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PREFACE

The following are the notes I wrote down for a course in Projective Geometry at Chalmers in the fall of 1989.

The course was intended to start more or less from scratch, and from one small corner of the Algebraic Geometrical Universe expand gradually the horizon of the students.

Thus I eschewed all ambitions of setting up a nice machinery from which the results could then follow nicely and smoothly. It is in fact my conviction that the presentation of a more or less formidable machinery may be good in hindsight but that it postpones motivation and ultimate gratification.

I have hence worked rather intuitivly, and I have not been ashamed of waving my hands, especially at the end. The idea being that the students should first be confronted with the problem before they start to develop the concepts to formulate and attack it. To this effect I have included a fair amount of exercises. Many of those are routine and many are computational, some fill in gaps in proofs, while others are pure digressions, of which there are endless opportunities. One learns by doing and a prospective reader is very much encouraged to try his or her hand at most of them, or at least those that catches her or his fancy.

On the other hand I have become aware of the frustrations such an approach may entail as well, and I regret many of the repetitions of standard arguments that I have employed over and over again. Nevertheless I think that the attempt is worthwhile.

As to the choice of topics. I have decided to concentrate on conics, as they are quite elementary and provide a bridge between linear algebra and algebraic geometry. In particular I have treated linear systems of conics, which are all more or less classified throughout the course of the lectures. The final section on nets is a bit sketchy and ought to be expanded in a revised version, in particular the stratification of all singular nets (following C.T.C Wall) should be presented in detail.I have also digressed on quadrics in $\mathbf{P}^3$, a topic which one cannot pass over easily, and also perhaps less of a necessity on cubics and elliptic curves, where I have presented a particularly hefty collection of exercises. This choice was motivated by the geometry of a non-singular net which is intimately connected with the discriminant cubic, although I have presented far more than is strictly necessary. This is also yet another indulgence.

One has to confine oneself, yet I feel that many topics are missing and ought to be included in a revision. E.g. higher dimensional quadrics, in particular a thorough discussion of the quadric line complex. Pencils of quadric surfaces may also be included, as to further justify the lengthy digression on elliptic curves. The cubic hypersurface is another topic which is always very tempting to include. It is also a shame not to have included anything on metrics on projective planes as compact manifolds. The geometry of Möbius transformations is discussed in some detail, but little on the geometry of $\mathbf{PGL}(3,\mathbb{C})$. Elementary toplogy is hinted at in the exercises, especially the Euler characteristics, but this being rather fundamental, it ought to be lifted out more in the open. In general one may discuss whether not many of the exercises would not deserve a fuller treatment in the text.

As to arithmetic applications there are some in the text, or mainly in the exercises. One could include a little bit more on finite geometries, most of it is in the exercises; and also be more systematic on elementary diophantine discussions (almost all of what there is is to be found in the exercises), in particular it may not be amiss to prove the Hasse principle for Quadrics, although it may be too much of a digression. Finally connecting with the lengthy section on cubic curves, one may discuss quadratic equations, lattices, $\mathbf{SL}(2,\mathbf{C})$ and orders, as yet another aspect of quadrics.

The big omission in this version is of course the pictures. I have not yet been able to produce nice pictures of what I want, nor have I figured out how to insert those in the text. When this has been done another version will appear. I have never seen a net of conics, and I definitely intend to include such pictures in the future (hoping not to be disappointed). It is also my opinion that only nice graphics, which to my knowledge is not so much available in classical geometry, would justify yet another elementary text.

But basically this has been a self-indulgence, and it may just remain so. Any suggestions are of course welcome, and advice well-taken if not always followed.

Göteborg
December 20, 1989

Preface to the extended version

To the notes of the fall 1989 I have added (in November 1990) three chapters , on Quadrics in general, the Quadratic Line complex and finally on pencils of Quadric surfaces. Yet the work is not complete, the chapter on the Quadratic line complex can easily be expanded. And there should be a chapter on nets of quadric surfaces, Whether to include a chapter on the cubic hypersurface is open, but as the intersection of two quadrics in $\mathbb{P}^4$ would be natural, the temptation is great. The chapter on the Quadratic Line complex easily lends itself to a discussion of quadratic complexes and Kummer surfaces, thus K-3 surfaces, Abelina surfaces and Jacobians of genus two curves naturally come into play. Those may be added at some later date.

Still there is no discussion of metric properties of projective spaces, nor any continued discussion of $\mathbb{P}GL(3, \mathbb{C})$ and higher groups. This may also be included.

And the illustrations are still missing (except for the title page!)

Göteborg
February 27,1991

Introduction

Projective Geometry deals with properties that are invariant under projections. Hence angles and distances are not preserved, but collinearity is. In many ways it is more fundamental than Euclidean Geometry, and also simpler in terms of its axiomatic presentation. Projective Geometry is also "global" in a sense that Euclidean Geometry is not. In Euclidean Geometry lines may or may not meet, if not, this is an indication that something is "missing". In Projective Geometry two lines always meet, and thus there is perfect duality between the concepts of points and lines.

Synthetic Projective Geometry was a timehonored subject in Secondary Schools in the past, and its ancestry goes back to the Old Greeks (*Pappus* and *Appolonius*), with a renewed interest during the Renaissance. It was not however systematically developed until the 17^{th}- and 18^{th}- century (*Desargues, Poncelet* and *Monge*) and reached its apogee during the last century. Although we will only peripherally touch upon the axiomatic and synthetic aspects, the general notions of projective spaces constitute the basic setting for Algebraic Geometry.

