Synopsis for Tuesday, January 29

What everyone should know about Riemann surfaces

Topological classification

Given two compact real surfaces S_1 and S_2 we can define the operation of topological sum $S_1 \# S_2$, by removing from each surface a disc and glueing them along the two disjoint discs (or if you prefer joining the holes with a hose, i.e. a cylinder). It is trivial to see that $e(S_1 \# S_2) = e(S_1) + e(S_2) - 2$. In this way we make surfaces into a semi-group. The neutral element is given by S the sphere, while the torus T generates the sub semigroup of orientable surfaces. In particular if $T_g = T \# T \ldots \# T$ (g copies of T) we have that $e(T_g) = 2 - 2g$ and thus orientable surfaces are classified by their eulernumbers or equivalently there genera (number of holes). In particular the genus of T_g is g. Adding a torus to a surface is usually referred to as attaching a handle.

> **Remark 1** How many holes does one of those contraptions for climbing have that you see at Childrens playgrounds? The easiest thing is to compute the eulernumber and then get g.

In general we also have non-orientable surfaces, the real projective plane P being the prime example. Attaching a projective plane is the same thing as to removing a disc and then attaching a Moebius strip to its boundary (recall that the projective plane is simply the union of a disc and a Moebius strip joined at ttheir boundaries) which is the same thing as making a real blow up. We have that $P \# P$ is the Klein-bottle but $P \# P \# P$ we can reduce to $T \# P$ and thus we get a complete classification, where there can be at most two P summands, and where an empty summand refers to the sphere.

A Riemann surface is a real surface endowed with a complex structure. As such it will be automatically orientable and hence we can concentrate on the T_g.

> **Remark 2** If we have z and w two complex co-ordinates we can write $z = u(x,y) + iv(x,y)$ (where $z = u + iv, w = x + iy$ is the decomposition into real and imaginary parts). The Jacobian matrix of the real co-ordinate change will hence be given by
>
> $$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$
>
> Now z being a holomorphic function of w means that the matrix would be of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ which preserves angles, from which

we immediately get the Cauchy-Riemann equations. In particular the determinant is given by $a^2 + b^2 > 0$

We can thus state that Riemann surfaces are classified by their genus. In general (i.e. with the exception of the Riemann sphere) there will be many complex structures on a given surface. Those vary continuously by so called moduli, and we can even write down spaces $\mathcal{M}_g$ parametrizing curves of genus g, so called Modulispaces.

Rough classification of curves

Curves come in three types - rational, elliptic and the rest, depending on the degree of the canonical divisor being negative, zero or positive.

Remark 3 This trichtomy is a general theme in mathematics. We see it already in the classification of conics into ellipsis, parabolas and hyperbolas, refering to defects, equality and excess. The terminology is transfered to the classical theory of partial differential equations with the three very different kinds of elliptic, parabolic and hyperbolic equations. Also the more general case of classification of varieties hinges on the canonical divisor, as being negative, essentially trivial, or positive, although we are not able to provide as precise informations as in the 1-dimensional case.

Let us present the following table

genus	structure	moduli	automorphismgroup	vectorfields	1-forms
$g = 0$	$S^2 = \mathbb{C}P^1$	0	PGL(2,$\mathbb{C}$)	3-dim	none
$g = 1$	$\mathbb{C}/\Lambda$	1	E plus finite	1-dim	trivial
$g \geq 2$	U/Γ	$3g - 3$	finite	none	plenty

We should note that there are three simply connected 1-dimensional complex manifolds, namely the Riemann sphere $\mathbb{C}P^1$, the complex plane $\mathbb{C}$ and the unit disc U, thus the universal cover of any compact Riemann surface has to be one of those three types which complies beautifully with the other basis for distinction.

