Synopsis for Thursday, October 18

Blow-ups

Global picture

Consider the projection $\pi : P^2 \to P^1$ given by $(x_0, x_1, x_2) \to (x_1, x_2)$. This is well-defined except at the center of projection $(1, 0, 0)$. If we ignore that point we can consider the graph Γ of the well-defined map $P^2 \setminus \{(1, 0, 0)\} \to P^1$ inside $P^2 \setminus \{(1, 0, 0)\} \times P^1$. If $(x_0, x_1, x_2; y_0, y_1)$ is a point in $P^2 \times P^1$ then clearly $y_0x_2 - y_1x_1 = 0$ for points in Γ, because that is just another way of saying that $(x_1, x_2) = \lambda(y_0, y_1)$ for some $\lambda \neq 0$. But this polynomial equation makes sense for the whole product and its zeroes make up the closure $\overline{\Gamma}$ of the graph. Now there is a projection $\pi : \overline{\Gamma} \to P^1$ given by the projection onto the second factor, which is defined on the whole of $\overline{\Gamma}$ as it is defined on the whole of $P^2 \times P^1$. But what is the relation of $\overline{\Gamma}$ to P^2? There is of course a projection of $P^2 \times P^1$ to the first factor that can be restricted to $\overline{\Gamma}$. What are its fibers? If $(x_0, x_1, x_2) \neq (1, 0, 0)$ then not both x_1, x_2 can be zero, thus there is a unique point $(y_0, y_1) \in P^1$ such that $(x_0, x_1, x_2; y_0, y_1) \in \overline{\Gamma}$. Thus the projection is $1 : 1$. However over $(1, 0, 0)$ there will be no restriction on y_0, y_1 so the fiber is the entire line P^1! Thus $\overline{\Gamma}$ is the same as P^2 except over the point P where it has been replaced by an entire line. We say that the point P has been blown up! And we will refer to $\overline{\Gamma}$ as the blow-up $B_p(P^2)$ of P^2 and the inverse image of the projection center the exceptional divisor.

Remark 1 : This procedure of blowing-up to make maps defined can of course be generalized. If we have a pencil of conics in the plane, there will be in general four base points, where the map

$$(x_0, x_1, x_2) \to (Q_0(x_0, x_1, x_2), Q_1(x_0, x_1, x_2))$$

is not well-defined. We can consider the graph of the map where it is well-defined and then take its closure. In this case it means that we look at the equation $y_0Q_1(x_0, x_1, x_2) - y_1Q_0(x_0, x_1, x_2) = 0$ which incidentally is bi-homogenous of degree $(1, 2)$. Once again the projection of the closure of the graph is $1 : 1$ except at the base points, where it is similarly blown-up.

Local picture

What is really going on? By taking away the line $x_0 = 0$ we can also look at the map $\pi : \mathbb{C}^2 \to P^1$ simply given by $(x, y) \mapsto \frac{x}{y}$. We see that it is constant on the lines through the origin, and that its value is given by the slope of the line. It is not well-defined at the origin, as that lies on all the lines. We are there reduced to the classical problem of assigning a value to $0/0$. This cannot be done as we see, unless we replace the origin with all the possible values.

Another way of putting it is this. Consider all pairs (P, L) such that P is a point on the line L through the origin. When P is not the origin, the line L is uniquely defined, but when P is the origin L can be any line. Thus what we want to do is to replace the origin with all the directions through it. Those are given by lines through it, and they are parametrized by P^1.

Remark 2 : We can of course do this for a pencil of conics as well. We simply consider pairs (P, C) where P is a point lying on the conic C in the pencil. The conic C is uniquely-defined by P unless it is a base-point, in that case any C in the pencil will do, and the base point is replaced by P^1 which parameterizes all conics in the pencil.

Now to return to the first case. This defines a line-bundle over P^1 by considering all the lines through the origin, but by having the lines separated there. We know all the line-bundles over P^1 and they are classified by their degree. What is the degree of this line-bundle?