As for the axiomatic and synthetic aspects of Projective Geometry there exist a host of classical references. The most elegant and least involved is probably **Hartshorne** : *Projective Geometry* (Benjamin); while works by e,g, Coxeter (with predictable titles) go into more detail. We will however not be overly concerned with those aspects.

The Projective Plane

As is wellknown two lines may or may not meet. If not they are said to be parallel. The notion of parallel is easily seen to be an equivalence relation among lines. We may then "force" two lines always to meet by "postulating" a missing point at "infinity". Infinity will consists formally of the equivalence classes of lines (with respect to parallelity) and each line will be augmented by its equivalence class. And in addition we will consider infinity as a line. In this way we have formally forced every pair of lines to meet, and still through two points there will be a unique line. The ensuing construct will be called the projective plane.

Such a construct is of course quite unsatisfactory. It appears forced and unnatural and very contrived, although from a formal point of view it may be impeccable. However it is not very geometric and it assigns to the missing points ("the line at infinity") a special status, that it does not deserve. A more geometric presentation is to consider the task of say a Renaissance artist using perspective to represent three dimensional reality onto a flat canvas.

We may consider him standing on a floor (F), ideally extending indefinitely, painting on an equally vast canvas (C). If his eye is represented by the point O (for simplicity we may think of him as cyklop) then to every point P on the floor we may associate the point P' on the canvas which is given by the intersection of the line OP with C. This sets up a correspondence between the points on F and its pictorial representation on C. (Of course the image on the canvas is not limited by what is on the plane floor, but to every point P (except O of course) we may associate P' as above. In this case to each point on the canvas corresponds an entire ray in space). However a moments reflection reveals that not all points on the canvas corresponds to points on the floor. If we consider the plane through O parallel to

F, its intersection H with C will constitute a line corresponding to the "missing" points of F. H is of course the horizion, and the images of parallel lines on F will all be lines meeting at H. H is hence the natural "compactification" of the plane F, the line at infinity. Conversely, not all points on F will be represented on C, the line M on F given by the intersection with a plane through O parallel to C, will have no image on C. And lines on F meeting on M will be mapped onto parallel lines on C. (One may observe that in a real picture the horizon bisects the canvas, and only what is in front of the artist will be depicted, while what is behind is ignored, and its image is usurped by say the sky above. The consistent application of this method of projection would bisect any object lying across the line M, with the part behind the artist (center of projection) shown upside down!).

It should now be clear how to give a formal and yet natural definition of the projective plane.

Consider a 3-dimensional real vectorspace V, with distinguished point 0 (the origin O), and identify two non-zero vectors v and v' iff $v = \lambda v'$ for some $\lambda \neq 0$. (They lie on the same ray, and hence have the same "image").

If we choose coordinates on V then the coordinates (X_0, X_1, X_2) of a point p in the projective plane $\mathbf{P}(V)$ will be called homogenous coordinates. And two sets of coordinates (X_0, X_1, X_2) and (X_0', X_1', X_2') represent the same point iff there is a λ different from zero such that

$$X_0 = \lambda X_0'$$
$$X_1 = \lambda X_1'$$
$$X_2 = \lambda X_2'$$

(note that we must exclude $(0, 0, 0)$)

Homogenous coordinates (apparently of as recent origin as Klein) have implicitly popped up in elementary linear algebra. Any line in the plane is given by an equation of the form

$$AX + BY + C = 0$$

where the triple (A, B, C) is only defined up to a multiple. However this only defines a line if either A or B is nonzero. (if $A = 0$ then the line is parallel to the x-axis, if $B = 0$ it is parallel to the y-axis). The case $A = B = 0$ does not make sense in the plane, however it does in the projective plane. The nonsense equation $C = 0$ represents the line at infinity!

To make less nonsense out of this we will homogenize and set $X = X_1/X_0, Y = X_2/X_0$ and clearing denominators we end up with

$$AX_1 + BX_2 + CX_0 = 0$$

with $X_0 = 0$ to be the line at infinity.

A line (rearranging the letters) is given by a linear form

$$AX_0 + BX_1 + CX_2 = 0$$

in homogenous coordinates. Which on the level of $\mathbb{R}^3$ could be thought of the plane

$$AX_0 + BX_1 + CX_2 = 0$$

through the origin.

A line is a projective space on its own, the projective line, and maybe thought of as a circle. The line completed at infinity.

From now on we will, for reasons to become consistent later, denote the projective plane by $\mathbb{R}P^2$ and refer to it as the real projective plane. While the lines are denoted by $\mathbb{R}P^2$

The topology of $\mathbb{R}P^2$

Consider the quotient map

$$\mathbb{R}^3 \setminus 0 \to \mathbb{R}P^2$$

by simply considering the coordinates (X_0, X_1, X_2) as the homogenous coordinates for the points on the projective plane. The fibers of this maps are given by the rays $\mathbb{R}^*$.

