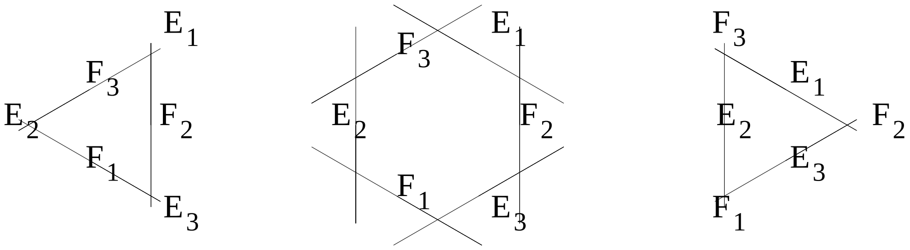
Synopsis for Tuesday, November 13

Cremona transformations

Let N be a net with 3 base-points. It gives a generically one-to-one map of P^2 into N^* blowing up the base-points and blowing-down the lines joining any two base-points.



A classical example is the net spanned by xy, yz, xz which blows up the points $(1,0,0),(0,1,0),(0,0,1)$ and blows down the co-ordinate lines given by $xyz = 0$. One suggestive way of writing it is $(x,y,z) \mapsto (\frac{1}{x}, \frac{1}{y}, \frac{1}{z})$ which exhibits its involutionary nature, but it is more convenient to write it as $(x,y,z) \mapsto (yz, xz, xy)$. Thus if $F(x,y,z) = 0$ is a curve of degree n we will consider the total transform $F(yz,xz,xy)$ of degree $2n$. However if F passes through a base-point the total transform will contain a suitable multiple of the corresponding exceptional divisor, in fact given by the multiplicity of the singularity of F at the point. (Multiplicity 0 means that it does not pass through the point, multiplicity 1 that it passes simply through the point, i.e. is non-singular there).

Thus if F passes through with multiplicity m_i at base point p_i the proper transform of F will be $F - m_1E_1 - m_2E_2 - m_3E_3$. In particular the lines $F_i (= H)$ through two of the base-points p_j, p_k will have proper transforms $2H - E_j - E_k$. If the degree of F is m (i.e. $F = mH$) then

$$(F - m_1E_1 - m_2E_2 - m_3E_3)(H - E_1 - E_2) = m - m_1 - m_2$$

and similar for the two others. Furthermore the curves $2H - E_1 - E_2 - E_3$ will play the role of the lines in the image.

We can then verify the identity

$$\begin{gathered}(2m - (m_1 + m_2 + m_3))(2H - E_1 - E_2 - E_3) - (m - (m_1 + m_2))(H - E_1 - E_2) \\ -(m - (m_1 + m_3))(H - E_1 - E_3) - (m - (m_3 + m_2))(H - E_3 - E_2) \\ = mH - m_1E_1 - m_2E_2 - m_3E_3\end{gathered}$$

and thus we will get the scheme

$$(m; m_1, m_2, m_3) \mapsto (2m - m_1 - m_2 - m_3, m - m_1 - m_3, m - m_2 - m_3, m - m_2 - m_1)$$

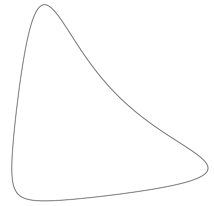
which is easy to understand as the proper transform will intersect the 'new' lines $2H - E_1 - E_2$ in $2m - m_1 - m_2 - m_3$ points and have singularities at the 'new' pooints F_3, F_2, F_1 of multiplicities $m - (m_1 + m_2)$ etc.

Example 1 Lines avoiding all the base-points will be mapped to conics passing through the new base points.

Lines passing through one base-point will be mapped to lines passing through the 'opposite new' basepoint.

Conics avoiding the base-points will be mapped onto quartics passing doubly through the 'new' base-points, conics passing through one base point will be mapped on singular cubics passing through the 'opposite new' base-point. If intersecting the opposite line transversally in two points, we will get a node, otherwise a cusp.

