Synopsis for Tuesday, November 6 and Thursday, November 8

Linear Systems of Conics

As we have already noted, the linear space of global sections $\Gamma(2H)$ of the divisor $2H$ on P^2 consists of all the conics. This is a linear space spanned by all the quadric monomials in three variables. They give a map

$$\Theta : P^2 \to P^5$$

given by $(x_0, x_1, x_2) \mapsto (x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_2x_1)$.

If we take a hyperplane H in P^5 given by

$$A_{00}X_{00} + A_{11}X_{11} + A_{22}X_{22} + A_{01}X_{01} + A_{02}X_{02} + A_{12}X_{12} = 0$$

and cut it with the image V of Θ a so called hyperplane section, we get a curve $H \cap V$ on V. This curve is clearly the image of all the points $x \in P^2$ such that $\Theta(x) \in H$ or in other words

$$(x_0, x_1, x_2) : A_{00}x_0^2 + A_{11}x_1^2 + A_{22}x_2^2 + A_{01}x_0x_1 + A_{02}x_0x_2 + A_{12}x_1x_2 = 0$$

From this we see that all possible conics appear as hyperplane sections with the image.

Now this construction works for any sub linear space of what is referred to as the complete linear system. If the linear system is not complete, the hyperplane sections will not cut out all the curves linear equivalent to a given.

Digression on duality

There is also a co-ordinate-free way of presenting a map associated to a linear system. Recall that every vector space V has a dual V^* of linear forms. This has a direct geometric analog. To every projective space P^n we can associate its dual P^{n*} of hyperplanes. To every set of homogenous co-ordinates $(x_0, x_1, x_2, \ldots x_n)$ we have a point. And to every $(\lambda_0, \lambda_1, \Lambda_2 \ldots \lambda_n)$ we have the hyperplane $\sum_i \lambda_i x_i = 0$.

Now given a linear system L on a variety X which we can think of as a vector-space. Then to every point $p \in X$ we consider the hyperplane of elements in L which vanish at p. This gives a map $\Theta : X \to P(L^*)$.

To return to our initial example. Given a point $p = (p_0, p_1, p_2)$ let us look at all conics $\sum_{i \geq j} A_{ij}x_ix_j = 0$ vanishing at p. Those will be given by homogenous co-ordinates A_{ij} such that $\sum_{i \geq j} A_{ij}p_ip_j = 0$, i.e. by a hyperplane of conics given by the homogenous coefficients p_{ij}

Projections

Given a linear subsystem $M \subset L$ what is really the relationship between the maps $\Theta_M : X \to P(M)^*$ and $\Theta_L : X \to P(L)^*$?

A map $M \to L$ gives rise to a map $L^* \to M^*$. If the former is an inclusion the latter will be a projection. Thus we get a factorization

$$\Theta_M = \pi \Theta_L$$

where π is the projection $\pi : P(L)^* \to P(M)^*$.

Let us look at π more closely. A hyperplane H of L restricts to a hyperplane of M iff $M \not\subset H$. Those hyperplanes that contain M form a linear subspace K of L^*. More specifically if $l + 1 = \dim L$ and $m + 1 = \dim M$ the dimension of that subspace is $l - m$ and is clearly the dimension of the kernel of the projection $L^* \to M^*$. Clearly at points $p \in P(K)$ the projection is not well-defined. It is refered to as the center of the projection. The actual projection is given by considering linear subspaces of dimension $l - m + 1$ containing $P(K)$ and which are parametrized by $P(M)^*$. We may represent the latter by chosing a subspace of the same dimension in $P(L)^*$ and disjoint from the center. Each linear subspace containing the center as a hyperplane will then meet the given representation in exactly one point.

Example 1 : If $P(K)$ is a point P we chose a hyperplane H disjoint from P and consider all the lines through P. This gives a projection with fibers P^1

Example 2 : If $P(K)$ is a line L we chose a subspace M of codimension 2 disjoint from L and consider all planes containing L. Those planes will be paranetrized by M and the projection will have as fibers P^2.

