Synopsis Thursday, September 27

Basic facts and definitions

We have one one hand ideals $\mathcal{I}$ in the polynomial ring $k[x_1, \ldots x_n]$ and subsets V of k^n. There is a natural correspondence.

$$\begin{array}{c|c} \mathcal{I} & V(\mathcal{I}) = \{(k_1, k_2, \ldots k_n) : f(k_1, k_2, \ldots k_n) = 0, \forall f \in \mathcal{I}\} \\ V & \mathcal{I}(V) = \{f : f(k_1, k_2 \ldots k_n) = 0, \forall (k_1, k_2 \ldots k_n) \in V\} \end{array}$$

Note that $V \subseteq V(\mathcal{I}(V)$ and the inclusion can be strict. Thus not all V occur as sets of zeroes of polynomials. We can think of $V(\mathcal{I}(V))$ as the algebraic closure of V and those algebraic sets are those with which we will be interested.

Furthermore we have likewise that $\mathcal{I} \subseteq \mathcal{I}(V(\mathcal{I}))$ where the former can be a proper subset.

Hilberts Nullstellensatz *If k is algebraically closed (i.e. $k = \bar{k}$) then if $f \in \mathcal{I}(V(\mathcal{I}))$ then for some n we have $f^n \in \mathcal{I}$.*

Given an ideal $\mathcal{I}$ we can associate the radical $\sqrt{\mathcal{I}}$ defined as the set of all f such $f^n \in \mathcal{I}$ for some n. It is easily seen that the radical is an ideal a swell. Ideals equal to their own radicals we could call radical ideals, and then the point of the Nullstellensatz is to set up a 1-1 correspondence between radical ideals and algebraic sets.

Note: The proof of the Nullstellensatz is not easy, although for $\mathbb{C}$ there is a quick short-cut.

The following facts are easily verified

$$\begin{array}{ccc} \mathcal{I}_1 \subseteq \mathcal{I}_2 & \Rightarrow & V(\mathcal{I}_1) \supseteq V(\mathcal{I}_2) \\ V(\mathcal{I}_1 \cap \mathcal{I}_2) & = & V(\mathcal{I}_1) \cup V(\mathcal{I}_2) \\ V(\mathcal{I}_1 + \mathcal{I}_2) & = & V(\mathcal{I}_1) \cap V(\mathcal{I}_2) \end{array}$$

Thus in particular algebraic sets are closed under union and intersection.

To each ideal algebraic set V corresponding to $\mathcal{I}$ we can associate the ring of regular functions $R(V) = k[x_1, x_2 \ldots . x_n]/\mathcal{I}$. It can be thought of as the ring of all polynomial functions on V. Conversely any ring homomorphism $\psi : R \to k$ is determined by specifying $x_i \mapsto \Lambda_i \in k$. The restrictions on λ_i is given exactly by specifying that $(\lambda_1, \lambda_2, \ldots /lambda_n) \in V$. The kernel of such maps are maximal ideals and can thus be identified with points in V.

An algebraic set V is called reducible if it can be written in a non-trivial way as the union $V1 \cup V_2$ i.e. when none of the V_i is equal to V.

The following is relatively easy to verify

Noetherian rings *The following conditions are equivalent* *i) Any ascending chain of ideals $\mathcal{I}_1 \subseteq \mathcal{I}_2 \ldots \subseteq \mathcal{I}_k \subseteq \ldots$ stabilizes, i.e. $\mathcal{I}_n = \mathcal{I}_k$ whenever $n \geq k$ for some k.*

ii) Any ideal is finitely generated

Hint 1 Given an ideal $\mathcal{I}$ and choose a sequence of elements $f_i \in \mathcal{I}$ such that f_{i+1} does not belong to the ideal generated by $f_1, f_2 \ldots f_i$. This cannot continue indefinitely. Conversely set $\mathcal{I} = \bigcup_k \mathcal{I}_k$ this is finitely generated, and eventually all those generators are picked up in the sequence.

The crucial observation is given by the following

Hilbert *It A is a Noetherian ring so is $A[x]$*

Hint 2 The leading coefficients of the polynomials in $\mathcal{I} \subset A[x]$ make up an ideal in A

As a field k is obviously Noetherian, any polynomial ring $k[x_1, x_2, \ldots x_n]$ is Noetherian.

In any Noetherian ring any set of ideals contain maximal members, in particular any ideal is contained in a maximal ideal. On the level of algebraic sets we should simply replace maximal with minimal. Minimal algebraic sets are obviously given by single points.