One way of looking at this is to say that $\mathbb{R}P^2$ parametrizes all lines through the origin. Now to each line there are two directions. After all $\mathbb{R}^*$ splits up into two connected components (the positive and the negative numbers). All directions are parametrised by the sphere. (Spherical coordinates; whose "ugliness" is due to the clumsy effort of trying to represent the sphere as a rectangle!). Antipodal points on the sphere correspond to opposite directions, but to the same line. Thus the projective plane is formed by identifying antipodal points on a sphere.

More formally, if S denotes the unit sphere $x^2 + y^2 + z^2 = 1$ then the fiber of the qoutient map consists of the two points (x, y, z) and $(-x, -y, -z)$. (We are normalizing the homogenous coordinates in the vain hope of finding a canonical representative).

In fancier language we are considering an action of the group $\mathbb{Z}_2$ on the sphere defined by $e(x, y, z) = -(x, y, z)$ for e the generator of the group. The qoutient $S/\mathbb{Z}_2$ is identified with $\mathbb{R}P^2$ and the fibers are the orbits (pairs of points).

This is an example of a double (unramified) covering. To each point of the projective plane do we correspond two points. Those will make up a sphere. This covering is non-trivial. i.e. we cannot keep track of the two covering points by say labeling one blue and the other green in a continous fashion. (Of course we could do it by brute force, but then the colors of the points would have to change drastically -see exercise 5.).

This is further an example of a universal covering. The sphere S is simply connected, and the qoutient $S/\mathbb{Z}_2$ is not, but has fundamental group equal to $\mathbb{Z}_2$. Each closed path on $\mathbb{R}P^2$ lifts to S, if it still is closed the path can be contracted, if not it provides a nontrivial loop in the fundamental group.

Lifting a loop on $\mathbb{R}P^2$ means chosing a point in the fiber in a continous way. If this can be done consistently, the loop can be contracted; if not, then the lifted path does not close upon itself and we have an illustration of the phenomena of monodromy.

A line in $\mathbb{R}P^2$ gives an example of a loop that cannot be contracted. In fact the inverse images of lines are given by great circles on the sphere. The identification of antipodal points on a circle will still yield a circle (cf exercise 8.) Just taking half of a great circle, i.e. one of the arcs that joins two antipodal points, the image will be a line and the arc a non-connected lifting. (We see how we have continously changed one point to its antipodal)

It is hard to visualize $\mathbb{R}P^2$. If we disconnect S by removing say the equator, we may take either hemisphere as a representative for $\mathbb{R}P^2 \setminus$ line at infinity. This is just topologically identifying the plane $\mathbb{R}^2$ with the disc. The hard part is to sew on the equator back. If we fatten it into a circular band we see that the quotient by antipodes gives us a Möbius strip. The boundary of a Möbius strip is a connected, in fact a circle, and if we glue it to the boundary of a disc we have the desired plane. This operation is however impossible to do inside $\mathbb{R}^3$ unless one is willing (and able) to let the surface intersect itself. (Such models can be made, but are not particularly illuminating -see e.g. **Hilbert,Cohn-Vossen** *Anschauliche Geometrie*).

$\mathbb{R}P^2$ is an example of a so called non-orientable surface. To see this consider a line (if you so want the line at infinity). Give it an orientation (i.e let us trace it in one of two possible directions) and assume that the surface is orientable, i.e. we are able to choose a direction making a positively oriented set of directions with the direction of the line. This means that we can keep track of which side is left and right. But this is impossible, as you can see tracing along the centerline say of a Möbius strip. When we come back to our starting point left and right has switched places!

Projective Spaces in General

The definition of $\mathbb{R}P^2$ as $\mathbf{P}(V)$ for V a 3-dimensional real vectorspace has the advantage that it formally generalizes directly.

We may take for V any vectorspace of any (finite) dimension and over any field, in fact over any skewfield.

We define $\mathbf{P}(V)$ simply as the quotient of V^* $(= V \setminus 0)$ by the equivalence relation $v \equiv v'$ iff $v = \lambda v'$ for $\lambda \neq 0$.

If the (skew)field is denoted by $\mathbf{K}$ and $dim_{\mathbf{K}} V = n + 1$ we may also write $\mathbf{K}P^n$ for $\mathbf{P}(V)$. This is the n-dimensional projective space over $\mathbf{K}$.

Homogenous coordinates are simply given by coordinates on V and all the formal manipulations go through. Two lines may of course no longer meet, if dimension is at least three, as they may now be skew. But the right generalization to higher dimensions is given by the following.

Proposition. *If V and W are two linear subspaces of dimension m and p of a projective space $\mathbf{P}^n$ then the dimension of their intersection is at least $m + p - n$. In particular they are bound to meet if $m + p \geq n$. Furthermore through any $p + 1$ points not lying on a subspace of dimension $p - 1$ there is a unique subspace of dimension p containing them.*

Proof:. Lift everything to the corresponding vectorspaces one dimension higher. As they all have at least one point in common (0) we just apply the wellknown lower bound $(m + 1) + (p + 1) - (n + 1)$ and subtract one dimension during the projectivization. The second statement is just elementary linear algebra.

Yet in the deeper properties of projective spaces the dependence on the base field is very important. We will now consider a few typical cases, with the first case, that of the complex numbers, being the most important.