Example 3 : Given the graph of any projection $P^n \to P^m$ the closure will be an algebraic subset of $P^n \times P^m$ given by a bilinear equation and define the blow-up of P^n along the center of projection. We are already familiar with the case $n = 2, m + 1$.

The Veronese Embedding

Let us look closer at the map $\Theta : P^2 \to P^5$ given by the complete linear system of conics. The image is isomorphic to P^2. The system separates points, meaning that there are no base-points, because given a point it is trivial to find a conic not passing through it. It also separates infinitely-close points, because if we look at conics passing through a given point, all possible tangent directions occur. This is enough to ensure that we have an embedding. We say that the family of conics making up the complete linear system of $2H$ is very ample. The image of Θ we will denote by V and refere to as the Veronese.

Now every linear subspace of P^5 of co-dimension two is given by the intersection of two hyperplanes. In general that intersection will consist of a finite number of points, the degree of the Veronese. As the hyperplane sections are given by conics, they will intersect in four points on the original P^2 and hence the degree of the Veronese is four.

On P^2 we have a 2-dimensional family of lines (i.e. the dual of P^2). Each of those lines are mapped into P^5 by considering the restriction of the linear system of conics to the line. Those will be given by binary quadrics and the image will be of degree two, hence a conic. Thus the Veronese has a 2-dimensional family of conics, each two of which intersect at one point. Any conic in a projective space determines a unique plane, namely the plane spanned by it. Thus to the Veronese we have a 2-dimensional family of planes. Through each point of such a plane we will have a secant to the variety pass, in fact any line in t hat plane will be a secant, as it will intersect the conic in two points. Conversely given any secant to the Veronese, it will give arise to two points on it, or equivalently two points on the original P^2. Through two such points there is a unique line, thus to every secant there is a unique conic intersecting it in two points. Thus every secant belongs to a unique plane. Thus the union of all the secants to a Veronese surface make up a 4-dimensional variety. This is onle less than you expect of a surface, because every choice of two points make up a variety of dimension 4 and add to that the 1 dimensions of the secants themselves.

Remark 1 One can show that given any algebaric surface X we can embed it in some projective space P^n by finding a very ample divisor. Then we can look at the variety spanned by its secants. That is 5-dimensional. Thus we can project successively down to P^5 and still keep in embedding. In the same way we can show that any n-dimensional variety can be embedded in $2n+1$. In particular any curve C can be represented in P^3.

Because the dimension of secants is so low, it means that through each point there will be a 1-dimensional family of secants. Now, what kind of 4-dimensional variety is spanned by all the secants of the Veronese? We have already encounterd it, namely the discriminant variety of degree 3 parametrising all singular conics in the space P^5 of conics, and whose singular locus is in fact he Veronese? Any secant of the singular locus will intersect the discriminant in 4 points counting multiplicities $(2+2=4)$. The degree being 3 it must be contained. And in fact all the conics $\lambda X^2 + \mu Y^2$ splits into two linear factors. Conversely any singular conic can be written in that way 9as a sum of squares) and thus lies on a secant to the singular locus.

Projections of the Veronese

If we chose a point p that lies on no secant of the Veronese, and project onto P^4 we get a non-singular surface of degree 4 isomorphic to P^2. This is an interesting surface and it is not so easy to find equations that define it, just as it is not so easy to characterize the points on the Veronese in P^5 in terms of equations.

However, if we chose a point p on a secant, it will lie on many secants, and the image will have a line of double points.

If we chose a point p on the Veronese and project, the image will drop its degree by one, and be isomorphic to P^2 blown up one point. This is beacsue the linear system we are looking at will consist of all the conics passing through one point p on P^2 and we need to blow it up. Thus we can think of the system as actually a complete linear system on $B_p(P^2)$ corresponding to $2H-E$ where E is the exceptional divisor corresponding to the blow-up of p. The elements of that linear system intersect at three three (residual) points. Formally $(2H-e)^2 = 4H^2 + E^2 = 4 - 1 = 3$.