We can now state

Decomposition *Any algebraic set is the union of irreducible sets*

PROOF: Assume not. Let V be a minimal set which is not the union of irreducible sets. Such a set cannot be irreducible, hence it is the union of two strictly smaller sets, each of which by definition must be a union of irreducible sets.

Definition: An ideal that corresponds to an irreducible set is called a prime ideal. Prime ideals thus cannot be the intersection of two ideals which properly contain it.

Prime ideals can also be defined as ideals $\mathcal{P}$ such that $A/\mathcal{P}$ does not have any zero-divisors. Such rings are exactly those which can be embedded in fields of fractions.

Definition. An irreducible algebraic set will be called a variety.

Note that to each variety V there is a function field $K(V)$, namely the fields of fractions of its regular ring of functions. The transcendence degree over k is defined as the dimension of the variety V.

Derivations Let $D : K \to K$ be a k-linear map, (where K is a field-extension of k). We say that D is a k derivation if $D(fg) = fDg + (Df)g$.

It is easy to see that the derivations make up a vector-space over K simply by multiplication by elements of K, also if f is algebraic over k then necessarily $Df = 0$.

Hint 3 Show that $D1 = 0$ and hence that D vanishes on the constants. If P is a polynomial, show that $D(P(f)) = P'(f)Df$.

Thus it should not be too much of a surprise to learn that the dimension of the derivations is equal to the transcendence degree of K over k as for a transcendental element f say we can choose Df with no restrictions.

The tangent space of a variety V (or more generally for any algebraic set) can be defined externally as a sub-linear space cut out by all the gradients of the polynomials in the associated ideal $\mathcal{I}$. More specifically the tangent space $T(V)_a$ at $a = (a_1, a_2, \dots a_n)$ is given by

$$T(V)_a = \{(x_1, x_2, \dots x_n) : \sum_i \frac{\partial f}{\partial x_i}(a)(x_i - a_i) = 0 \quad \forall f \in \mathcal{I}(V)\}$$

If the tangent space has higher dimension at a point than is given by the dimension of the variety, we say that the point is singular.

Obviously the partial derivatives of a set of generators f the ideal $\mathcal{I}$ can be put in a matrix, and the dimension the rows cut out will be in terms of the rank of the matrix, which is given by sub-determinants, which are nothing but polynomials in the derivatives. Thus the singular locus can be cut out by polynomials and constitute an algebraic set.

Example 1 The gradient of xyz is given by (yz, xz, xy), thus where at least two of the co-ordinates are non-zero, the hypersurface is non-singular. Describe the singular locus of it!

Projective varieties

To each affine space k^n we can associate its projective compactification $P^n(k)(kP^n)$. We can define it as the space of all lines of k^{n+1} through the origin. Alternatively we can see it as given by homogenous co-ordinates $(x_0, x_1, x_2, \dots x_n) \neq (0, 0, \dots 0)$ where we make the identification

$$(x_0, x_1, x_2, \dots x_n) \sim (\lambda x_0, \lambda x_1, \lambda x_2, \dots \lambda x_n)$$

for $\lambda \in k^*$. If we have $x_0 \neq 0$ we can normalize the first co-ordinate to 1 by multiplying with $1/x_0$ and consider $(t_1, t_2, \dots t_n) \in k^n$. If $x_0 = 0$ then we can consider $(x_1, x_2, \dots x_n)$ as homogenous co-ordinates for P^{n-1}. Thus we have the basic inductive step

$$P^{n+1}(k) = k^{n+1} \cup P^n(k)$$

We say that we compactify the affine space with a hyperplane at infinity

Example 2 $(\mathbb{R}P^1 =)P^1(\mathbb{R}) = S^1$ where we one-point compactify the real line. (Note that if we would compactify it with two points $\pm\infty$ we would not get a manifold, but a manifold with boundary, in fact homeomorphic with a closed interval.

Example 3 $(\mathbb{C}P^1 =)P^1(\mathbb{C}) = S^2$ This is the Riemann-sphere, the complex plane compactified with ∞. If we have homogenus co-ordinates (Z_0, Z_1) we can set $z = Z_1/Z_0$ and $w = Z_0/Z_1$ and then we have two copies of $\mathbb{C}$ one using the complex co-ordinate z the other w. When $z, w \neq 0$ we have $z = 1/w$ which gives the glueing of the two complex planes along there common $\mathbb{C}^*$. This is usually the way the Riemann sphere is presented when you first encounter it.

