \det \begin{pmatrix} a_0 & a_1 & * & * & * & a_n & & & & \\ & a_0 & a_1 & * & * & * & a_n & & & \\ & & a_0 & a_1 & * & * & * & a_n & & \\ & & & * & * & * & * & * & * & \\ & & & & a_0 & a_1 & * & * & * & a_n \\ b_0 & b_1 & * & * & * & b_m & & & & \\ & b_0 & b_1 & * & * & * & b_m & & & \\ & & b_0 & b_1 & * & * & * & b_m & & \\ & & & * & * & * & * & * & * & \\ & & & & b_0 & b_1 & * & * & * & b_m \end{pmatrix}$$

Now if we consider f, g as homogenous polynomials, then the condition $a_0 = b_0 = 0$ simply means that they have ∞ as a common root.

The polyniomial $R(a,b)$ is said to be the resultant of f and g. Note that $R(\lambda a, b) = \lambda^m R(a,b)$ and $R(a, \mu b) = \mu^n R(a,b)$ and is thus bi-homogenous of bi-degree (m,n).

Now consider two homogenous polynomials F, G in three variables x, y, z. We can write them as $F(x,y,z) = a_0(y,z)x^n + a_1(y,z)x^{n-1} + \dots a_n(y,z)$ and $F(x,y,z) = b_0(y,z)x^m + b_1(y,z)x^{m-1} + \dots b_m(y,z)$ respectively, where $a_k(y,z), b_k(y,z)$ are homogenous of degree k. The crucial observation is that then the resultant $R(f,g)$ becomes homogenous of degree mn in y, z. It will then have mn roots, which will correspond to mn common intersection points of the plane curves $V(F)$ and $V(G)$.

if time we will also discuss a topological invariant

Euler characteristics

Eulernumber: This is an invariant $e(X)$ of a topological space under homeomorphism which has the following properties
i) $e(p) = 1$ if p is a point
ii) $e(X \times Y) = e(X) \times e(Y)$
iii) $e(X \cup Y) = e(X) + e(Y) - e(X \cap Y)$

Thus e is a counting function, and for finite sets X we simply have that $e(X)$ is its cardinality.

We note that if $X \to Y$ is a fibration which is locally a direct product with fiber F then $e(X) = e(F)e(Y)$. An example of a fibration which is not a direct product is given by $S^1 \to S^1$ given by $z \mapsto z^2$ with S^1 identified with $|z| = 1$. In this case F consists of two points. Hence $e(S^1) = 2e(S^1)$ from which we conclude that $e(S^1) = 0$. As S^1 is the one-point compactification of an open interval I we find $e(I) = -1$ and thus $e(I^n) = (-1)^n$. If X is triangulated by simplexes then it is a disjoint union of their open interiors, and s_k is the number of $k-$simplexes then $e(X) = \sum_k (-1)^k s_k$. Recall that the chain-complex C_k used to define (co)-homology of a space X will have dimension s_k and that the alternating sum above hence also can be considered as the alternating sum of its dimensions, which is the same as the alternating sign of the dimensions of the (co)homology groups, which is usually the first definition of the euler characteristics you encounter but is misleading as the components - the Betti numbers, are far more difficult to compute than their alternating sum.

The idea of the axiomatic approach above is to subdivide a topological space into simpler parts, and our ability to make the computation depends on our ability to do a suitable subdivision into parts we are already familiar with.

Eulernumbers of projective spaces

We have the following inductive decomposition

$$P^n(k) = k^n \cup P^{n-1}(k)$$

where the union is disjoint. We also have that $P^0(k) = (1)$ a simple point, hence $e_o(k) = e(P^0(k)) = 1$ always. We thus get the following recursive formula

$$e_n(k) = e(k)^n e_{n-1}$$

and all we need to know is how to compute $e(k)$.

If k is a finite field $\mathbb{F}_q$ (with $q = p^n$) this is simple and we get

$$e(P^n(\mathbb{F}_q) = \sum_i^n q^i = \frac{q^{n+1} - 1}{q - 1}$$

The latter has a nice interpretation, as $\mathbb{F}_q^{n+1}$ has $q^{n+1} - 1$ elements, and we divide out by $\mathbb{F}_q^*$ which has $q - 1$ elements.