Another way of looking at it, is to consider the Segre embedding of $P^2 \times P^1$ into P^5 given by

$$(x_0, x_1, x_2; y_0, y_1) \mapsto (x_0y_0, x_0y_1, x_1y_0, x_1y_1, x_2y_0, x_2y_1)$$

this is clearly an example of a map associated to a linear system on $P^2 \times P^1$ namely the system of bihomogenous polynomials of bidegree $(1, 1)$ also known as bilinear forms on $\mathbb{C}^3 \times \mathbb{C}^2$. The corresponding divisor on $P^2 \times P^1$ is given by $L = \pi_2^*(H) + \pi_1^*(h)$ where π_i are the obvious projections onto the factors P^i and H is a hyperplane on P^2 and h one on P^1.

The degree of this image is computed by L^3. How do we compute it? Write $L = L_2 + L1$ (with $L_i = \pi_i^*(H)$) and note that $L_1^2 = L_2^3 = 0$ (because points on P^1 are distinct, and three lines on P^2 do not generally meet simultaneously). Furhermore L_2^2 is the inverse image of a point on P^2 and thus $L_1L_2^2 = 1$. Now formally consider $(L_2 + L_1)^3$ the only surviving term is $3L_2^2L_1$ which comes out as 3.

Now our original definition of B_pP^2 was as the closure of the graph of the rojection $\pi : P^2 \setminus \{(0, 0, 1)\} \to P^1$ given by $(x_0, x_1, x_2) \mapsto (x_0, x_1)$ which was given by the bilinear polynomial $x_0y_1 - x_1y_0$. But that is just one member of the linear system above. Hence $B_p(P^2)$ is a hyperplane section of the Segre embedding of $P^2 \times P^1$ and as such it sits as a surface ofdegree 3 in P^4.

We can also look at linear systems given by two base-points P_1, P_2, those will map P^2 blown up the two base-points to a quadric in P^3. But this is not a $1-1$ map, because given any point P on the line L through P_1, P_2. It will correspond to a conic passing through P, P_1, P_2 and as they lie on a line L this line will be a component of any such conic. Thus the system of conics passing through P will be the same for any other point P' on L! Thus the line L will be blown down under the map. And this is what we have been led to expect as we have noted before that $P^1 \times P^1$ is given by P^2 by blowing up two points and blowing down the line through them.

The linear system of circles

As notes circles for a web, i.e. a 3-dimensional linear system, as they can be characterized by passing throughthe two so called circular points $(1, \pm i, 0)$ at infinity[1]. Thus a web given by two base-points. Let us restrict attention to the reals, and considering 'real' circles. The linear system will then give a map into $\mathbb{R}P^3$ as a quadric. This quadric will consist in the blow down of the line

[1] Sometimes referred to by the pet-names of Isaac and Jacob in elementary instructions.

at infinity (as the circular points are not real, we will not get any blow-ups of them on the real locus). Topologically this is a sphere S^2.

How can we write down this web explicitly? Any circle is of form $(x-a)^2 = (y-b)^2 = r^2$. Here we see the 3 parameters explicitly. But how to make it linear? We can look at a basis given by x^2+y^2, xz, yz, z^2 thus we see that we also need to include conics of the type $(ax+by)z$. Those corrrespond to infinite circles degenerating into lines. Some of them degenerate into the double line at infinity.

How do we find this quadric explicitly? We can set $X_0 = x^2+y^2, X_1 + xz, X_2 = yz, X_3 = z^2$ and consider all the 10 possible qudratic monomials in those variables. They will correspond to trinary quartics (i.e. polynomials of degree 4 in x, y, z). They make up a linear space of dimension 15. (An easy combinatorial count). But those quartic pass not only through the two base points, but doubly through them, which means that at each point we will have three conditions (A local inspection of $a + bx + cy + \ldots$). Together we get 6 conditions and hence a 9 dimensional family. Hence there must be a linear condition among the monomials, which is the same thing as a quadric.

Example 4 In our example this is easy to do (write down the monomials X_iX_j strike out those who contain unique monomials in x, y, z and repeat the process, in the end we need only to deal with X_0X_3, X_1^2, X_2^2) in fact $X_1^2 + X_2^2 - X_0X_3$ will do. This can be rewritten as

$$X_1^2 + X_2^2 + (\frac{X_0 - X_3}{2})^2 = (\frac{X_0 + X_3}{2})^2$$

which gives the equation of a sphere in $\mathbb{R}P^3$ or in fact in $\mathbb{R}^3$ if we choose $X_0 + X_3 = 0$ as a line at infinity.