Remark 4 In the case of U one usually endows that with the hyperbolic metric of constant curvature -1 (which means that the angular sum of a triangle is less than π the defect being given by the area of the triangle). There are many interesting groups Γ which can operate on that space, each giving rise to fascinating tesselations. In the case of $\mathbb{C}$ we endow it with the flat matric, i.e. constant curvature equal to zero (the Euclidean plane), while in the case of the sphere we have the constant curvature 1 with angular sum being in excess of π the amount of excess being equal to the area of the triangle.

As to groups, the first one is well-known, it is the group of Moebius transformations, which operates triply transitive, meaning that any three distinct points can be mapped to any other three distinct points (usually normalized as $0, 1, \infty$. It is also initimately associated to the complex vector fields on the Riemann sphere.

On an elliptic curve we always have the automorphism given by translation, thus parametrized by the points of E and isomorphic to it as a group. In addition to that we have the involution $z \mapsto -z$. For two special elliptic curves, the finite part can be extended to $\mathbb{Z}_4$ and $\mathbb{Z}_6$ respectively.

Finally for curves of general type (meaning the rest) the automorphism group is always finite (and typically trivial), and its order can be shown to be bounded linearly in g (in fact $84(g-1)$ the so called Hurwitz bound).

Review of Linear Systems

Let us for simplicity reduce to the case of curves, although most of what is being said has general validity.

Recall that we have the notions of line-bundles and divisors. A divisor is simply a formal sum of points $\sum_P n_P P$ on a curve X[1]. And we say that the divisor is effective if $n_P \geq 0$.

> **Remark 5** In the case of curves we can easily talk about the degree of a divisor $\sum_P n_P P$ as simply $\sum_P n_P$. Every rational function ψ gives rise to a divisor (ψ) by simply counting its zeroes and poles with appropriate multiplicities. It is a fundamental fact that the degree of any divisor associated to a rational function is zero, this follows from a residue calculation of the intergal $\int \frac{\psi(z)}{\psi'(z)} dz$. Thus in particular linearly equivalent divisors have the same degree.

While a line-bundle is formally a locally trivial fibration of lines $\mathbb{C}$ over a curve X. We write $\pi : L \rightarrow X$. To describe a line-bundle we need to have an open covering U_i over which $\pi^{-1}(U_i) = U_i \times \mathbb{C}$ is trivial. To describe the twisting we are given transition functions $\theta_{ij} : U_i \cap U_j \rightarrow \mathbb{C}^*$ subjected to $\theta ij \theta ji = 1$ on $U_i \cap U_j$ and $\theta_{ij}\theta_{jk}\theta_{ki} = 1$ on $U_i \cap U_j \cap U_k$.

> **Remark 6** One may extend the notion of a line-bundle to that of a vector-bundle, by letting the transition functions become invertible matrices. If those can be diagonalized, we say that the vector-bundle splits into a direct sum of line-bundles.

The important thing is that a line-bundle is determined by the data given by the transition functions, and that data behaves well under natural transformations, such as pull-backs (in particular restrictions) while this is not the case of divisors.

[1]In general we need to replace points P by subvarities of codimension one, those are in general harder to come up with and get an overview of, this is why the situation for curves is so simple and elementary.

Now every divisor gives rise to a line-bundle and every line-bundle gives rise to divisors.

As to the latter, we need the notion of a section of a line-bundle. By that is meant a holomorphic map $s : X \to L$ such that $\pi s = id_X$, in other words for each fiber we pick an element, and this has to be done in a holomorphic way. Note that every line-bundle has a trivial section, namely the zero-section. More generally a holomorphic section is given by holomorphic functions s_i such that $s_i = \theta_{ij} s_j$. Note that unless the line-bundle is trivial, a holomorphic section is not a function on X. On a compact complex manifold the only holomorphic functions are the constants, but a line-bundle may have non-constant sections.

Now every holomorphic section s gives rise to an effective divisor (s) defined as the formal sum of its zeroes. Note that because $\theta_{ij} \neq 0$ a zero of s_i is automatically a zero of s_j and vice-versa, and the multiplicities are the same.

> **Remark 7** We can also talk about meromorphic sections, by considering meromorphic functions s_i instead of holomorphic. The associated divisors will no longer be effective in general.