The Riemannsphere and Complex Projective Spaces

Recall the definition of the Riemannsphere. The sphere S is bisected by a plane P through the equator. It is projected onto P from the Northpole (n) and Southpole

(s) respectively. This gives identifications of P with S$\backslash n$ and S$\backslash s$ respectively. If we identify P with the complex plane $\mathbb{C}$ in the first projection and with the conjugate $\bar{\mathbb{C}}$ in the other. (We look at $\mathbb{C}$ from above and below) then if z and z' denote the images of a point p on the sphere distinct from the poles then by using similar triangles we see that

$$z = \frac{1}{z'}$$

In this way we have given S a complex structure, using the two charts S$\backslash n$ and S$\backslash s$ and the above transition function.

This we may compare with the construction of $\mathbb{C}P^1$ given by homogenous coordinates (z_0, z_1). Let us denote by n $(0,1)$ and by s the point $(1,0)$; and let $z = z_1/z_0$ be the complex coordinate on $\mathbb{C}P^1\backslash$n (=$\mathbb{C}$) and $z' = z_0/z_1$ the coordinate on $\mathbb{C}P^1\backslash$s (=$\mathbb{C}$). We then obviously have

$$z = \frac{1}{z'}$$

as well. Thus we see that the two constructions are isomorphic.

Using homogenous coordinates on the Riemannsphere illuminates the linear character of the Möbius transformations (broken linear transformations). Indeed a projective transformation of $\mathbb{C}P^1$ is given by

$$(z_0, z_1) \mapsto (dz_0 + cz_1, bz_0 + az_1)$$

Setting $z = z_1/z_0$ we obtain the more familiar form

$$z \mapsto \frac{az+b}{cz+d}$$

Möbius transformations are given by matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which are elements of GL(2,$\mathbb{C}$) with the proviso that two matrices which are multiples of each other define the same transformation. ("Homogenous matrices" in analogy with homogenous coordinates) This can also be expressed by $A = \lambda A'$ for two matrices, where λ can be identified by the scalar matrices λI with I the identity, i.e. the center of GL(2,$\mathbb{C}$). The corresponding quotientgroup will be denoted by PGL(2,$\mathbb{C}$).

In analogy with the covering of $\mathbb{R}P^2$ by the sphere, the complex projective spaces may be covered by spheres. Indeed inside $\mathbb{C}^{n+1}\backslash 0$ consider the sphere S^{2n+1} defined by

$$|z_0| + \ldots |z_n| = 1$$

Now the point $\lambda(z_0, \ldots, z_n)$ will lie on the sphere iff $|\lambda| = 1$, thus the fibers of the projection map will be circles (1-dimensional circles) not just two points (0-dimensional circles!). We have now a S^1 action (not just a $\mathbb{Z}_2$ action) on the spheres, and the projection will hence define a fibration with S^1 fibers. This is called the Hopf fibration, and is of particular interest for the case n=1, when we present S^3 as a fibration over S^2 by S^1. (see exercise 15)

What can be said about the topology of complex projective spaces? One obvious and very important fact is that they are compact. This follows from the fact that they are quotients of compact spaces (spheres).

Less obvious is that they unlike real projective spaces are simply connected. This has been checked for n=1 and the general fact can be checked heuristically (with some hand waving) as follows.

Given $\mathbb{C}P^n$ we remove a hyperplane (a $\mathbb{C}P^{n-1}$) the complement is $\mathbb{C}^n$ hence simply connected (in fact contractible). Now a hyperplane has real codimension two, hence we may "wiggle" a loop so as to avoid it, as it then will sit inside something simply connected we are done.

Axiomatics and Finite Geometries

The notion of a projective plane (and of course higher dimensional projective spaces as well) may be axiomatized. One may introduce the primitive concepts of lines and points, and the primitive relation of incidence. Thus a point incident with a line simply means that the "point lies on the line". We may then write down various axioms.
A1) Given two distinct points there is a line incident to both
A2) Given two distinct lines there is a point incident to both
A3) Given two lines there is a point incident to none of them

Note the symmetry (duality) between the first two axioms. The third is thrown in just to exclude a trivial possibility.

One may think that the axioms are too simple to have any interesting consequences. Yet one may state the following assuming that the plane is finite

Proposition. *Any two lines of a finite projective plane contain the same number of points* $(q+1)$ *and any such plane contains* q^2+q+1 *points.*

Proof:. Given the two lines L and L' choose a point p not incident to either (which is possible due to A3)). This point sets up a 1-1 correspondence between the two lines through "projection". Namely to each point r on L consider the "intersection" r' (the unique point incident to) of the line <r,p> (the unique line through r and p) and L'. Denote the number by $q+1$. Now any point in the plane except p lies on a unique line through p. The lines are parametrised by L (or L' or any line not through p) thus there are $(q+1)$ of them, each containing q points (removing p) then adding p at the end giving $q(q+1)+1 = q^2+q+1$

But it is not so trivial to construct finite projective planes. It is equivalent to finding a collection of subsets each with $(q+1)$ elements of a set with q^2+q+1 elements such that any two distinct subsets have exactly one element in common, and for any two distinct elements there is exactly one subset in the collection containing them both.

However if $q = p^n$ this can be solved. In fact let us recall the basic facts about finite fields.