Our explicit parametrization will be

$$(x, y, z) \mapsto (\frac{2xz}{x^2+y^2+z^2}, \frac{2yz}{x^2+y^2+z^2}, \frac{x^2+y^2-z^2}{x^2+y^2+z^2})(= (u, v, w))$$

whose image will satisfy $u^2 + v^2 + w^2 = 1$. Furthermore if we set $z = 0$ to be the line at infinity we get a parametrization of $S^2 \setminus P$ by $\mathbb{R}^2$ given by

$$(x, y) \mapsto (\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1})(= (u, v, w))$$

which will blow down the line $z = 0$ to the point P given by $(0, 0, 1)$. This is a parametrization with which you should be familiar from calculus of several varaibles[2]

Now a hyperplane section $Au + Bv + Cw = D$ will correspond to the conic

$$(x + \frac{A}{C-D})^2 + (y + \frac{B}{C-D})^2 = \frac{A^2+B^2+C^2-D^2}{(C-D)^2}$$

[2] At least if you have had me for a teacher

when $C \neq D$ and otherwise to

$$Ax + By = C$$

which correspond to planes passing through the northpole $(0, 0, 1)$.

From the top equation we see that we get non-empty intersections iff the distance of the plane to the origin is at most one (as expected).

Let us now embed the parametrizing plane into $\mathbb{R}^3$ by $(x, y) \mapsto (x, y, 0)$. It now follows that the determinant

$$\det \begin{pmatrix} x & y & 0 \\ \frac{2x}{x^2+y^2+1} & \frac{2y}{x^2+y^2+1} & \frac{x^2+y^2-1}{x^2+y^2+1} \\ 0 & 0 & 1 \end{pmatrix}$$

vanishes.

This means that the parametrization is given by taking a point p on the equator plane and joining it with the North Pole $P = (0, 0, 1)$ and looking at the residual intersection.

As the images of circles will be circles (as the intersection of a plane with a sphere is a circle (if anything), it means that the projection of circles on the sphere will be circles on the plane (except those that pass through the North Pole and will be projected onto lines). This happy state of affairs is due to a judicious choice of co-ordinates. One should not confuse the curve given by a hyperplane section with its pullback onto the parametrizing surface, a mistake easy to make if both are planes.

Nets of Conics

A net of conics is a 2-dimensional linear system of conics. In otherwords we are looking at conics $\lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2$. Let us denote it by N. Inside N there is the discriminant curve - the intersection with the discriminant locus. It is a cubic curve, in general non-singular. The geometry of the net is in many ways encoded in the cubic curve, which has a lot of structure.

Recall that we have a map $\Phi : P^2 \to N^*$ which to each point p associates the sub-pencil of conics passing through p. This is a line in N and hence a point in N^*. For this map to be defined everywhere, we need that the net has no base points.

Now this pencil of curves has four base points, each of which define the same pencil. Hence they are the points of the fibers of Φ which is hence $4 : 1$. We say that a point $p^* \in N^*$ is a branch-point if the points of the corresponiding fiber $\Phi^*(p^*)$ are not distinct. This means that the corresponding pencil is a line tangent to the discriminant cubic, thus it is equal to its dual

Digression on dual curves and polars

Let $C \subset P^2$ be a plane curve. For each non-singular point $p \in C$ we can associate the tangent $T_p \in P^{2*}$. Taking the closure of the set we get a plane curve C^* inside the dual projective space and referred to as the dual curve. The process can be continued and the double dual will simply be the curve itself, as can be see as the intersection of nearby tangents is close to the tangency points.