If (x, y) are the co-ordinates of $\mathbb{C}^2$ then they become homogenous co-ordinates on the line of all directions through the origin, i.e. homogenous co-ordinates on the exceptional divisor. Now we can think of x, y to be fiber-coordinates on the fibers, which are simply the lines through the origin. We can also find two co-ordinate patches on P^1 by chosing $z = y/x$ and $w = x/y$. The relation between the fiber co-ordinates then becomes

$$x = (x/y)y$$

or

$$x = z^{-1}y$$

which means that the degree is -1. Thus the only section of this line-bundle is the zero-section. That there cannot be any other sections is obvious, because any such would map down to a compact connected complex submanifold of $\mathbb{C}^2$ and any such must be reduced to one point. (the co-ordinate functions will assume maximal values and hence be constant.)

We can further localize this construction by introducing local-coordinates, namely set

$$\begin{array}{ll} u = x & u' = x/y \\ v = y/x & v' = y \end{array}$$

Remark 3 Another way of putting it is to consider the open patches $(u, v) \in \mathbb{C}^2$ and $(u', v') \in \mathbb{C}^2$ and the map π defined by $\pi(u, v) = (u, uv)$ and $\pi(u', v') = (u'v', v')$ which implicitly define the relation between the local coordinates. Working it out explicitly we get of course

$$\begin{array}{ll} u = u'v' & u' = 1/v \\ v = 1/u' & v' = uv \end{array}$$

Note that the local equations for the exceptional divisor E will be given by $u = 0$ and $v' = 0$ respectively, and local coordinates for E will be given by v and u' with the relation $v = 1/u'$. Now if we think of u, v' as fiber coordinates we get $u = v^{-1}v'$ thus once again degree -1.

The topology of a blow-up

First we should note that we can localize the construction by instead of considering big open subsets $\mathbb{C}^2$ which cannot in general be accommodated in a 2-dimensional complex manifold, we can choose an arbitrary small open neighbourhood U around a point p (usually a small open ball) and then define a small tubular neighbourhood around P^1 as in remark . What that means is that we define a small tubular neighbourhood of the zero section in the linebundle over P^1 of degree -1. Explicitly we cover the Riemann sphere with two hemi-spheres (the northern and the southern) by $|z| \leq 1$ and $|w| \leq 1$ which meet at the equator $|z| = |w| = 1$. Now we thicken the hemi-spheres by muliplying them with discs of radius ϵ, or using fiber-coordinates we look at (z, ζ) with $|z| \leq 1, |\zeta| \leq \epsilon$ and similarly for (w, ξ). Now because of the relation $\zeta = z^{-1}\xi$ on the equator, those are compatible. The point is that if you puncture the tubular neighbourhood by removing the zero-section what you get is isomorphic to a punctured ball in $\mathbb{C}^2$. A blow-up is a special kind of surgery in which you replace a small ball around a point with a small tubular neighborhood of P^1 in the above line-bundle.

From this we see right away that if X is a 2-manifold and BX is a blow-up of a point then $e(BX) = e(X) + 1$ because we replace a point P with $e(P) = 1$ with a sphere P^1 with $e(P^1) = 2$. The point is that we can use this process inductively and blow up any number of times.

We can also picture a blow-up in the real case. Then we can think of the lines through the origin as a spiral stair-case. Consider the plane $\mathbb{R}^2$ and a vertical z-axis at the origin. Now each line $y = ax$ with slope a through the origin is lifted up to the line $z = a$ and $y = ax$, where $a = y/x$ becomes a loca coordinate along the $z - axis$ the exceptional divisor.

Now the tubular neighbourhood of the exceptional divisor in the real case is just a Moebius strip, whose boundary is connected, in fact a circle S^1. Blowing up means that you remove a disc from a surface and glue on a Moebius strip along its boundary. Clearly we lower the euler-number by one.