Example 4 A linear map on the Riemann sphere is given by

$$(Z_0, Z_1) \mapsto (cZ_1 + dZ_0, bZ_0 + aZ_1)$$

if we dehomogenize we get the broken linear transformation $(1, z) \mapsto (1, \frac{az+b}{cz+d})$ recognized as the classical Moebius transformations.

Example 5 $(\mathbb{R}P^2 =)P^2(\mathbb{R})$ is the real projective plane. It can be thought of as the sphere S^2 identifying antipodal points. It is an example of a non-orientable surface. We can see it topologically by considering a small band around the equator along with the two hemispheres on either side. One of them is identified with the other, hence making up an ordinary disc, while the band around the equator is turned into a Moebius strip. Thus we can think of $\mathbb{R}P^2$ as taking a Moebius strip and glueing a disc along its border, which is connected. This is hard to accomplish in 3-space, but possible in 4-space. The Moebius strip is the normal neighbourhood of the line at infinity. When you cross it, you do not leave the projective plane, as you would have done if you had simply compactified the open unit disc with its boundary, but you re-enter the plane in the opposite direction.

We will now consider homogenous polynomial, i.e. polynomials in $x_0, x_1, \ldots x_n$ where all the monomials have the same degree. Thus a homogenous polynomial $F(x_0, x_1, x_2 \ldots x_n)$ satisfies the homogeneity relation

$$F(\lambda x_0, \lambda x_1, \lambda x_2 \ldots \lambda x_n) = \lambda^d F(x_0, x_1, x_2 \ldots x_n)$$

where d is the degree of the polynomial. For a homogenous polynomial F the zero-set $F = 0$ is well-defined, and more generally if we consider only sets of homogenous polynomials they define well-defined subsets, which we call projective varieties.

We can mirror what we did before by considering the so called homogenous polynomial ring $k[x_0, x_1, \ldots x_n]$ and considering only homogenous ideals $\mathcal{I}$ which split up as direct sums $\bigoplus \mathcal{I}_d$ where $\mathcal{I}_d$ consists of homogenous polynomials of degree d and $\mathcal{I}_d \otimes \mathcal{I}_e \subseteq \mathcal{I}_{d+e}$.

There is a simple way of going from a homogenous polynomial to a dehomogenized one, simply by setting $x_0 = 1$ and conversely you can homogenize any polynomial in $f(x_1, \ldots x_n)$ by considering $F(x_0, x_1 \ldots x_n) = x_0^d f(x_1, \ldots x_n)$ where d is the degree of f (i.e. the highest degree of a monomial). We say that $V(F)$ gives the closure of $V(f)$ in $P^n(k)$ while $V(f)$ is the intersection of $V(F)$ in the finite open part k^n of $P^n(k)$.

Example 6 In $\mathbb{R}P^2$ any two parallel lines meet at a point at infinity. Examples of two parallel lines are $y = ax+b_1$ and $y = ax+b_2$, there homogenous forms will be given by $y = ax + b_1z$ and $y = ax + b_2z$ and those lines meet the line at infinity $z = 0$ at the same point $(1, a, 0)$

The whole procedure can of course be done on the level of ideals by homogenizing everything in sight. The point of projective spaces is that they are compact. This is interesting globally, but many notions such as tangentspace and dimension is defined locally, and this will then automatically apply to projective spaces.

Segre The product $P^n \times P^m$ of any two projective spaces is a projective space. In fact can be embedded in $P^{(n+1)(m+1)-1}$

PROOF: If $(x_i]$ are homogenous co-ordinates for a point $x \in P^n$ and y_j for a point $y \in P^m$ we can consider the $(n+1)(m+1)$-tuple z_{ij} given by $z_{ij} = x_iy_j$ as representing the point $z = (x, y)$ those will constitute homogenous co-ordinates for an appropriate projective space, and the map is well-defined (if $(x_i) \sim (x'_i)$ and $(y_j) \sim (y'_j)$ then $(z_{ij}) \sim (z'_{ij})$. Furthermore the co-ordinates for z satisfy the quadratic relations $z_{ij}z_{kl} = z_{il}z_{kj}$.

Example 7 The case of $n = m = 1$ gives a nice embedding of $P^1 \times P^1$ into P^3 with the image a quadric. Over the complexes any non-singular quadric can be put under the form $XY - ZW$ and thus any non-singular complex quadric has this product structure. Over the reals this is only possible if the index of the quadratic form is zero, which corresponds to the one-sheeted hyperboloid.