Synopsis for Thursday, October 11

Divisors and Linebundles

Divorsors

By a divisor $D \subset X$ in a variety is simple meant a finite linear combination of hypersurfaces. In particular if X is a curve, then a divisor is simply a finite linear combination $\sum_p n_p P$ of points. A typical eaxmple of a divisor is given by a meromorphic (rational) function ϕ. It can be written as $(\phi) = \phi^{-1}(0) - \phi^{-1}(\infty)$. The first part is referred to as the zeroes of ϕ the second part to the poles. It is clear that when we multiply rational functions we add their corresponding divisors. Divisors form an additive group, in fact a huge one, as any non-trivial variety over $\mathbb{C}$ is uncountable.

An important invariant for divisors over curves is their degree. It is defined simly as $d(\sum_P n_p P) = \sum_P n_p$ and is clearly an integer.

First we must once and for all understand the basic

Proposition *If ϕ is a rational function on a projective curve X then $d(\phi) = 0$.*

PROOF: If γ is a simply closed curve and ϕ is a meromorphic function then

$$\frac{1}{2\pi} \int_\gamma \frac{\phi'}{\phi} = N - P$$

where N is the number of zeroes and P the number of poles inside the region enclosed by γ (assumed positively generated, otherwise we have of course a change of sign). Now let γ be a very small curve enclosing no zeroes and poles in its interior. But what is its interior? There is the small region enclosed by it, and the big one outside of it. The latter contains all the zeroes and poles, and we can easily compute their number, with appropriate signs, which is nothing but the degree of the divisor. As the integral by continuity can be made arbitrarily small we are done.

Now we come to the central definition.

Definition: Linear equivalence *Two divisors D, D' are said to be linearly equivalent if there is a rational function ϕ such that $D - D' = (\phi)$*

What is the meaning of this? First of all we say that a divisor is effective if we have that the multiplicities n_D are all non-negative. A necessary condition for effectiveness is that the degree is non-negative, but this is not sufficient. So assume that D, D' are effective. Then being equivalent means that D is say the zeroes of ϕ and D' the poles. The rational function ϕ gives a map $\phi : X \to P^1$ and we get a linear deformation of the divisor D via $D_t = \phi^{-1}(t)$ that moves D_0 to D_∞.

Line bundles

A line bundle L over X is a fibration $\pi : L \to X$ whose fibers are lines ($\mathbb{C}$) and such that the fibration is locally trivial. So what does that mean? We should consider X to be covered with open subsets U_i over which π is trivial. We can then write $\pi^{-1}(U_i) = U_i \times \mathbb{C}$. Those need to be glued together, and how do we express glueing data? We can think of local co-ordinates as (z_i, t) for $\pi^{-1}(U_i)$ where z_i is so to speak the local fiber co-ordinate. Now if we have two open covers intersecting in $U_{ij} = U_i \cap U_j$ we should have $z_i = \phi_{ij}(t) z_j$ where $\phi_{ij}(t) \neq 0$ and defined on the intersection U_{ij}. It is natural to think of ϕ_{ij} as holomorphic functions if we want L to be a holomorphic bundle. Now those ϕ_{ij} are referred to as the transition functions, and they cannot be arbitrarily chosen. If $U_{ijk} = U_i \cap U_j \cap U_k$ is non-empty we have many identifications over points in U_{ijk} and they have to be compatible. What is the compatibility condition? It is easily seen to be given by $\phi_{ik} = \phi_{ij}\phi_{jk}$ where we make the natural convention that $\phi_{ij} = 1/\phi_{ji}$ and hence $\phi_{ii} = 1$.

Now those transition functions are not uniquely determined by the linebundle, after all we can effect changes of the fiber co-ordinates. Chosing non-zero holomorphic functions f_i on U_i we can introduce new co-ordinates $z'_i = f_i z_i$ and the new transition functions ϕ'_{ij} will be transformed accordingly $\phi'_{ij} = f_i/f_j \phi_{ij}$. It is clear that if ϕ_{ij} satisfies the compatibility condition so does ϕ'_{ij}. In particular if we can find functions f_i such that $\phi_{ij} = f_i/f_j$ then we can use those to get a global trivialization of L, namely to write L as $\mathbb{C} \times X$.