The degree of the dual curve is called the order of the curve and is the number of tangents through a point. The tangent of a curve given by a homogenous polynomial C at a point p is given by

$$\frac{\partial C}{\partial x}_{|p} x + \frac{\partial C}{\partial y}_{|p} y + \frac{\partial C}{\partial z}_{|p} z = 0$$

Thus the points $p \in C$ such that their tangents pass through a fixed point $(\zeta_0, \zeta_1, \zeta_2)$ lie on the curve

$$\frac{\partial C}{\partial x}\zeta_0 + \frac{\partial C}{\partial y}\zeta_1 + \frac{\partial C}{\partial z}\zeta_2 = 0$$

which is called the polar to the curve C at the point $(\zeta_0, \zeta_1, \zeta_2)$, and the corresponding tangency points are given by its intersection with C itself.

As it is clear that if $\deg C = n$ the polar has degree $n-1$ we conclude that the order of a plane curve of degree n is $n(n-1)$. Note also that if the point $p = (\zeta_0, \zeta_1, \zeta_2)$ lies on C then the polar is tangent to C at p.

Example 5 If $n = 2$ we see that the polars of a conic are lines, and through each point there are hence two tangent lines (as the order is two as well). Those two tangent lines coincide when the point is on the conic itself, and forms the polar. A conic is self-dual.

In general the order of a curve is bigger than the degree of the curve and how come that the order of the dual curve is not even bigger belying the fact of duality. The point is that the computation of order only works when the curve is non-singular. If there are singularities there will be 'false' tangents which will have to be discarded. In general taking the dual will create singularities. Obviously a bitangent will give rise to a node and maybe less immediate, a flexed tangent will give rise to a cusp.

Example 6 As we will see later, a non-singular cubic has nine flexed tangents and its dual will be a sextic with nine cusps

Resumption

The branch curve on N^* will hence be a plane sextic with nine cusps. The cusps will occur where the fiber contains a point with multiplicity 3. Those corresponds to pencils which are flexed to the discriminant cubic in N.

Now above the sextic branchcurve on N^* we will have multiple points of the fibers. Those will be referred to as the ramification points. A degenerate fiber corresponds to a tangent line to the discriminant cubic and a pencil with just two degenerate fibers. The degenerate fiber corresponding to the tangency point on the cubic, will be the one whose singular point coincides with one of the basepoints - the ramification points. Those points clearly corresponds to the singular points of the singular conics parametrized by the discriminant cubic. They will trace out a curve in P^2 which will be isomorphic to the discriminant, and be referred to as the Jacobian of the net. At a point of the Jacobian all the conics of the net will have a common tangent. When the point will be of order 3 all the conics will be flexed to each other.

Remark 2 If we plot conics of the net, the Jacobian will emerge as a shaded curve.

What is the degree of the Jacobian? We can compute it explicitly. A point p is a singular point of a conic in the net, iff we can find a non-trivial set $(\zeta_0, \zeta_1, \zeta_2) \in N^*$ of parameters such that the system of equations

$$\begin{aligned}
\lambda_0 \tfrac{\partial Q_0}{\partial x} + \lambda_1 \tfrac{\partial Q_1}{\partial x} + \lambda_2 \tfrac{\partial Q_2}{\partial x} &= 0 \\
\lambda_0 \tfrac{\partial Q_0}{\partial y} + \lambda_1 \tfrac{\partial Q_1}{\partial y} + \lambda_2 \tfrac{\partial Q_2}{\partial y} &= 0 \\
\lambda_0 \tfrac{\partial Q_0}{\partial z} + \lambda_1 \tfrac{\partial Q_1}{\partial z} + \lambda_2 \tfrac{\partial Q_2}{\partial z} &= 0
\end{aligned}$$

For this to have a solution we need that

$$\det \begin{pmatrix} \frac{\partial Q_0}{\partial x} & \frac{\partial Q_0}{\partial x} & \frac{\partial Q_0}{\partial x} \\ \frac{\partial Q_0}{\partial y} & \frac{\partial Q_0}{\partial y} & \frac{\partial Q_0}{\partial y} \\ \frac{\partial Q_0}{\partial z} & \frac{\partial Q_0}{\partial z} & \frac{\partial Q_0}{\partial z} \end{pmatrix} = 0$$

which gives a cubic.