The sections of a line-bundle L make up a linear vector space which we will denote by either $\Gamma(L)$ or $H^0(L)$ the associated divisors will all be linearly equivalent, because if you have two sections s, t then the quotient $\psi = s/t$ will become a meromorphic function on X and thus for the associated divisors D_s and D_t will we have $D_s = D_t + (\psi)$. Those divisors form a projective space, called the linear system $|L|$ and we have a canonical map $\Theta_L : X \to |L|^*$ defined by $\Theta_L(x) = \{D : x \in D\}$ or less canonically, we chose a basis $\phi_0, \ldots \phi_n$ of $H^0(L)$ and consider the map $\Theta_L = (\phi_0(x), \ldots \phi_n(x)$ into $PH^0(L)$. I this way we have an abstract curve X and a map into projective space, such that the divisors D correspond to hyperplane sections of the image.

Conversely a divisor D gives rise to a line bundle. Simply chose a covering of open sets U_i such that the divisor is given by local equations ϕ_i. Then whenever $U_i \cap U_j \neq \emptyset$ we define $\theta_{ij} = \phi_i/\phi_j$. Those will be non-zero functions and satisfy the critera to qualify as transition functions, and hence define a line-bundle, where the ϕ_i define a section, whose divisor is just D. This is pretty tautologous.

> **Remark 8** In the case of curves we can chose for each P appearing in the divisor a small neighbourhood U_P containing no other point, and the open set U given by X minus the points of the divisor (i.e. the complement of its support). On U_P we set $\phi_P = z^{n_P}$ and on U we set $\phi = 1$. This will give rise to transition functions defined on the punctured neighbourhood U_P^* given by z^{n_P}, note that those transistion functions are indeed holomorphic even if $n_P < 0$ because the poles are excluded in the punctured neighbourhood.

We can now side-step the reference to the linebundle and for every divisor D define the linear system $|D|$ simply by all the divisors $D + (\psi) \geq 0$. The rational functions ψ that occur form a vector-space. In fact all the rational functions ψ

which have zeroes at least of order $-n_p$ for the pole-part of the divisor D and poles at most of the order m_P for the zero-part of the divisor. Note that the sum of two functions having a zero of order m at a point P has a zero at P at least of order m, while if they have poles of order n the sum will have a pole at most of that order.

The Riemann Sphere

Given a divisor D of degree zero we can easily write down a rational function ψ such that $(\psi) = D$, namely

$$\frac{\prod_P (z-P)^{n_P}}{\prod_Q (z-Q)^{m_Q}}$$

the condition $\sum_P n_p = \sum_Q m_Q$ ensuring that the degrees of the numerator and denominator agree and hence that the expression defines a function.

This means that for the Riemann Sphere $\mathbb{C}P^1$ it is also sufficient for two divisors to be linearly equivalent that they have the same degree.

The divisors of $\mathbb{C}P^1$ are hence classified up to linear equivalnce by their degrees, and we can normalize the line bundles to be given by the transition functions $z^n (= \frac{z_1}{z_0}^n)$ on the two open sets U given by $(z_0, 1)$ and V given by $(1, z_1)$

> **Remark 9** The line-bundles are represented by $n0$, chose V and z^n as defining it, while U is the complement of its support. The transition function will hence be z^n on the annulus $U \cap V$.

Only if $n \geq 0$ can there be effective divisors, and the linear spaces $H^0(n0)$ non-trivial. One may take in that case as sections the polynomials of degree n those have dimensions $n+1$.

We can then state Riemann-Roch for the Riemann Sphere as

$$\begin{array}{ll} \dim H^0(D) = \deg D + 1 & \deg D \geq 0 \\ \dim H^0(D) = 0 & \deg D < 0 \end{array}$$

Furthermore for each $n > 0$ we get a map of $\mathbb{C}P^1$ into P^n. The case of $n = 1$ is just the identity (in the sense of being a biholomorphic map), while the case of $n = 2$ is its representation in the projective plane as a conic. Any line intersects it in two points and every two points on it define a line. Thus incidentally the set of unordered pairs of points on $\mathbb{C}P^1$ (i.e. effective divisors of degree two) is parametrized by $\mathbb{C}P^2$. For $n = 3$ we get the twisted cubic in P^3 and for $n = 4$ a smooth rational curve of degree four in P^4 etc.