Example 1 In a stereographic projection of a sphere onto a plane, typically by taking the lines from the northpole intersecting a plane through the equator, we get a 1-1 correspondence between the sphere minus the projection point (the north pole) and the plane. (In fact this is the way we get a local patch $\mathbb{C}$ on the Riemann sphere, and if we do it from the southpole we get another patch, and the relation between the local coordinates z, w is given by $z = 1/w$). But if we want to extend this to involve the projection center as well? Then it is not well-defined, as we will consider lines tangent to the sphere at the center, and those will correspond to points at the line at infinity. But if we do a real-blow up, this will be well-defined, so $\mathbb{R}P^2$ is simply the sphere with a real blow-up.

Remark 4 In the terminology of real surface topology we are adding a cross-cap. The result will be something non-orientable, as the Moebius strip is non-orientable. Note also that this is compatible by thinking of the real projective plane as the glueing of Moebius strip along a disc, which we naturally get by considering the projective plane as the sphere quotient out by the antipodal involution. Think of the sphere as a cylinder ($S^1 \times I$) along the equator and two discs on either side. By doing the antipodal map the discs are identified and the cylindrical strip is mapped onto a Moebius strip.

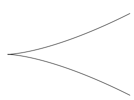
Remark 5 This makes sense in the finite setting as well. Then we replace a point by $q+1$ points.

Blowing up

As we have seen blowing up is a way of making maps well-defined by separating the level curves where they intersect. Blowing up has an even more important application in resolving singularities. So let us look at this a little bit systematically.

The node and the cusp

We have two fundamental examples of so called plane double points, namely the ordinary node that can be exemplified by $y^2 = x^2$ and the cusp $y^2 = x^3$. The first one is easily seen as two lines intersecting at the origin as $y^2 - x^2 = (y+x)(y-x)$ while the second is a bit more subtle, and the real picture is as below

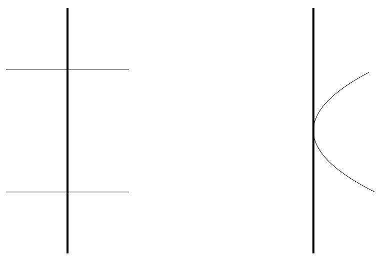


The complex picture is far more complicated, note however that the cusp has a simple parametrization by $t \mapsto (t^2, t^3)$ which establishes a homeomorfism between $\mathbb{C}$ and the cuspidal cubic given by the equation. This map is not a diffeomorphism, the cusp has a sharp 'cusp'. If we intersect the cusp with the boundary S^3 of a ball, we will get a curve lying on a torus, and winding two times around one axis and three times around another. In the case of the two lines, each will intersect the S^3 in a circle, and be a cone over that circle, and the two circles will be linked.

Algebraically things are much easier. If we replace x by u and y by uv we will get the equation $u^2(1-v^2)$ in the case of the node and $u^2(u-v^2)$ in the case of the cusp. Those will be the local equations of the pullbacks of the curves to the blow up $\pi : BP^2 \to P^2$ and will be denoted by $\pi^*(C)$ for any curve C and refered to as the total transform of C. We note that the exceptional divisor E will occur with multiplicity 2 as that divisor is given by $u = 0$. That the multiplicity is two illustrates the fact that we are talking about double points. If we factor out the contribution of the exceptional divisor we get something more interesting, and which we refer to as the proper transform. We can think of this as the closure of the inverse image of $C \setminus \{P\}$.

In the case of the node we get $v^2 - 1 = 0$ thus $v = \pm 1$ which corresponds to the two directions of the two lines, which are now separated in the blow up.

In the case of the cusp we get the proper transform $u = v^2$ which is a parabolic arc which is tangent to the exceptional divisor E given by $u = 0$.

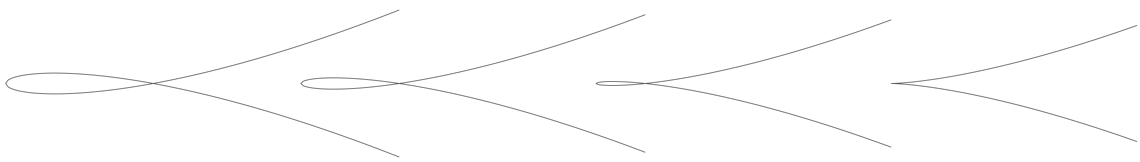


In both cases the singularity has been resolved.