Digression on Czech cohomology

By a Czech cochain is meant functions taking values in some sheaf $\mathcal{F}$ and depending on intersections of open sets of an open covering. I will not give a formal definition of a sheaf, it is easy enough, but you should think about it as functions defined on open subsets of a topological set X. If the functions are constant, we think of the sheaf as one of locally constant functions.

By a 0-cochain we can think of as functions f_U defined on an open set U in the covering, while a 1-cochain is given by functions f_{UV} defined on $U \cap V$ while a 2-cochain is similarly given by functions f_{UVW} defined on intersections $U \cap V \cap W$. Now we can define a boundary operator ∂ defined as follows

$(\partial f)_{UV} = f_U - f_V$ and $(\partial f)_{UVW} = f_{UV} - f_{UW} + f_{VW}$ and so on.

What does it mean that $\partial f = 0$ for a 0-chain? Simply that $f_U = f_V$ on $U \cap V$. In other words that we can find a function F globally defined such that $F_U = f_U$ we say that the local data patches up to a globally defined function. What does it mean that $\partial f = 0$ for a 1-chain? Namely that $f_{UV} = f_{UW} + f_{WV}$ (Note that $f_{XY} = -f_{YX}$). This is almost the same thing as the compatibility condition for transition functions, except that it is written additively instead of multiplicatively. Furthermore a 1 boundary is of the form $f_u - f_v$ which is the additive interpretation of the multiplicative presentation of transition functions which are trivial.

If $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on X (it means simply that we have for open sets U in some open cover, holomorphic functions defined on U). As there are no global holomorphic functions, save the locally constant on a compact complex manifold we see that $H^0(\mathcal{O}_X$ is simply a vectorspace of dimension equal to the number of components of X. If the latter is connected. the dimension is simply given by 1.

We can even introduce the sheaf $\mathcal{O}_X^*$ which consists of non-vanishing holomorphic functions on X and should be treated multiplicatively (or by taking logarithms seen as additive). Recall that whenever U is an simply connected open set, we can write an element $f \in \mathcal{O}_x^*$ as $f = e^{2\pi i g}$ where $g \in \mathcal{O}_X$. We can now interpret the set of linebundles as given by $H^1(\mathcal{O}_X^*)$

Sections of line-bundles and linear systems

By a section s of a linebundle is meant a map $L \leftarrow X$ such that $\pi s = I_X$. A section thus maps every point $x \in X$ to a point $s(x)$ in the fibre $\pi^{-1}(x)$ of x. A section is given by local sections s_i such that $s_i = \phi_{ij} s_j$. Note that the section s_i may very well be zero on U_i so they do not trivialize the transition functions. There is always a trivial section, namely the zero section. However if there is a section that is always non-zero, i.e. never meeting the zero-section, we do get that the line bundle has to be trivial. A trivial line bundle over a compact space has only constant sections. Thus if we have a non-constant section it has to meet the zero-section non-trivially, and in fact the s_i define a divisor by looking locally at $s_i = 0$ at U_i. As $\phi_{ij} \neq 0$ those zero-sets coincide, whether we look at them at U_i or U_j and thus are globally defined. Now if we have two sections s_0, s_1 then s_1/s_0 is well-defined as a meromorphic function as the local pieces satisfy the same transition factor. Thus the divisors D_0 and D_1 associated to s_0 and s_1 respectively are linearly equivalent, as surely $D_1 - D_0 = (s_1/s_0)$. Thus any line-bundle defines linearly equivalent divisors. Conversely if D is any effective divisor linearly equivalent to a divisor D_0 associated to a section s_0 of a linebundle, then it is the divisor of some section. In fact if $D - D_0 = (\psi)$ then $s_0\psi$ is a section. The key-point is that ψ is a bone-fide function on X and thus we have locally $\psi_i = \psi_j$, hence $s_0\psi$ satisfies the same transition functions as s_0 and hence is a section of L. All the effective divisors linearly equivalent to a divisor D is called the complete linear system associated to D. It can be thought of as a vector space of meromorphic functions, namely the space of meropmorphic functions ψ such that $(\psi) + D$ is effective, or with obvious notation $(\psi) + D \geq 0$. It is a finite-dimensonal space, provided X is compact. We will denote it by $H_0(D)$.