The basic fields are the prime fields. There is one for each prime p, namely the fields $\mathbb{Z}/p\mathbb{Z}$ with p elements (also denoted by $\mathbb{F}_p$). (And of course $\mathbb{Q}$ in case of characteristic 0). Any finite field $\mathbb{F}$ is an extension of its prime field, thus a vectorspace over it of finite dimension n. Thus the cardinality is given by p^n. Furthermore $\mathbb{F}^*$ (the invertible elements of $\mathbb{F}$, i.e all non- zero elements) is a group under multiplication, as its order is $q-1$, we get that

$$x^q - 1 = 1 \text{ if } x \neq 0$$

thus $x^q = x$ for all elements in $\mathbb{F}$. This shows that $\mathbb{F}$ is the splitting field of the equation $x^q = x$ over the primefield $\mathbb{F}_p$ thus they are all isomorphic up to the number of elements q, and maybe denoted by $\mathbb{F}_q$.

If we now take the field **K** to be F_q and V a vectorspace of dimension 3 we obtain a projective plane with $(q^3-1)/(q-1)$ elements. This is of course q^2+q+1. Furthermore each line will contain $q+1$ elements. (The field plus the element at "infinity")

As far as I know, no finite projective planes have been constructed other than those above.

The simplest projective plane correspond to $q=2$ and correspond to 7 points, which in homogenous coordinates maybe listed as

$$(1,0,0),(0,1,0),(0,0,1),(0,1,1),(1,0,1),(1,1,0)\mathrm{a}nd(1,1,1)$$

(One may think of a triangle with its vertices and its "medians" intersecting in a point (1,1,1). The three midpoints (1,1,0),(1,0,1) and (0,1,1) do not usually lie on a line, but they do exactly if the characteristics is equal to two!)

In many ways it is a bit absurd to talk about geometries in the finite setting, yet the general methods are as relevant here as in the classical cases; and although the problems may always be stated in elementary combinatorical fashions, in order to solve them one need to fit them into the general geometric scheme.

Furthermore many diophantine geometrical problems can be reduced modulo p and stated in finite terms where they can be solved.

This section would be incomplete without mentioning two classical theorems.

The first is

Desargues Theorem. *If two triangles ABC and A'B'C' are in perspective, i.e. the lines AA',BB' and CC' all go through a point O, then the three pairwise intersections (AB)(A'B') (AC)(A'C') and (BC)(B'C') lie on a line.*

Proof:. If the two triangles do not lie in a plane, this is easy as the three intersection points must lie on the intersection of the two planes spanned by the two triangles. If the projective plane can be embedded in a three dimensional projective space (as all the planes we have so far encountered) the degenerate case can be reduced to this. (see exercise 24)

One may construct projective planes which are not Desarguian; those planes are by necessity rather contrived, thus Desargues theorem is usually added to the axioms.

The second is

Pappus theorem. *If two triplets of points A,B,C and A',B',C' lie on two lines, then the three pairwise intersections (AB')(A'B), (AC')(A'C) and (BC')(B'C) lie on a line.*

In fact Pappus theorem is true for a projective plane of type $\mathbf{P}(V)$ over a skewfield **K** iff **K** is commutative.

Duality and Conics

Given a projective plane there is complete duality between points and lines. Two lines determine a point and two points determine a line. The Axioms A1) and A2) are dual to each other, while the dual of A3) is " given any two distinct points there is a line incident to none of them". One may reinterpret lines as "points" and points as "lines" and the notion of incidence being "selfdual" the new model would be another projective plane.

The notion of duality enables one to state a dual version to each theorem and even dualizing each proof, thus cutting down the work to half. (Except that of course some theorems may be selfdual).

On the less axiomatic level, we have already encountered duality. Given a plane then each line maybe written in the form

$$Y_0X_0 + Y_1X_1 + Y_2X_2 = 0$$

where (Y_0, Y_1, Y_2) form the homogenous coordinates for the lines.

Thus the dual projective space of $\mathbf{P}(V)$ is given by $\mathbf{P}(V^*)$ where V^* is the dual vectorspace of V.

Geometrically a point p in $\mathbf{P}(V)$ determines a line $\check{p}$ in $\mathbf{P}(V^*)$ by considering all the lines ("points" of $\mathbf{P}(V^*)$) through p.

And of course a line l determines a point $\check{l}$ in the dual plane, the lines through which corresponds to the points of l.

There is however no canonical duality between a projective plane (or more generally space) and its dual, no more than there is a canonical identification between a vectorspace and its dual.

To give such an identification is equivalent to give a nondegenerate bilinear form

$$B(X_0, X_1, X_2; Y_0, Y_1, Y_2)$$

Explicitly each point (Y_0, Y_1, Y_2) determines the linear form $B(*; Y_0, Y_1, Y_2)$.

We say that the line $\check{p}$ corresponding to the point p is the polar of p. Now we may look at the locus of all points p such that $p \in \check{p}$. This is given by the vanishing of the quadratic form $B(X, X) = 0$ the points which form a conic. Conversely given a conic we may not determine the bilinearform uniquely, unless we insist that $B(X, Y)$ should be symmetric. Geometrically this means that if $p \in \check{q}$ then $q \in \check{p}$.