Example 2 We can obviously restrict the Cremona transformation to the finite part of the plane. We then get $(\frac{x}{z}, \frac{y}{z}) \mapsto (\frac{z}{x}, \frac{z}{y})$. The picture below shows what happens to the circle $(x - \frac{3}{2})^2 + (y - \frac{3}{2})^2 = 1$.



Those will be the real points of a quartic with three nodes with complex conjugate nodal directions and hence be invisible, except for the singularities which will show up as isolated points. Note the bitangent on the real oval. For this to be possible there must be two real flexes between the two tangency points.

An explicit equation is given by

$$(2z - 3x)^2 y^2 + (2z - 3y)^2 x^2 - 4x^2 y^2 = 0$$

and it will have its node in the three points $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.

Now for a homogenous polynomial P of even degree, points where it is positive on the real projective space are well defined as $P(\lambda x, \lambda y, \lambda z) = \lambda^{2n} P(x, y, z)$. The polynomial above will have isolated zeroes at the isolated nodes, and they all belong to the same component after we have removed the sero set given by the oval above. By the form it is positive on the co-ordinate axis thus positive everywhere outside the oval. Thus if we perturb it a little bit with a negative number, the isolated nodes will disolve into small ovals, and we will have the real points of a non-singular quartic, which will form four disjoint components. The local equations will be given by $4y^2 + 4x^2 + \ldots, 4z^2 - 12xz + 14x^2 + \ldots, 4z^2 - 12yz + 14y^2 + \ldots$ respectively. In the first case perturbation will lead to small almost circles, in the second case to almost ellipses.

Dual Curves and Plücker formulas

Let $C \subset P^2$ be a non-singular plane curve and associate to each p its tangent line $T_p \in P^{2*}$ this defines a plane curve $C^* \subset P^{2*}$. The degree of the dual curve is called the order of the curve C and gives how many tangents to the curve C passes through a point p. As we have already noted the order is easily computed as the number of points of intersection of the curve with its polar at the point p and thus is given by $n(n-1)$. In particular the dual of a conic is a conic in the dual space. The dual of the dual of the conic is the original conic, which can easily be seen by observing that the intersection of nearby tangents is a point close to the conic. This argument also indicates that this is true in general. But how could it be true? If $n > 2$ then the degre of the dual is strictly bigger than that of the curve itself? The point is that in general the dual is singular. Two types of singularities occur. If there is a flex point the tangent lines trace out a cusp in the dual space, and if there is a bitangent two different points will have the same tangent so the dual curve will have a node.

Thus we need to extend the notion of dual curves also to possibly singular curves. This is easily done, we restrict the tangents T_p to non-singular points and then we take the closure. This is a procedure that is very similar to what we did when we defined proper transforms. In fact if we have a node say, any line through it is in a formal sense a tangent, as it intersects the curve in two (coinciding) points, but there are two lines that stick out, namely the two tangents to the two branches, those are limiting lines of the family of tangents to the non-singular points. They also have the property that they intersect the curve in three (coinciding) points. In the case of a cusp, we have similarly a distinguished tangent, namely the cuspidal tangent.

But if we allow nodes and cusps the order of the curve has to be modified. The polar will always pass through the singular points and we need to compute their intersections.

Local calculations

Asssume that we have a node at $(0,0,1)$ then we can write the curve in the form

$$xyz^{n-2} + f_3(x,y)z^{n-3} + \dots$$

where $x = 0, y = 0$ give the tangents to the branches, the so called nodal tangents.

The polar at a point $p = (\lambda_0, \lambda_1, \lambda_2)$ will have the form

$$\lambda_0 yz^{n-2} + \lambda_1 xz^{n-2} + \dots$$

and if we avoid the point $(0,0,1)$ the polar is non-singular at the singularity and intersect it with multiplicity two (unless of course it lies on one of the nodal tangents). Thus there will be two-false tangents.