Now $y^2 = x^3$ is a cuspidal cubic, as a projective variety it has the equation $zy^2 = x^3$. If we intersect it with the line at infinity given by $z = 0$ we see that it intersects it with multiplicity three at the point $(0, 1, 0)$. The line at infinity is a so called flexed tangent to the cubic. If we make another dehomogenization setting $y = 0$ as the line at infinity, we get the curve $z = x^3$.

Now $y^2 = x^2$ is of course not a cubic, but if we multiply the right-hand side with a factor $(x - \lambda)$ it becomes one, namely $y^2 = x^2(x - \lambda)$. The real picture will be as follows, depending on the sign of λ

Note that when we let $\lambda \to 0$ the node will approach a cusp



If we look at a strand of hair, when our line of sight is a secant to the hair then we see a node, but if it is tangent we see a cusp.

Resolving plane singularities

If we dehomogenize a homogenous polynomial $F(x_0, x_1, x_2)$ to $f(x, y) = F(1, x, y)$ we can write it as

$$f(x, y) = f_0(x, y) + f_1(x, y) + f_2(x, y) + \dots f_n(x, y)$$

where f_i is the homogenous part of degree i.

f_0 is a constant and $f_0 = 0$ iff the corresponding curve goes through the origin (which we can always normalize as to ensure) and which we will assume from now on.

if $f_1(x, y)$ is not identically zero, it defines a line $f_1(x, y) = 0$ which is in fact the best linear approximation of the curve at $(0, 0)$ in other words the tangentline. Thus $V(F)$ is non-singular at $(0, 0)$ iff $f_1(x, y) \not\equiv 0$.

The smallest m such that $f_m(x, y) \not\equiv 0$ is called the multiplicity of the singularity, and we say that $V(F)$ has a double, triple, quadrupel etc point at the origin if $m = 2, 3, 4 \ldots$.

If we take the total transform of an m-tuple point, we can write the equation as

$$\pi^* f(u, v) = u^m \overline{f}(u, v)$$

where $\overline{f}(u, v)$ is the equation of the proper transform.

If $m = 2$ and $f_2(x, y)$ has two distinct roots, we have the case of an ordinary node. One blow up resolves it. Then there are a whole slew of double points in which $f_2(x, y)$ has a double root, by change of coordinates we can reduce it to the case $f_2(x, y) = y^2$.

Making the pullback we get $u^2(v^2 + uf_3(1, v) + \ldots)$. If $f_3(1, 0) \neq 0$ then the proper transform has a non-vanishing linear term and is thus non-singular after the blow-up and we have a cusp. If $f_3(1, 0) = 0$ we need to look at the next higher order term and we look at $v^2 + uvf_3'(1, v) + u^2 f_4(1, v)$ if this has two distinct roots, the proper transform has an ordinary node and we say that the original curve had a tacnode. As an example we can look at $y^2 = x^4$ which is given by two parabolic arc which are tangent, but which intersect transversally after one blow up.

We can actually classify all double points in the plane inductively. We will call an ordinary node by a_1 and a cusp by a_2. If a double point resolves to a double point of type a_k on the proper transform, we will call it a a_{k+2} point. How do we know that this process terminates and thus makes sense. Patience! We will shortly return to it.

The Neron-Severi group

To divisors on curves we may assign in a natural way the degree of the divisor, but this is not the case of divisors on surfaces, which are linear integral combinations of curves on surfaces. What we have instead is an intersection product which we can see as a generalization of Bezouts theorem.

The preliminary definition of the Neron-Severi group is as a lattice generated by an intergal basis of divisors up to linear equivalence together with a quadratic form - the intersection product. We will denote it by $\mathbb{NS}(X)$ for a surface X.