We also have another interresting curve associated to the net. Each point in the discriminant gives rise to two lines in P^2 hence two points in P^{2*}. Those form a curve, what is the degree of that curve? A line in P^{2*} corresponds to a pencil of lines in P^2, which is just the same thing as the lines going through a point p. We are asking how many singular fibers in the net go through that point. All such singular fibers belong to the subpencil of the net going through the point and we know that there are three such. Hence this curve, which is a double cover of the discriminant cubic, is also a cubic.

Jacobian Nets

Let C be a cubic curve, then we can consider the net spanned by the partials $\frac{\partial C}{\partial x}, \frac{\partial C}{\partial y}, \frac{\partial C}{\partial z}$ which is nothing but the polars of C. Thus we have identified N with P^2 itself, as any point $p \in P^2$ we associate a conic, namely its polar with respect to C.

What is its discriminant locus and what is its Jacobian locus? The latter is simpler to describe, and we have already encountered it above. We only have to replace Q_i in the expression with the partial $\frac{\partial C}{\partial X}$ etcgetting that $\frac{\partial Q_0}{\partial x} = \frac{\partial^2 C}{\partial x^2}$

etc and obtain the Hessian given by the determinant

$$\begin{matrix} \frac{\partial^2 C}{\partial x^2} & \frac{\partial^2 C}{\partial x \partial y} & \frac{\partial^2 C}{\partial x \partial z} \\ \\ \frac{\partial^2 C}{\partial x \partial y} & \frac{\partial^2 C}{\partial y^2} & \frac{\partial^2 C}{\partial y \partial z} \\ \\ \frac{\partial^2 C}{\partial x \partial z} & \frac{\partial^2 C}{\partial y \partial z} & \frac{\partial^2 C}{\partial z^2} \end{matrix}$$

For the discriminant, it corresponds to points (x, y, z) such that the polar is singular. Let us look at the partials more carefully. The point is that the coefficients of the cubic C can be expressed in terms of third derivaties. We have e.g. $a_{111}x^3$ and we recapture it as $\frac{1}{6}\frac{\partial^3 C}{\partial x^3}$ while $a_{123}xyz$ is recaptured by $\frac{\partial^3 C}{\partial x \partial y \partial z}$ We can then write

$$\begin{aligned} \frac{\partial C}{\partial x} &= \frac{1}{2}\frac{\partial^3 C}{\partial x^3}x^2 + \frac{\partial^3 C}{\partial x^2 \partial y}xy + \frac{\partial^3 C}{\partial x^2 \partial z}xz + \dots \\ \frac{\partial C}{\partial y} &= \frac{1}{2}\frac{\partial^3 C}{\partial x^2 \partial y}x^2 + \frac{\partial^3 C}{\partial y^2 \partial x}xy + \frac{\partial^3 C}{\partial x \partial y \partial z}xz + \dots \\ \frac{\partial C}{\partial z} &= \frac{1}{2}\frac{\partial^3 C}{\partial x^2 \partial z}x^2 + \frac{\partial^3 C}{\partial x \partial y \partial z}yxxy + \frac{\partial^3 C}{\partial z^2 \partial x}xz + \dots \end{aligned}$$

We are then setting up a symmetric matrix of the form

$$\begin{pmatrix} \frac{\partial^3 C}{\partial x^3} + \frac{\partial^3 C}{\partial x^2 \partial y}y + \frac{\partial^3 C}{\partial x^2 \partial z}z & \frac{\partial^3 C}{\partial x^2 \partial y}x + \frac{\partial^3 C}{\partial y^2 \partial x}y + \frac{\partial^3 C}{\partial x \partial y \partial z}z & \dots \\ \frac{\partial^3 C}{\partial x^2 \partial y}x + \frac{\partial^3 C}{\partial y^2 \partial x}y + \frac{\partial^3 C}{\partial x \partial y \partial z}z & \frac{\partial^3 C}{\partial y^2 \partial x}x + \frac{\partial^3 C}{\partial y^3}y + \frac{\partial^3 C}{\partial y^2 \partial z}z & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

Now we can use Eulers formula

$$F = n \sum_i \frac{\partial F}{\partial x_i} x_i$$

for any homogenous form of degree n. It is easy to prove, as it follows from verifying it on monomials.