In the case of a cusp at $(0,0,1)$ we can write

$$x^2 z^{n-2} + f_3(x,y)z^{n-3} + \dots$$

and similarly get the polar

$$\lambda_0 2xz^{n-2} + \lambda_1 \frac{\partial f_3}{\partial y} z^{n-3} + \lambda_2 x^2 z^{n-3} + \dots$$

which unless $\lambda_0 = 0$ (i.e. the point p lies on the cuspidal tangent) will have intersection-multiplicity three.

Thus we get that the order N of a plane curve with δ nodes and κ cusps is given by

$$N = n(n-1) - 2\delta - 3\kappa$$

Duality

Now let C be a plane curve of degree n with δ nodes and κ cusps and b bitangents and f flexes. There will be two relations involving the four variables thus given two the other two can be determined. This means in practice that given n, δ, κ we can compute b, f.

There are two ideas.

a) duality

which means that

$$n = N(N-1) - 2b - 3f$$

b) the resolutions of C and its dual C^* has the same genus, because there is a $1-1$ correspondence between their smooth points. This leads to

$$n(n-3) - 2\delta - 2\kappa = N(N-3) - 2b - 2f$$

To actually work out the formulas is a bit messy, except that we get a nice formula for f by observing that

$$f = N(N-1) - N(N-3) - n + n(n-3) - 2\delta - 2\kappa = 2N + n^2 - 4n - 2\delta - 2\kappa$$

as $N = n(n-1) - 2\delta - 3\kappa$ this simplifies to

$$f = 3n(n-2) - 6\delta - 8\kappa$$

where we recognize $3n(n-2)$ to be the intersection with the Hessian, which gives the flexes in the non-singular case, but as we noted above, each node decreases by 6 and each cusp with 8.

Examples

Example 3 Let $n = 4, \delta = \kappa = 0$ in other words a smooth quartic. We then get

$$4 = 12 \times 11 - 2b - 3f$$
$$4 = 12 \times 9 - 2b - 2f$$

This is easily solved and we get $b = 28$ and $f = 24$

Example 4 Now consider a quartic with three nodes it will have order $12 - 2 \times 3 = 6$ and hence

$$4 = 6 \times 5 - 2b - 3f$$
$$4 - 2 \times 3 = 6 \times 3 - 2b - 2f$$

and we get $b = 8$ and $f = 6$

Example 5 A quartic with three cusps will have order $12 - 3 \times 3 = 3$ and thus its dual will be a cubic. We will get

$$4 = 3 \times 2 - 2b - 3f$$
$$4 - 2 \times 3 = -2b - 2f$$

which gives $f = 0$ and $b = 1$ (a nodal cubic cannot have any cusps)

Example 6 A quintic has genus 6 and can have 6 nodes but can it have 6 cusps? If so its dual has order $20 - 3 \times 6 = 2$ and this is absurd. Thus no? Can it have 5 cusps and 1 node? then the dual will have order 3 and we once again know all the duals of cubics, and a quintic does not appear among them. Or

$$5 = 3 \times 2 - 2b - 3f$$
$$10 - 2 \times 6 = -2b - 2f$$

gives the solution $f = -1$ which is absurd.

What about 4 cusps and 2 nodes. The dual will be a quartic with 1 cusp and 2 nodes and such a curve is easy to construct using a Cremona transformation.

Example 7 An irreducible curve of degree n can at most have $\frac{(n-1)(n-2)}{2}$ double points. It will then be rational. Such examples can easily be manufactured by taking a generic projection onto a plane of a complete linear system of binary forms of degree n although it is not so easy to exhibit explicit equations. We then get that the order of the curve is $n(n-1) - (n - 10(n-2) = 2(n-1)$ amd thus

$$n = 2(n-1)(2n-3) - 2b - 3f$$
$$-2 = 2(n-1)(2n-5) - 2b - 2f$$

with $f = 3(n-2)$ and $b = 2(n-2)(n-3)$