Example 2 The projective plane P^2 has a Neron-Severi group isomorphic to $\mathbb{Z}$ with a generator H (corresponding to a hyperplane, i.e. a line) and with $H^2 = 1$. In other words every divisor is lineraly equivalent to a line with appropriate multiplicity. The space of global sections corresponding to nH with $n \geq 0$ is naturally identified with all ternary forms (i.e. homogenous forms in three variables) of degree n. The fact that $nH\dot{m}H = nmH^2 = nm$ is just Bezouts theorem to the effect that a curve of degree n meets a curve of degree m in nm points (with intersections counted with the appropriate multiplicities).

Remark 6 Why is this so? One idea is to think of P^2 as the union $\mathbb{C}^2 \cap P^1$ where all the linebundles on $\mathbb{C}^2$ are trivial, thus any line-bundle can be 'pushed-off' to the line at infinity and reduce to the case one lower. This is a vague argument and it is not so obvious how to make it precise.

Example 3 There are many other compactifications of $\mathbb{C}^2$ (unlike the case of curves when $\mathbb{C}$ can only be compactified in one way) another example is the quadric $P^1 \times P^1$. This surface has a Neron-Severi group equal to $\mathbb{Z}^2$ with a basis given by curves in the two fiberings of the surface. Let us call them F_1 and F_2. Obviously $F_1^2 = F_2^2 = 0$ while $F_1 F_2 = 1$. Thus we have the intersection matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

. If $m, n \geq 0$ then the space of global sections to the divisor $mF_1 + nF_2$ corresponds to bi-homogenous forms of bi-degree (m, n)

Now how should we define the intersection form? Given two curves C_1 and C_2 it is intuitively clear how we do it (although admittedly there may be some problems with the appropriate multiplicities of the intersection points). In particular if C_1 and C_2 are disjoint we ought to have $C_1 C_2 = 0$ (as we assumed above). Now to do it more formally, let L_1 and L_2 be the associated linebundles of which the C_i $(i = 1, 2)$ are sections. Fix C_1. We can restrict the transition functions of L_2 to C_1 and get a linebundle on C_1. This correspond to linearly equivalent divisors on that curve, which actually correspond to intersections with linearly equivalent sections of L_2 i.e. divisors linearly equivalent to C_2. In particular the degree of those is an invariant. But what if we take another linearly equivalent divisor C_1' associated to L_1? We just reverse the role of L_1 and L_2. Fix a divisor C_2 of L_2 and intersect it divisors linearlty equivalent to C_1. The degrees are invariant.

Remark 7 If we fix a divisor C, we get a finer invariant than just the degree when we restrict a line-bundle. However, when C is allowed to move, this finer distinction does no longer make sense, only the crude degree. Thus we get a numerical value, when we try to intersect two line bundles.

Remark 8 The group of divisors up to linear equivalence may often be far too big, so one introduces a cruder invariant, referred to as numerical equivalence. Two divisors D_1, D_2 are numerically equivalent iff $D_1 D = D_2 D$ for all divisors D. For many elementary surfaces we will be encountering, numerical equivalence turns out to be the same as linear equivalence (just as on P^1 being of the same degree is not only necessary but also sufficient for linear equivalence).

Example 4 If X is a surface and BX is a blow-up of that surface at a point, we can relate the lattices $\mathbb{NS}(BX)$ and $\mathbb{NS}(X)$. In fact if $\pi : BX \to X$ denotes the blow-up we can naturally embed $\mathbb{NS}(X)$ in $\mathbb{NS}(BX)$ via the total transform π^8 we have discussed before. Intuitively we avoid the blown-up point P, and then what we do with divisors in $X \setminus P$ is the same what we do in $BX \setminus E$. However E is the 'new' divisor in BX, and we have in fact that $\mathbb{NS}(BX) = \mathbb{Z}E \oplus \mathbb{NS}(X)$ thus whenever $C \in \mathbb{NS}(X)$ we have $CE = 0$ as we can think of C as 'moving away' from a point. What about E^2. We have seen E as sitting as the unique zero-section of a line-bundle of degree -1. This means that $E^2 = -1$. As we have seen E is rigid, we cannot perturb it.