In partiuclar we find

$$\begin{aligned} \frac{\partial^3 C}{\partial x^3} + \frac{\partial^3 C}{\partial x^2 \partial y}y + \frac{\partial^3 C}{\partial x^2 \partial z}z &= 2\frac{\partial^2 C}{\partial x^2} \\ \frac{\partial^3 C}{\partial x^2 \partial y}x + \frac{\partial^3 C}{\partial y^2 \partial x}y + \frac{\partial^3 C}{\partial x \partial y \partial z}z &+ 2\frac{\partial^2 C}{\partial x \partial y} \\ \dots &= \dots \text{ etc} \end{aligned}$$

and the determinant that gives the cubic discriminant turns out to be the Hessian as well!

Thus we see that the Jacobian cubic and the Discriminant cubic coincide for Jacobian nets.

Remark 3 Given the Euler formula we can of course use it inductively. This gives us

$$F = n(n-1)\sum_{i,j} \frac{\partial^2 F}{\partial x_i \partial x_j} x_i x_j$$

$$F = n(n-1)(n-2)\sum_{i,j,k} \frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k} x_i x_j x_k$$

etc

Remark 4 Another application of the Euler formula is that a point lies on its polar iff it lies on the curve.

In fact if $(\zeta_0, \dots z_n)$ by $\sum_i \frac{\partial C}{\partial x_i}\zeta_i = nC(\zeta_0 \dots \zeta_n) = 0$ we see that lying in one implies lying in the other(and this is geometrically obvious in the case the point is chosen on the curve as as it then lies on its tangent).

In this case we can compare the tangents of the curve (hypersurface) and that of the polar. They are given respectively by

$$\left(\frac{\partial C}{\partial x_0}, \dots, \frac{\partial C}{\partial x_n}\right)$$

and

$$\left(\frac{\partial^2 C}{\partial x_0^2}\zeta_0 + \dots + \frac{\partial^2 C}{\partial x_0 \partial x_n}\zeta_n, \dots, \frac{\partial^2 C}{\partial x_0 \partial x_n}\zeta_0 + \dots + \frac{\partial^2 C}{\partial x_n^2}\zeta_n\right)$$

but the first clearly is equivalent to the second.

Remark 5 Recall that if C is non-singular the order of the curve C is given by $n(n-1)$ which gives the amount of ramifications. Thus $e(C) = 2n - n(n-1)$. But if we let the projection point lie on the curve, the degree goes down by one, and then the number of tangent lines (which are given by the intersection with the polar) should go down by two. Those two must be absorbed by the point itself, thus given a tangency.

Hessian and Flexes

A cubic which has tangent $x = 0$ at the point $p = (0, 0, 1)$ can be written in the form $xz^2 + ax^2z + bxyz + cy^2z + f_3(x, y)$ whree f_3 is a binary cubic. We can compute its polar at the point p this is simply its partial derivative with respect to z and gives $2xz + ax^2 + bxy + cy^2$. When is this quadratic singular, i.e. when does p lie on the Hessian as well. We simply compute its gradient which turns out to be

$$(2z + 2ax + by, bx + 2cy, 2x)$$

This has a non-trivial solution iff $c = 0$ in which case we have $x(2z + ax + by)$. In this case if we set $x = 0$ we get the coefficient of y^3 as the only surviving one, which means that the tangent is a flexed tangent.

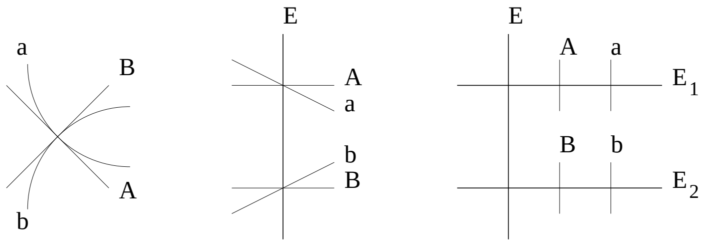
Thus we have established that the intersections of a cubic with its Hessian correspond to its flexes (points at which the tangent is flexed). As the Hessian of a cubic is also a cubic, we have established that a non-singular cubic has 9 flexes.