There is, provided that the characteristics of the field is different from two, a one-to-one correspondence between symmetric bilinear forms $B(X, Y)$ and quadratic forms $Q(X)$ given by

$$Q(X) = B(X, X)$$

and

$$B(X, Y) = \frac{1}{2}(Q(X + Y) - Q(X) - Q(Y))$$

Geometrically we may proceed as follows. Given a conic C and a point p on C, then $\check{p}$ passes through p and may cut the conic in some other point q. As $q \in \check{p}$ we have that $p \in \check{q}$, as both $\check{p}$ and $\check{q}$ pass through p and q two distinct points, we have that the two lines are equal, hence $p = q$; thus we see that $\check{p}$ is tangent to the conic.

Let L be a line then L intersects C in two points (which may coincide) this can be seen as follows. A line can be parametrised by

$$s(A_0, A_1, A_2) + t(B_0, B_1, B_2)$$

plugging this into the conic Q(X) we obtain a binary quadric

$$q_0(A, B)s^2 + q_1(A, B)st + q_2(A, B)t^2$$

which (over the complex numbers) has two roots corresponding to the two intersectionpoints.

Let the two points be p_1 and p_2, as they both belong to L their polars contain the polar to L, which is hence formed as the intersection of the the two tangents.

Conversely we have shown that given any point p, there will be two tangents through p to the conic. (This can also be seen directly see exercise 28) the corresponding tangency points determine a line the polar $\check{p}$.

The tangents to a conic form a curve in the dual projective space which intersects each line in two points, and in fact constitute a conic.(cf exercise 29) The dual conic, which establishes the isomorphism between the dual plane and its dual (which is canonically isomorphic with the original plane)

In the real case a line may of course not meet a conic, nor need there be two tangents from a point. In fact we may abstractly define the outside and the inside of a conic as follows.

A point p is inside a conic C iff no tangents to C pass through p it is outside iff two tangents pass through it and of course it is on C iff there is only one tangent through it.

Any real conic may of course be considered as a conic in complex projective space. (The real structure will be canonical as the complex structure will be given by tensoring with $\mathbb{C}$ cf exercise 13) We may then consider all the complex points that lie on it. Those points will be invariant under complex conjugation and those fixed will of course be the real points that we see. If the real point happens to be inside the conic then the two complex tangents are conjugate (If a line is a tangent so is its conjugate) and their tangency points likewise. Thus the line joining them is real (and can be seen). In the same vein if the real line lies outside the conic, the corresponding complex line will intersect it in two conjugate points, whose corresponding tangents are of course conjugate, and hence meet in a real point.

Exercises

1 Given $\mathbb{R}^3$ and let $z = 0$ denote the floor (F) and $x = 0$ the canvas (C) and let (1,1,1) (O) be the position of the artists (seeing) eye.
a) Find the horizon
b) find the equations on the canvas corresponding to the tiling given by $x = n, y = m \quad n, m \in \mathbb{Z}$
c) find the image (on C) of a general line ax+by=c (on F)
d) find the images of the parabola $y = x^2$ and the circle of radius 2 centered at (1,1)

2 (Same as above) We can identify the floor with the canvas by letting $p = (x, y, 0) \mapsto (0, x, y) = p'$. Joining the points p and p' we get lines how can you characterize those?

3 (As above) By the identification of F with C as in exercise 2 any point P in $\mathbb{R}^3$ defines a projective transformation of say C via projection from P. In fact to each p' associate the intersection p" of the line Pp with C. Such transformations are called perspectives.
a) how should P be chosen to get the identity transformation?
b) what happens if P lies on either C or F?

4 (As Above) Let $(0, x, y)$ have the homogenous coordinates $(-1, x-1, y-1)$
a) Write down the projective transformation corresponding to projection from the point (a, b, c) according to the convention of exercise 3 in (homogenous) matrix form.
b) Give an example of a projective transformation of the projective plane that is not gotten in this form (i.e not a perspective).
c) Give an example of two perspectives whose composition is not.
d) Give an example of a projective transformation which is not the composition of two perspectives.
e) Is every projective transformation a composition of three perspectives?

5 Observe that the sphere cannot be decomposed into two disjoint compact subsets; use this fact to show that the double covering of $\mathbb{R}P^2$ by the sphere cannot be trivial.

6 Would it be possible to subdivide the earth (the sphere) into land and water, such that each hemisphere contains equal amount of both?

7 The universal covering of a circle S^1 is given by the real line $\mathbb{R}$ with fundamental group $\mathbb{Z}$. Thus $\mathbb{R}/\mathbb{Z} = S^1$.

In fact show that the map $\mathbb{R} \to S^1$ given by $\theta \mapsto e^{2\pi i\theta}$ exhibits the covering with the action of $\mathbb{Z}$ given additively by $\theta \mapsto \theta + 2n$

Naively chosing an angle means making a choice up to an additive multiple of 2π. If one wants to do this consistently and continously one will end up with a discrepancy for the angle at the initial and the final point. The discrepancy tells us how many times we have "looped" around.

8 Show that the map $S^1 \to S^1$ given by $z \mapsto z^2$ identifies antipodal points and hence exhibits $\mathbb{R}P^1$ as a circle. Write down this map on the level of the universal covers $\mathbb{R} \to \mathbb{R}$

9 If you remove a line from $\mathbb{R}P^2$ the complement is still connected. Why? Thus show that Jordans theorem does not hold on the real projective plane. A simple closed curve does not disconnect the plane in an outside and an inside. Give examples of similar phenomenas say on a torus.