Normal bundles and self-intersections

It might be relevant to insert a little digression here. Given a curve C in the surface X, it gives rise to a line-bundle L which we can restrict to C. Recall that if C is defined locally by $z_i = 0$ then z_i/z_j give the transition functions for L and restricting them to C gives a line-bundle on C which is referred to as the normal bundle of C in X. But a linebundle over a curve C can be seen as a surface on its own merits. Admittedly it is not compact, but we can always compact it at 'infinity' which we will do later, but anyway this is of no real importance. As a curve in its normal linebundle it also has a normal linebundle in that surface. But this is a kind of tautological construction, clearly its normal line-bundle in its normal line-bundle will just be the origial line-bundle. However, if we ask the stronger question about tubular neighbourhoods, we get a different more interesting story. The tubular neighbourhood around the section in its normal line-bundle may be different from its tubular neighbourhood in the original surface. The line-bundle can be thought of as the first order approximation of it. However it will turn out that for an exceptional divisor they are the same! Thus we will see that given any smooth rational curve E with $E^2 = -1$ has a tubular neighbourhood which is isomorphic to a punctured ball. This allows us to reverse the blow up operation and allow us to blow down so called exceptional curves.

Functorial properties of the canonical divisor

We will study maps $f : Y \to X$ and see how we can recapture the canonical divisor K_Y of Y from the map f and the canonical divisor K_X of X.

We will be interested in three cases i) f is the inclusion of a hypersurface Y in X ii) f is the blow-up of X in one point and iii) f is a finite covering of X.

Adjunction formula

Let $z_1, \ldots z_n$ be local cooridnates for X and let Y be cut out by z_1. Now recall that $-K_X$ is given by the Jacobians $\det|\frac{\partial z_i}{\partial Z'_j}|$. If we restrict those transition functions to Y the first row of the determinant consists of zeroes, except the very first term which is given by $\frac{\partial z_1}{\partial z'_1}$. By l'Hospitals rule that term can be written as $\frac{z_1}{z'_1}$ which give the restriction of the transition functions of the divisor Y in X. The corresponding minor is simply the transition functions of $-K_Y$. Thus we can write

$$(-K_X)_{|Y} = Y_{|Y} + K_Y$$

which is refered to as the adjunction formula, and is more often written in the form

$$K_Y = (K_X + Y)_{|Y} = (K_X + Y)Y$$

where we have tried to make sense of the last product in the previous section.

Example 5 As a first application of this we can compute the canonical divisor of P^2. All the divisors of P^2 are of the form nH for some n and H denotes a hyperplane. Note that $H_{|H} = H^2$ denotes a point on H (two lines meet in a point) and we can actually write down $H^2 = 1$. We should have by adjunction that $(nH + H)_{|H} = (n+1)H^2 = K_H$. As $K_H = -2$ because H is a P^1 we obtain $n + 1 = -2$ thus $n = -3$. Hence the canonical divisor of P^2 is given by $-3H$.

Example 6 If we do the same thing on $P^1 \times P^1$ we make the 'Ansatz' $K = mF_1 + nF_2$ and use that both F_1 and F_2 are P^1. Hence we have both $((m+1)F_1 + nF_2)F_1 = -2$ and $(mF_1 + (n+1)F_2)F_2 = -2$ which translates into $m = n = -2$.

There is also another way of getting to the same result recalling that $P^1 \times P^1$ can be seen as a quadric Q in P^3. The canonical divisor in P^3 can be seen as $-4H$ (Any divisor is of the form nH, hardly surprising by now, and use the adjunctionformula on H which is a P^2). Now use adjunction on $Q = 2H$ and we get $K_Q = (-2H)_{|Q}$. Now $H_{|Q} = F_1 + F_2$ (Choose H to be tangent to Q!) and we are done again.