Elliptic Curves

One may define linebundles on elliptic curves by coverings, but it is more convenient to use the structure of an elliptic curve E as $\mathbb{C}/\Lambda$. Any linebundle L then lifts up to $\mathbb{C}$ and becomes trivial. Its sections thus will be bonafide entire functions with quasi-periodic behaviour. (Note that an entire function which is doubly periodic becomes bounded and hence by Liouville constant, or equivalently descends to a holomoprhic function on compact E and hence has local maxima and minima.) Those quasi-periodic functions can be normalized in many ways, classically it is done in a specific way and referred to as theta-functions. We have already doe that in detail in previous sections, so it will suffice to point out the salient features of this procedure.

a) The linebundles are implicitly defined by their linear spaces of sections, so called theta-functions. Those are subjected to certain quasi-periodic behaviour, encoded by non-zero multipliers, which can be used to define the line-bundles on an elliptic curve instead of via transition functions.

b) We may normalize the theory to one particular theta-function $\theta(z)$ which is an entire function with simple zeroes at the lattice Λ and no other zeroes. Out of this function other theta-functions can be constructed, in particular by translation.

There will now be an analogue to the representation of a rational function namely

$$\frac{\prod_P \theta(z-P)^{n_P}}{\prod_Q \theta(z-Q)^{m_Q}}$$

But in order for the numerator and denominator to live in the same vector space, i.e. having identical multipliers, it is not sufficient that $\sum_p n_p = \sum_Q m_q$ we also need that $\sum_p n_p P = \sum_Q m_q Q$ in terms of the group addition.

Thus for elliptic curves we have an elegant criteria for linear equivalence for divisors, in addition that the degrees are the same we also need to require that the formal sums interpreted as real sums in the sense of addition on the curve, must agree as well.

We can then state Riemann-Roch for elliptic curves as follows

$$\begin{array}{lr} \dim H^0(D) = \deg D & \deg D > 0 \\ \dim H^0(D) = 1 & D \sim 0 \\ \dim H^0(D) = 0 & D \not\sim 0 \\ \dim H^0(D) = 0 & \deg D < 0 \end{array}$$

As to projective maps, we note that for each $d > 1$ there is a map given by D with $\deg D = d$ to P^{d-1}.

If $d = 2$ we get a double cover onto P^1. Such a double cover must be ramified at exactly four points. Three of those points can be normalized to $0, 1, \infty$ while the fourth is determined by this normalization and denoted by $\lambda \neq 0, 1, \infty$.

Remark 10 This illustrates the case that elliptic curves depend on one continuous parameter, its so called moduli. Different λ can give rise to the same elliptic curve, because we can chose the normalization differently. In fact any permutation of S_3 is represented by a suitable Moebius transformation effecting this permutation on the points $0, 1, \infty$. Explicitly by $z, \frac{1}{z}, 1 - z, \frac{z-1}{z}, \frac{z}{z-1}, \frac{1}{1-z}$. One may write down a rational function that is invariant under those transformations, namely

$$j(\lambda) = \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(1 - \lambda)^2}$$

the so called j-invariant that classifies elliptic curves.

Note also that the four ramification points are also the 2-torsion points on the elliptic curve. As such they make up a 2-dimensional vector space over $\mathbb{Z}_2$ fixing an ordered basis for that, corresponds to a specific normalization, and then λ classifies elliptic curves with that additional structure.

If $\deg D = 3$ we get a map of E as a cubic in P^2. A natural way to do so for an elliptic curve is to fix $D = 3O$, the zero is then mapped onto a flexed point and all the flexes correspond to points P such that $D = 3P$. In this case the divisor D is characterized by having sum zero.