Line-bundles associated to divisors

Given a divisor D. Then it is locally given by equations $f_i = 0$. For it to be well-defined we need that the quotients $\phi_{ij} = f_i/f_j$ are everywhere non-zero on $U_i \cap U_j$. Those ϕ_{ij} define transition functions and hence line-bundles.

It is now easy to see that linearly equivalent divisors give rise to isomorphic line-bundles, and can be recaptured from those by considering zeros of meromorphic sections. If they are effective, then they can be recaptured as zeroes of holomorphic sections.

Key Point. The crucial data are given by transition functions.

Those can be manipulated very easily, much more easily than individual sections and corresponding divisors. Inherent in them is the 'moving property' and as such they will give an easy proof of Bezout as we will see later.

Functorial properties of Line-bundles

If we have a map $\theta : X \to Y$ we can lift any line-bundle L of Y to one denoted $\theta^*(L)$ above X. The corresponding line-bundle is called the pull-back and is almost trivially defined by considering its transition functions $\theta^*(\phi_{ij}) := \phi_{ij}(\theta)$ by composition.

If θ is an inclusion, then we simply restrict the transition functions from Y to X.

Maps associated to linear systems

If we have two sections s_0, s_1 to the same line-bundle then we have seen that s_1/s_0 is a well-defined meromorphic function, and hence a map into P^1. We can also represent it as the map $x \mapsto (s_0(x), s_1(x)$. More generally, let V be a linear subspace of $H^0(D)$. Such a space is referred to as a linear system. If we choose a basis $s_0, s_1, \ldots s_n$ for it we get a map of X into P^n in the obvious way.

Now we should be careful. If all $s_i(x) = 0$ we say that x belongs to the base locus of the linear system, and it is only outside the base locus that we have a well-defined map.

We can make the maps into projective space co-ordinate free, provided that we map into their duals. In fact if x does not belong to the base-locus, the sections vanishing at x form a hyperplane.

Given a map $p : X \to P^n$ associated to a line-bundle L. We can then consider the hyperplanes H of P^n and look at $p^*(H)$ those are referred to as the hyperplane sections, and they correspond to divisors of sections of L and all such occur in this way.

If we consider an arbitrary linear system i.e. a linear subspace of the complete linear system, the map onto that can be thought of as a projection from the complete P^n.

Pencils, which we already have encountered, are simply 1-dimensional linear systems. They give rise to fibrations over P^1

2-dimensional systems are called nets classically, while 3-dimensional are referred to as webs.

Linebundles on $\mathbb{C}P^1$

It is a fact that every linebundle over $\mathbb{C}$ is trivial. Hence the line bundles on the Riemann sphere can be thought of as a gluing of two copies of $\mathbb{C}^2$. More specifically we have U_0 with co-ordinates (ζ, z) and U_1 given by (ξ, w) where z, w are co-ordinates on $\mathbb{C}P^1$ satisfying $z = \frac{1}{w}$ and ζ, ξ fiber-coordinates. In this case the transition functions are reduced to a single one ϕ_{01}.

Fact: Any holomorphic map $f : \mathbb{C}^* \to \mathbb{C}^*$ has a winding number

PROOF: Let γ be a simply closed curve around the origin in the source. As it is compact and f is continous $f(\gamma)$ is also a closed curve in the target, and as such it has a winding number. This winding number depends continuously on γ and being integer valued it has to be locally constant, hence constant and independent of γ

From this we conclude that $\phi_{0,1}$ can be written for a unique n in the form $z^n\phi'$ such that we can take the logarithm ψ of ϕ'. Consider the Laurent expansion of ψ this means that we can write $\psi(z) = \psi_0(z) - \psi_1(w)$ where ψ_0, ψ_1 are non-vanishing analytic in U_0 and U_1 respectively. We can thus write the transition function as $z^n f_0(z) f_1(w)$ which means that we can normalize it to z^n.