Canonical divisors under finite coverings

Given a finite covering $f : Y \to X$ and let z_i be local cooordinates on X, what about $\pi^*(z_i)$? Obviously if we consider them on $\pi^{-1}(U)$ where U constitute open coordinate patches on X they cannot be local coordinates because they will not separate points (unless the finite covering is trivial!). But obviously we can try and rectify this by refining the $\pi^{-1}(U)$. If this works the covering is said to be unramified, and this means exactly that the number of points in the fibers stay constant. We have a so called topological covering, and we have disjoint open sets around each point in the fiber, such that $\pi : Y \to X$ restricted to each of those is $1:1$. In the un-ramified case we clearly have

$$K_Y = \pi^* K_X$$

in particular if the degree of the covering is n and X is a curve then $\deg(K_y) = n\deg(K_X)$

However in general the map may not be unramified, and then we can write

$$K_Y = \pi^* K_X + R$$

where R is the ramification divisor, which can be tautologically defined as $K_Y - \pi^* K_X$. Although claiming that R can be chosen effective adds some meat to the assertion.

In order to compute R we will restrict ourselves to the case of curves. Then ramification comes about by having $\pi^*(z) = u^n$ where z is a local variable on X and u a local variable around a point P' in the fiber above a critical image point P in X. The point is that there are only a finite number of such fibers whose cardinality is not the maximal. Topologically it means that around the point P the fibers contain n distinct points that come together at $P' \in \pi^{-1}(P)$. Now we do a formal calculation $dz = nu^{n-1}du$. How should we interpret it? If we have a section of the canonical linebundle of K_Y it seems to pick up an extra zero of multiplicity $n-1$ at the point P' with ramification index n. This would lead us to suggest that $R = \sum_P (n_p - 1)P$ for all the points $P \in Y$ (note that for all but a finite number of points P we have $n_P = 1$ so the sum is actually only finite). Let us go bravely ahead. We would get that $\deg(K_Y) = n\deg(K_x) + \sum_P (n_p - 1)$ which we would like to compare with the easy formula for eulernumbers $e(Y) = ne(X) - \sum_P (n_p - 1)$. In particular we note that if $\deg(K_X) = -e(X)$ for some curve X that would hold for all finite coverings of X. Now it is easy to inspect that it holds for $X = P^1$ as the dgree of the canonical divisor is -2 while the eulernumber is the same as that of the sphere, thus equal to 2. Now every curve with a rational function f on it gives a map $f : Y \to P^1$ and we are done (modulo the fact that every curve does have rational functions which will be a consequence of Riemann-Roch) and we can state the fundamental

Theorem:('Gauss-Bonnet') If X is a curve than $\deg(K_x) = -e(X)$

Canonical divisors under blowing up

Now we will restrict ourselves to a blowup $\pi : Y \to X$ a surface. This is almost a trivial map, except that it is dramatically ramified above the blown-up point. It is also clear that anything new will happen as regards to the two new open sets introduced above the small ball around the blown up point. Recall the coordinate changes

$$\begin{array}{ll} u = u'v' & u' = 1/v \\ v = 1/u' & v' = uv \end{array}$$

We can write down the Jacobian

$$\det \left| \begin{array}{cc} \frac{\partial u}{\partial u'} & \frac{\partial u}{\partial v'} \\ \frac{\partial v}{\partial u'} & \frac{\partial v}{\partial v'} \end{array} \right| = \det \left| \begin{array}{cc} v' & u' \\ -\frac{1}{(u')^2} & 0 \end{array} \right| = \frac{1}{u'}$$

Those the transition fuction for the canonical divisor along E is given by u'. Now E is defined by $u = 0, v' = 0$ and hence the linebundle associated to E is given by $u/v' = u'$. Hence we have $K = E$ locally, or more generally we get

$$K_{BX} = K_X + E$$

which is a fundamental formula.