What happens if the cubic has singularities? Let us make a brute force local calculation. Set $C = f_2 z + f_3$ which has a double point at $(0, 0, 1)$. We easily compute the second partials of C and set up the determinant and collect terms in powers of z. We then get

$$(2\frac{\partial^2 f_2}{\partial x \partial y}\frac{\partial f_2}{\partial x}\frac{\partial f_2}{\partial y} - (\frac{\partial^2 f_2}{\partial y^2}(\frac{\partial f_2}{\partial x})^2 + \frac{\partial^2 f_2}{\partial x^2}(\frac{\partial f_2}{\partial y})^2))z + (2\frac{\partial^2 f_3}{\partial x \partial y}\frac{\partial f_2}{\partial x}\frac{\partial f_2}{\partial y} - (\frac{\partial^2 f_3}{\partial y^2}(\frac{\partial f_2}{\partial x})^2 + \frac{\partial^2 f_3}{\partial x^2}(\frac{\partial f_2}{\partial y})^2))$$

From which we conclude that the Hessian also passes through the point $(0, 0, 1)$ and has a double-point (at least) there.

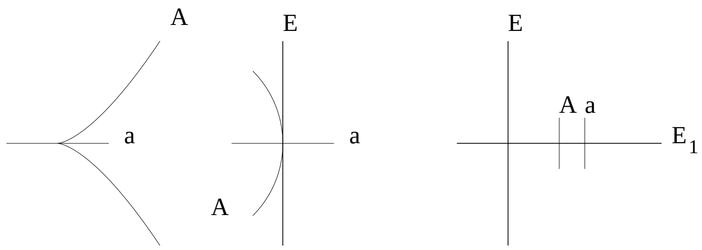
Assume that we have a node. Then we can choose co-ordinates x, y such that $f_2 = xy$. The above then simplifies to $2xyz + \dots$ Thus the Hessian has a node as well and with the corresponding branches tangent (at least). We conclude that we need to blow up once, to desingularize at the curves and at two points to separate the branches.



Formally $(3H - 2E - E_1 - E_2)^2 = 9H^2 + 4E^2 + E_1^2 + E_2^2 = 9 - 6$. Thus the intersection with Hessian absorbes (at least) six points at the node. That it does not absorb more requires a more careful analysis.

If we instead have a cusp we can assume that $f_2 = x^2$ and we then see that the Hessian has a triple point. We can now choose analytic co-ordinates such that $f_3 = y^3$ and then the Hessian is given by $x^2 y$.

If we make a blow-up $x = uv, y = v$ where $v = 0$ gives the exceptional divisor E we get for the proper transform u^2+v for the cusp and u^2 for the Hessian. The latter has still a double point after the blow up, and the two curves still meet. They are separated after a second blow-up and we do the formal calculation $(3H - 2E - E_1)(3H - 3E - 2E_1) = 9H^2 + 6E^2 + 2E_1^2 = 9 - 6 - 2 = 1$ Thus eight points are absorbed in the intersection and only one remains.



Thus the cuspidal cubic has only one flex. Note that the flex of x^2z+y^3 lies on the line at infinity $z=0$ and has that line as its flexed tangent.

Example 7 The cubic $(x+y+z)^3-27xyz$ has a node at $(1,1,1)$ which is easily checked. The three lines $x=0, y=0, z=0$ are immediately seen to be flexed, and the three flexed points lie on the line $x+y+z=0$

Degenerate nets

Nets can degenerate either by acquiring base points or double lines, or both. The first corresponds to being tangent to the discriminant locus, the second of intersecting the singular locus of the latter.

Every base point imposes a singularity on the discriminant cubic as does every double line (supersingular conic). The base-points reduce the number of fibers in the mapping $\Phi : P^2 \to N^*$.

A Jacobian net has a base-point iff the cubic C is singular.

The Cremona Transformation

The simplest example of a net of conics is given by one with three base points. It gives a generically $1:1$ map to P^2. If we identify the source with the image we get an interesting bi-rational transformation known as the Cremona transformation.

Duality and Plücker formulas