If $\deg D = 4$ we get a map of E as a curve of degree four in P^3. What curve? let us choose sections $\phi_0, \ldots \phi_3$ and consider the ten monomials of degree two in those. They are sections of $2D$ and as $\deg 2D = 8$ only eight of those are linearly independant, hence we have a linear space of quadratic relations, in fact a pencil of quadrics in P^3 and its baselocus, the intersection of two quadrics is an elliptic curve. In fact it becomes a curve of bidegree $(2, 2)$ on a quadric, which clearly has genus one.

Remark 11 We all know how to represent the addition geometrically on a cubic, but what about a quartic elliptic? We can assume that $D = 4O$, then points P such that $D = 4P$ will play the role of flex points in the cubic case. They will correspond to points in which the hyperplane intersects in just one point, and hence with multiplicity four. (We can always find planes that intersect in three coinciding points, so called osculating planes or kissing planes). Those points will be sixteen in number and correspond to the $4-$torsion points.

Now given Two points P, Q and consider the plane through O, P, Q which will intersect in a fourth residual point R. We will have $P+Q+R = 0$. To get the involution $X \mapsto -X$ on the quartic we consider the tangent line L to O and the pencil of planes passing through L, it will intersect in residual points P, Q such that $2O + P + Q = 0$

If we split up D as $D = D_1 + D_2$ where $\deg D_i = 2$ each of those give maps onto P^1 and together they effect a map into $P^1 \times P^1$ (a quadric).

Riemann-Roch

The general form of Riemann-Roch for Riemann surfaces X is given by

$$\dim H^0(D) = \deg D + 1 - g + \dim H^1(D)$$

where g is the genus of the curve, and $H^1(D)$ is a vector space, whose dimension gives a positive fudge term. Now one may show that $H^1(D)$ is canonically isomorphic to $H^0(K-D)$ where K is the canonical divisor of X.

We say that a divisor D is special if it is part of the canonical divisor, meaning that one can find C effective such that $K = D + C$, or equivalently, there are holomorphic 1-forms which vanish on D.

For non-special divisors we have

$$\dim H^0(D) = \deg D + 1 - g$$

The case $g = 0$ shows that every effective divisor is non-special, hence $\dim H^0(D) = \deg D + 1$. In particular if $\deg D = 1$ we get a $1-1$ map to P^1. Thus every complex structure on the sphere is actually the Riemann sphere, a fact which would be non-trivial to properly formulate and prove from scratch.

In fact P^1 is characterized among curves of having a divisor of degree one of projective dimension one. In particular on a non-rational curve, no two distinct points (considered as divisors) are linearly equivalent.

In the case of $g = 1$ there is just one special divisor, namely K itself, or equivalently the trivial divisor, which needs special consideration.

The divisors of degree zero always make up a group for a curve X but if the genus is one, we can by a choice of special point O (the zero), identify those with X itself, and hence endow X with a group structure. Namely for each divisor D of degree zero we can consider $D + O$ which is of degree one. By Riemann-Roch we can thus represent this by a unique point P, thus every divisor of degree zero can be written in a unique way as $O - P$. We may then define $P \oplus Q$ simply by

$$(O - P) + (O - Q) = O - (P \oplus Q)$$

which turns out to be quite useful.

Applying Riemann-Roch to a divisor of degree three, we see that the complex structur of any curve of genus one, is indeed that induced by a smooth cubic in P^2. Now for higher genera, the analysis of the behavior of divisors becomes more involved as there will be more special divisors, and we will concentrate on two special cases $g = 2$ and $g = 3$.

Genus two curves

For non-special divisors D we get (setting $h^0(*)$ to denote $\dim H^0(*)$ according to a well-established convention).

$$h^0(D) = \deg D - 1$$

In particular the only divisor of degree two that gives a pencil is the only special divisor of degree two, namely the canonical divisior.