Calculating genera

We have seen that through the knowledge of the canonical divisor we can get hold of purely topological information, such as the eulernumber. The euler-number of a smooth orientable real surface, and hence of any smooth complez curve, is always even and can be written as $2 - 2g$ where g is referred to as the genus of the curve, and can be interpreted as the number of 'holes' of X (although for complicated curves, say like those climbing devices in playgrounds, it is not so clear how to identify separate 'holes' and thus to count them). As $e(X) = h^0 - h^1 + h^2$ where h^i is the i-th Betti-number i.e. $h^i = \dim(H^i)$ and $h^0 = h^1 = 1$ we see that $\dim(H^1(X)) = 2g$. As we will see later, we can split up $H^1(X)$ in two vectorspaces of the same dimension g of which one will be the space of complex 1-forms which is nothing but the space of global sections of the canonical divisor, which is denoted by $H^0(K)$. This is in fact the most elementary manifestation of the so called Hodge-decomposition.

Examples of calculating genera of smooth curves

Example 7 Let C be a smooth plane curve of degree n. By the adjunction formula the degree $2g-2$ of its canonical class is given by $((n-3)H)(nH) = n(n-3)$ from which we get $2g - 2 = n(n-3)$ or equivalently $g = \frac{(n-1)(n-2)}{2}$. This complies with what we have computed earlier. In particular we note

If $n = 1, 2$ we have $g = 0$ i.e. a line and a conic are isomorphic with P^1, but this we knew before.

If $n = 3$ then $g = 1$ and the curve is a torus. Such curves are called ellipic curves by a historical accident (it would be more logical to call P^1 elliptic, elliptic curves parabolic, and all the curves with genus $g > 1$ hyperbolic.) As noted those curves provide enough material for a year-long course meeting five times a week, four hours a day (with a two week recession around Christmas to make it a 1000 hour course).

If $n = 4$ then $g = 3$ and the curve has three holes.

Example 8 The case of a curve of bidegree (n, m) in $P^1 \times P^1$ will be left to the reader as an exercise.

Example 9 If we consider the complete intersection C of two hypersurfaces of degree n and m in P^3 we note that as the canonical divisor of P^3 is $-4H$ we get by a repeated application of the adjunction formula that

$$2g - 2 = (n + m - 4)nm$$

First use the adjunction formula on one of the surfaces, say that of degree n. The canonical divisor is the restriction of $(n-4)H$ to $X = nH$. The curve C sits in X as the restriction of mH to X. Now use the adjunction formula on X to conclude that the canonical divisor of C is given by the restriction of C of the divisor $(n+m-4)H_{|X}$. This gives $((n+m-4)H)(nH)(mH) = (n+m-4)nmH^3$ and as $H^3 = 1$ we are done.

In particular the complete intersection of two quadrics in P^3 is an elliptic curve.

Curves with singularities

Let C be a curve with an m-tuple point on a surface X. Blow it up and assume that the proper transform is smooth. What is its genera?

This now follows from a straightforward computation. Note that if C is a curvewith an m-tuple point to be blown up, its total transform contains the exceptional divisor E with muliplicity m. Thus the proper transform is given by $C - mE$.

We now use adjunction and get

$$2g-2 = (K_X+E+C-mE)(C-mE) = (K_X+C)C+m(m-1)E^2 = 2g_f-2-m(m-1)$$

as $CE = 0$ and $E^2 = -1$ and where g_f denotes the genus it would have had, had it already been smooth on X. In fact we can define g_f by $2g_f-2 = (K+C)C$ as a kind of formal genus, and the genus smooth perturbations of the curve would have.

This process is clearly inductive. As we have for an irreducible curve that $g \geq 0$ and each singularity reduces it strictly, the process cannot continue indefinitely, so some proper transform has to be smooth. This makes thus sense of our inductive definition of a_k.

Example 10 If $m = 2$ the genus is reduced by one. Thus nodes and cusps are resolved by one blow up and the formal genera is reduced by one. Hence a cubic with a node or a cusp becomes a P^1 after one blow up.

If we have an infinitely close double point, such as a tacnode requiring two blow ups, the genus is reduced by two. Thus a cubic with a tacnode (a_3) will get genus -1. This is impossible for an irreducible curve, and indeed no irreducible cubic can have a tacnode, but a reducible can. In fact a conic and a tangent line make up a cubic with a tacnode. When reoslved we get two disjoint P^1 which correspond to eulernumber $2 + 2 = 4$ which indeed corresponds to $g = -1$.