Take-home exam for Algebraic Geometry

February 11 - February 18, 2013

1 Find all rational solutions to the diophantine equations
a) $x^2 - 3y^2 = 1$
b) $x^2 + y^2 - z^2 = 1$

2 Find a rational solution to the cubic equation $y^2 = x^3 - x + 1$ such that either the numerator or the denominator (or both) has three digits.

3 Find all the primitive 2-torsion points to the elliptic curve $y^2 = x^3 - x$ with the origin as usual at $(0, 1, 0)$

4 Let Q be a quadric and C a cubic in P^3 and assume that both contain a line L. Compute the genus of their residual intersection, assuming that to be generic. In other words if $Q \cap C = L + D$ with D smooth, find $g(D)$.

5 A linear system of degree d and projective dimension r on a curve is classically called a g^r_d. Thus a g^1_d is a pencil and gives a map onto P^1, while g^2_d is a net, and gives a map into P^2.

a) Show directly, without using Riemann-Roch, that a genus two curve, can have at most one g^1_2. (In fact it always has one, namely the canonical divisor).

b) Show that if a curve has two different g^1_3 its genus is at most four. Give an example of two different g^1_3's on a quartic plane curve!

6 Try to find a genus two curve with a g^1_3 such that the corresponding map onto P^1 has four points of total ramification.

7 Find the dimension of the linear space of $(2, 3)$ curves on a quadric, and compute the degree of the hypersurface of singular such curves.