Every genus-two curve has a canonical involution defined by $p + \bar{p} = K$. To be careful. As the linear system $|K|$ moves in a pencil (meaning it is paramtrized by P^1) for any p there is a divisor D linearly equivalent to K such that $D = p+q$ and that q is uniquely determined as $D - p$ is of degree one. That unique q will be the image of the involution. The quotient of the involution will of course be P^1. Conversely given any involution τ such that $X/\tau = P^1$ it has to be the canonical involution, because the fibers make up linear equivalent divisors of degree two.

> **Remark 12** There could of course be other involutions. A double cover of an elliptic curve branched at two points will be a genus two curve.

Now a genus two curve being a double cover of P^1 has to be branched at six points. Three of those can be normalized, which shows that there are three free parameters, hence the moduli $\mathcal{M}_2$ has dimension three.

> **Remark 13** The actual description of $\mathcal{M}_2$ is more complicated, as in the case of elliptic curves, there will be group actions on the three parameters depending on the normalizations.

Note that if Φ is an automorphism of a genus-two curve (or any curve for that matter) then $\Phi^*(K)$ of the canonical divisor is still the canonical divisor (this is in the notion of 'canonical'). Note that this is not true in general, for non-rational curves any two distinct points are not equivalent. This has as a consequence that any automorphism of a genus-two curve descends to P^1 leaving the set of the six branch-points invariant. The six ramification points upstairs are important, they are exactly the six points p such that $K = 2p$, and are called Weierstrass points. Any automorphism must permute them, and any automorphism that fixes them, must either be the identity or the canonical involution.

> **Remark 14** Note that in this way we get a crude upper estimate for the order of the automorphism group, namely 2×720, and any group of automorphisms must modulo an involution be a subgroup of S_6. In particular there cannot be any automorphism of order seven, which incidentally shows that the Hurwitz bound above cannot be sharp in the case of $g = 2$.

Another approach to studying the automorphism group is to look at X/G which will have to be another curve of lower genus. Its eulernumber can be computed provided you have full knowledge of its stabilizers.

It is easy to give example of genus-two curves with nice automorphism, but it should be clear that a generic choice of the six points on P^1 will yield no automorphisms save the trivial and canonical.

Example 1 Chose the six points to be $0, e^{\frac{2\pi ik}{5}}$ with $k = 0, \ldots, 4$. This will allow a automorphism group of order ten. Taking the quotient by an element of order five, fixing one of the six Weierstrass points and permuting the others, will give a quotient of eulernumber two (why? there must be other fixed points, find them!).

Remark 15 As there is a complete classification of finite groups acting on P^1 we can use that to get a handle of what groups can occur for a genus two curve. We see in particular that we cannot lift the icosahedral group, because all its orbits have more than six elements. While the octahedral group has an orbit of order six, namely $0, \pm 1, \pm i, \infty$ which gives an automorphism group of order 48 upstairs. We cannot do better, as any cyclic or dihedral group with an orbit of six elements, can at most have twelve elements.

If $\deg D = 3$ the divisor has to be non-special ($K - D$ has negative degree), it thus gives $3:1$ mappings onto P^1. How many such mappings are there? Or how many divisors up to linear equivalence are there of degree three. If we disregard linear equivalence there is obviously a 3-dimensional family, linear equivalence cuts it down by one, thus a 2-dimensional family. In particular the generic divisor of degree three cannot be written on the form $3p$ because such divisors obviously make up a 1-dimensional family. So generically every genus two curve can be exhibited as a triple cover of P^1 without any total ramification, thus an euler-count shows that there will be eight ramification points. The obvious count $8 - 3 - 2 = 3$ gives another proof of the three moduli.

Example 2 A genus two curve can be represented as a curve of bidegree $(3, 2)$ on a quadric. The canonical class is cut out by the fibration that hits the curve twice, and the projection onto that factor gives the canonical involution. On the other hand the projection onto the other factor gives one of those triple covers onto P^1. Thus there is a 2-dimensional number of ways we can represent a genus-two curve as a smooth curve in a quadric. A count of the number of bi-homogenous forms of type $(3, 2)$ gives $(3 + 1)(2 + 1) = 12$. The group of automorphisms of a quadric has order $3 + 3 = 6$ (each automorphisms induces one the projections). And once again we get $3 = (12 - 1) - 2 - 6$.