10 If we draw three lines in the plane, not passing all through a point, we disconnect it into seven connected components. What happens if we do it in the projective plane instead?

11 If we have a polyhedron without holes, say a Platonic Solid, then it is well-known that #(Faces)-#(Edges)+#(Vertices) = 2. This is due to Euler. In general the lefthand side is called the Eulercharacteristic of the polyhedron.

Consider now a regular octahedron and identify antipodal points. (Vertices will correspond to vertices and edges to edges etc) We will then get a welldefined polyhedron, which is however a bit difficult to visualize and which will constitute a triangulation of the projective plane. Use this to compute the Eulercharacteristic of $\mathbb{R}P^2$.

12 Let V and W be two linear subspaces of dimension m and p in a projective space of dimension n. Let Z be the span of V and W, i.e. the smallest linear subspace that contains them both. Give the dimension of Z in terms of m,p,n and the dimension of the intersection of V and W.

13 By a real structure on a complex vectorspace V is meant a real subvectorspace $V_{\mathbb{R}}$ such that $V_{\mathbb{R}} \otimes \mathbb{C} = V$. E.g. If $V=\mathbb{C}^n$ then the natural real structure is $\mathbb{R}^n$, and all real structures are obtained in this way, i,e by imposing coordinates on V. With respect to a real structure we may define complex conjugation on V, the fixed locus of which will be the space $V_{\mathbb{R}}$. This notion of conjugation descends of course down to projective space. A real point of a complex projective space will hence be a point that can be given by real coordinates.

Show that two conjugate lines intersect in a real point and the line through two conjugate points maybe defined by real coefficients.

14 Consider $\mathbb{C}^2$ as $\mathbb{R}^4$ via $(x+iy, z+iv) = (x, y, z, v)$. A real plane is defined by two linear equations which of course are not uniquely determined. Anyway try to formulate some condition on the equations which is equivalent with the plane being a complex line. I.e. being defined by one complex equation. (*Hint: If we write down the 6 2x2 minors of a 4x2 matrix formed by two equations, those will then up to a uniform scalar multiple depend only on the plane. They are so called Plücker coor dinates.*)

15 As above, consider S^3 projected onto $\mathbb{R}^3$ $(v = 0)$ from the northpole $(0, 0, 0, 1)$. The circular fibers of the Hopf fibration will then become ellipses in $\mathbb{R}^3$ except the fiber through the northpole which will become a straight line. By slight abuse we will refer to those as the Hopf fibers in $\mathbb{R}^3$.

a) Find the equation for the Hopf fiber which is a straight line

b) Show that no two Hopf fibers intersect

c) Show that the Hopf fibers lie on planes and that their centers coincide with the origin

d) Does every plane through the origin contain a Hopf fiber?

e) The planes through the origin are parametrized by $\mathbb{R}P^2$, the Hopf fibers by S^2. Is it possible that each plane supports two Hopf fibers?

f) Are the Hopf fibers linked?

16 Let $\mathbb{H}$ denote the (Hamiltonian) quaternions. Any element can be written under the form $\alpha = a + ib + jc + kd$ with

$$i^2 = j^2 = k^2 = ijk = -1$$

The realsubspace Im$\mathbb{H}$ is formed by those for which $a = 0$ (purely imaginary) and we have conjugation

$$a + ib + jc + kd \mapsto a - ib - jc - kd$$

denoted by α^*. Letting $|\alpha| = \alpha\alpha^*$ be the norm we may consider the quaternions of norm 1, they form a subgroup of $\mathbb{H}^*$ and define a groupstructure on S^3. They operate on Im$\mathbb{H}$ via $x \mapsto \alpha x \alpha^*$ and define special orthogonal transformations (i,e orthogonal with determinant 1 the group denoted by $\mathbf{SO}(3)$).
a) Show that we get a covering $S^3 \mapsto \mathbf{SO}(3)$ which in fact is a double covering. $\mathbb{R}P^3$ is in fact $\mathbf{SO}(3)$!
b) In fact the map is a homomorphism of groups and find its kernel.
c) Show that $\mathbf{SO}(3)$ naturally covers $\mathbb{R}P^2$ (all lines through the origin) with fibers S^1
d) Fit a) and c) into a commutative square with the missing arrows provided by the Hopf fibration and the double cover of $\mathbb{R}P^2$ by S^2

17 Show that any point P in $\mathbb{Q}P^2$ can be written uniquely up to sign with integral homogenous coordinates with no common factor. Denote by h(P) the largest absolute value of those, and refer to it as the height of P. The higher the height the more complicated the point).
a) Compute the number of points with height equal to one
b) Compute the number of points with height at most equal to h
c) If L is a line whose height (as a point in the dual space) is at most h, give an upper bound for the height of the point with smallest height on it. (Lines which are not complicated should contain points which are not complicated)
d) If L and L' are two lines of height at most h, give an upper bound for the height of the intersection point. ("Simple" lines should intersect in simple "points")

18 Show that any line in a projective space contains at least three points.

19 Is it possible to find a projective plane with $q = 6$? Compute the number of different collections of subsets of a set with 41 elements that has to be checked.

20 Find the equations of the seven lines of the simplest projective plane, and list them by making a matrix with 7x7 entries (and no diagonal), each entry defined by two points.