Theorem: *The linebundles on the Riemannsphere are classified by the integers* $\mathbb{Z}$

Now we would like to determine the space of sections. We would like to find analytic functions $s_0(z)$ and $s_1(w)$ such that $s_0(z) = z^n s_1(w)$

As $w = \frac{1}{z}$ we can write the right-hand side as $\sum_{k \geq 0} a_k z^{n-k}$ and thus the condition $a_k = 0$ if $k > n$. Hence if $n < 0$ we only have the zero section. Otherwise the dimension of the linear space of sections is given by $n + 1$.

Proposition: If $V_n = \Gamma(L_n)$ is the vector space of sections of the line-bundle L_n of degree n then we have

$$\begin{array}{ll} \dim V_n = n+1 & n \geq 0 \\ \dim V_n = 0 & n < 0 \end{array}$$

Furthermore we can identify V_n for $n \geq 0$ with the vector space of binary forms of degree n.

PROOF: This we have already proved, except for the last. Given any binary form $F(x_0, x_1)$ of degree n we can dehomogenize it in two ways

$$x_0^n F(1, \frac{x_1}{x_0}) = F(x_0, x_1) = x_1^n F(\frac{x_0}{x_1}, 1)$$

set $z = \frac{x_1}{x_0}$ and $w = \frac{x_0}{x_1}$ we have $z = \frac{1}{w}$ and $F(1, z) = z^n F(w, 1)$.

Note that binary forms are not functions on $\mathbb{C}P^1$ except in the trivial case of $n = 0$ but that they are sections of appropriate line-bundles.

However the quotient of two forms of the same degree becomes a rational function, and all rational functions can of course be written in that way.

From that we conclude the fundamental

Theorem: Divisors on $\mathbb{C}P^1$ are linearly equivalent iff they have the same degree.

Clearly the degree of a divisor is equal to the degree of its line-bundle.

1-forms and vectorfields on the Riemann sphere

Now by the chainrule $\partial F/\partial z = (\partial F/\partial w)(dw/dz)$ thus a vector field given by $s_0(z)\frac{\partial}{\partial z} = s_1(w)\frac{\partial}{\partial w}$ will satisfy $s_0(z)(dw/dz) = s_1(w)$ or $s_0(z) = -z^2 s_1(w)$. Thus there will be plenty of holomorphic vectorfields on the sphere, in fact they are given by the binary quadrics.

The dual case of 1-forms is different. Clearly $dw = \frac{dw}{dz}dz$ and hence if we have $s_0(z)dz = s_1(w)dw$ we obtain $s_0(z) = -z^{-2}s_1(z)$ hence there are no global 1-forms on the sphere, except the trivial.

We have thus the fundamental

Theorem: *The degree of the canonical linebundle of $\mathbb{C}P^1$ is -2*

Maps of linear systems on $\mathbb{C}P^1$

The map $\mathbb{C}P^1 \to \mathbb{C}P^2$ given by all binary quadrics has as an image a curve of degree 2 i.e. a conic. And conversely all conics occur in this way as we have already noticed.

The map $\mathbb{C}P^1 \to \mathbb{C}P^3$ given by all binary cubics has as an image a curve of degree 3. It is biholomorphic to CP^1 and is thus a so called rational curve, referred to as the twisted cubic.

If we consider the linear system of cubics passing through a fixed point on $\mathbb{C}P^1$ this forms a net and hence gives a projection of the twisted cubic onto the plane. The image will be a conic, as we can write the cubics as quadrics multiplied with a constant linear factor cutting out the fixed point. Geometrically it means that for any point p on the twisted cubic C that curve is contained in a quadric cone. Given two quadric cones corresponding to the points p, q their intersection will consist of the twisted cubic along with the line passing through p and q. The twisted cubic is not a complete intersection, you need three quadrics to cut it out. Given any such three quadrics they will not only cut it out but also generate all quadrics which contain the twisted cubic, which constitute a net with the twisted cubic as a base locus.