Remark 16 This representation of genus two curves gives a handle to the question, can you find a genus two curve with a totally ramified triple cover over P^1? In that case there must be exactly four ramification points $-2 = 3 \times 2 - 2 \times 4$. It is easy to see that a binary cubic $Ax^3 + Bx^2y + Cxy^2 + Dy^3$ is the cube of a linear form iff $B^2 - 3AC = C^2 - 3BD = 0$. In our case A, B, C, D are binary quadrics, and hence there can at most be four such complete ramification points. It is a non-trivial exercise to find such explicit cases.

Now a divisor D of degree four gives a map into P^2 and thus a genus two curve can be realized as a quartic in P^2, but not as a smooth one, as smooth quartics have genus three. In fact by hindsight we know that we need a cusp or a node (and nothing worse) to get a genus two curve. Assume that $D \neq 2K$ then $D - k \neq K$ and hence the linear system is 0-dimensional (projectively), this means that there are unique points p, q such that $D - K \sim p + q$, thus if we look at the divisors D in the system that contains K they are all of the type $K + p + q$ (conversely if they contain p, q they contain a K). This means that the two points p, q map to the same point in the plane (if they happen to coincide we get a cusp, otherwise a node). Furthermore the canonical involution is given by the pencil of lines that pass through the double-point cutting out two residual intersections, which make up the canonical divisors.

Remark 17 What happens if $D = 2K$? Then it is not too hard to show that the map is $2 : 1$ to a conic (the canonical involution again). In fact if ϕ_0, ϕ_1 make up a basis of the sections of K we will get $\phi_0^2, \phi_0\phi_1, \phi_1^2$ linearly independent sections of $2K$, as the dimension of the latter is just three, they make up all of them.

If $\deg D = 5$ we get likewise a map into P^3. Playing the same trick as above, the ten monomials of degree two from a basis of four sections are sections of a divisor $2D$ of degree ten, and the dimension is nine of such a line-bundle. Hence the curve lies on a unique quadric and we have already identified it.

Remark 18 If we fix a point p on the curve, and look at the subsystem of D passing through p this is equivalent of looking at the linear system $D - p$ of degree four. Geometrically we are projecting from the curve of degree five of bidegree $(3, 2)$ (note $3 + 2 = 5$) onto the plane. There is a unique tri-secant through p namely the line in the quadric ruling going through p intersecting the curve in two residual points, hence the projection will have a node. Note that those two points can make up the canonical divisor.

Note also that by blowing up a point on $P^1 \times P^1$ and blowing down the two exceptional divisors stemming from the two lines of the two fibrations, we get P^2.

Finally we can consider divisors of degree zero. If D is a non-trivial divisor of degree zero, then $K + D$ has degree two and is not special, hence there are unique points p, q such that $K + D = p + q$. Now chose $K = p + \bar{p}$ or $K = q + \bar{q}$.

this gives $D = q - \bar{p}$ or $D = p - \bar{q}$. Thus any such divisor can be written under the form $p - q$, and this representation is unique up to the involution I given by $p - q \mapsto \bar{q} - \bar{p}$. This gives a representation of all divisors of degree zero by $X \times X/I$ with the diagonal blown down to a point. This is a compact 2-dimensional variety with an addition, a so called abelian variety.

> **Remark 19** An abelian variety has torsion points. If its dimension is n, points of torsion m make up a subgroup of order m^{2n}. In particular if $n = m = 2$ we expect a group of order 16. And in fact if $2p = 2q$ then by necessity p, q are Weierstrass points. There being 6 of them there are 15 distinct pairs (p, q) such that $(p - q)$ has order two. (The 16th point of order two is of course any non-trivial divisor of degree zero).