If $k = \mathbb{R}$ then $e(\mathbb{R}) = -1$ (as $\mathbb{R}$ is homeomorphic to the open interval I) end hence $e(P^{2n}(\mathbb{R})) = 1$ and $e(P^{2n+1}(\mathbb{R})) = 0$. (As $S^n \to P^n(\mathbb{R})$ is 2:1 we get that even-dimensional spheres have eulernumber 2 and odd-dimensional 0 which could be easily seen also as S^n is the one-point compactification of $\mathbb{R}^n$)

If $k = \mathbb{C}$ then $e(\mathbb{C}) = 1$ (as $\mathbb{C} = \mathbb{R}^2$) and we get simply that $e(P^n(\mathbb{C}) = n+1$

Topology of surfaces

Compact orientable manifolds of real dimension 2 are classified by the number of 'holes' which is referred to as the genus. Genus zero corresponds to the sphere S^2, while genus one is the torus $T^2 = S^1 \times S^1$, higher genus have more holes. We can build them up inductively by something called attaching handles. A handle is a cylinder with two circles as boundaries. Being $S^1 \times I$ its has euler number zero. Take a surface and cut out two discs, that reduces the euler number by two, then glue the cylinder along its two circular edges, and that does not affect the euler number. Thus every attachment of a handle reduces by two. As S^2 has euler number 2 we get that a surface with g handles has euler number $2-2g$.

Non-singular plane curves are 1-dimensional complex manifolds and 2-dimensional real manifolds. How can we figure out how many handles they have, and hence how they look? In other words what shape do they have?

To do so we should clearly get hold of their euler number, and to do so, we will exhibit a fibration on it. This is a handy way due to the following observation which will play a crucial role in most of our euler counts.

Euler characteristics of vibrations Let $X \to T$ be a fibration with a finite number of special fibers F_t. Those are fibers that are not necessarily homeomorphic with all the other fibers F. Then

$$e(X] = e(F)e(T) + \sum_t (e(F_t) - e(F))$$

PROOF: Remove all the special fibers, say s of them, we then get $e(X) = e(F)(e(T) - s) + \sum_t e(F_t)$ from which the formula follows directly.

Now if we take a projection of a plane curve of degree n onto a line, the general fiber will contain n distinct points. However if a tangent line passes through the projection point, the fiber will only have $n-1$ points. Thus we need to figure out how many tangents to a curve goes through a given point. This number is incidentally called the order of the curve.

Let us choose co-ordinates (x, y, z) such that the projection point is at $p = (0, 0, 1)$. The tangents are given by the lines $\partial f/\partial x, \partial f/\partial y, \partial f/\partial z$, and the tangents going through p are those for which $\partial f/\partial z = 0$. So we are interested in the intersection between f and $\partial f/\partial z$. As the curves are of degree n and $n-1$ respectively we get that the order is $n(n-1)$ and hence the euler number is given by $2n - n(n-1) = 2 - (n-1)(n-2)$ and thus for a non-singular plane curve of degree n we have

$$g = \frac{(n-1)(n-2)}{2}$$

Example 3 If $n = 1, 2$ we have $g = 0$. This is obvious for $n = 1$ because a line is just a $\mathbb{C}P^1$, for $n = 2$ we have shown that a quadric can be parametrized by binary quadrics.

Example 4 $n = 3$ corresponds to cubic curves which are topologically tori. There is a rich theory connected to them, enough easily to fill up a whole semester.

*Cubic curves are groups and can be represented as $\mathbb{C}/\Lambda$ where Λ is a rank two $\mathbb{Z}$-lattice, which can be normalized to $< 1, \tau >$ where $\tau \in \mathbf{H}$(the upper half-plane). The field of meromorphic functions of such a curve is given by all doubly-periodic meromorphic functions on $\mathbb{C}$ with respect to Λ, the most famous being the Weierstraßfunction $\wp(z) = \sum_{w \in \Lambda^*} \frac{1}{w^2} - \frac{1}{(z-w)^2}$ which together with its derivative $\wp'(z)$ generate all such functions. The two satisfies an algebraic relation of form $\wp^2(z) = 4(\wp')^3(z) + p(\Lambda)\wp'(z) + q(\Lambda)$ which recaptures it as a cubic in the plane. The coefficients can be written as explicit functions of τ and have invariant properties under the modular group $PSL(2, \mathbb{Z})$ (Moebius transformations with integral coefficients and determinant one) known as modular forms.......

Example 5 $n > 3$. *This corresponds to curves with negative euler number and whose universal covering is an open disc, which can profitably be given a hyperbolic metric.......