21 Let V be a vectorspace of dimension n over a finite field $\mathbb{F}_q$ with $q = p^n$ elements.
a) Compute the number of elements of V
b) compute the number of one dimensional subspaces of V
c) Compute in general the number of k-dimensional subspaces of V
d) Assume that $n \geq 3$ compute the number of lines that meet two given skewlines.
e) State and prove corresponding claims for projective spaces over $\mathbb{F}_q$

22 Consider the finite groups $\mathbf{PGL}(2,\mathbb{F}_q)$ which operate on the projective lines $\mathbb{F}_q P^1$ (with $q + 1$) elements.
a) compute their orders
b) identify those groups for q=2,3 and 5
c) An element of $\mathbf{PGL}(2,\mathbb{F}_q)$ acts as a permutation of $q + 1$ elements through its canonical action on the corresponidng projective lines. Give a condition (in terms of the determinant) of an element to be an even or odd permutation. (*Hint: You need to know about the concept of quadratic residues. Furthermore it may be helpful to realize that a Möbius transformation can be split up into translations $z \mapsto z + 1$, homotheties $z \mapsto az$ and inversions $z \mapsto 1/z$*)

23 Let A be a nxn matrix with entries in a finite field $\mathbb{F}_q$, compute the number of matrices B which commute with A. (*Hint: You need to do some linear algebra here, A has to be diagonalized if possible or at least put in Jordan canonical form. The answer will obviously depend on its canonical form. To do so you will need to look at some extension of* $\mathbb{F}_q$ *solve the problem over this new field and then go back somehow. It may be wise to treat the case* $n = 2$ *first*)

24 Prove Desargues theorem in case the projective plane can be embedded in a 3-dimensional projective space, by refering the degenerate case to the general. (*Hint: Construct two triangles in perspective whose projection gives the given perspective in the plane*)

25 Prove Pappus theorem by brute force. Use homogenous coordinates to normalize as much as possible (e.g the two lines can be given by $x = 0$ and $y = 0$, the line AA' by $z = 0$ the point B by $(1, 1, 1)$) and then compute. Use this setup also to give a counterexample to Pappus over a skewfield.

26 Prove that the dual of A3) follows from the three axioms

27 State the dual versions of Desargues and Pappus theorems

28 Show directly that through each point not on a conic we may draw two tangents. (*Hint: Let the point be* (A_0, A_1, A_2) *with* $A_2 \neq 0$ *and parametrize the lines through it by their intersections* $(B_0, B_1, 0)$ *with the line* $X_2 = 0$*. Plug it into a quadric and get a binary form whose coefficients depend on B, then consider the discriminant*)

29 Let (X_0, X_1, X_2) be homogenous coordinates on the projective plane and (Y_0, Y_1, Y_2) coordinates for its dual, thus corresponding to the line

$$Y_0X_0 + Y_1X_1 + Y_2X_2 = 0$$

. Given the conic

$$X_0X_1 + X_2^2 = 0$$

find the equation for the dual conic (in terms of (Y_0, Y_1, Y_2)). Try to do this in general.

30 Let S be the unit sphere in $\mathbb{R}^3$, then to each great circle we can associate the orthogonal line through the center. In this way we get an isomorphism between $\mathbb{R}P^2$ and its dual. What is the corresponding conic?

31 Let C be the unit circle in $\mathbb{R}^2$, and let p be a point at distance r from the center. Show that the polar of p with respect to C is the line perpendicular to the line joining p with the origin at a distance 1/r.

Using this show how we can transform a conic to another conic by the above inversion of its tangent lines. In particular show that any conic containing the unit circle is mapped inside the circle; and parabolas are characterized by their images going through the center.

Furthermore write down explicitly how the equations of the conics change.

32 Given the dual conic in real projective space, the conic can be recaptured as the boundary of the union of all the tangentlines. If the line at infinity does not cut it, then it is convex (obvious). In fact note that this works for the boundary of any smooth convex region.

33 If a line in real projective space does not meet a conic, show directly from the definition that all of its points are outside (Note: This is of course obvious, but not true in general for other fields!)

34 Consider the double covering $S^2 \to \mathbb{R}P^2$. Show that the inverse image of a conic splits up into two components, as does the inverse image of the inside. What happens to the inverse image of the outside?

Crossratio

The group **PGL**(2,**C**) of Möbius transformations act on the Riemann sphere, it is then customary to talk about the *"natural"* conventions regarding ∞ . But if we simply consider the sphere as $\mathbf{CP}^1$ and∞ given by (0,1) we need not allude to any ad-hoc treatment of the point at infinity.

As is wellknown, the group of Möbius transformations, acts triply transitive. In fact given any three distinct points z_0, z_∞ and z_1 there is a unique Möbius transformation sending them to $0, \infty$ and 1 respectively. Namely

$$z \mapsto \frac{(z - z_0)(z_1 - z_\infty)}{(z - z_\infty)(z_1 - z_0)}$$

(*Note that this works over any field, not just* **C**, *see exercise 35*)

This is a more precise statement than to say that **PGL**(2,**C**) is three-dimensional. We do not expect **PGL**(2,**C**) to act transitively on four elements, and thus we may ask when can we map a fourtuples (z_0, z_∞, z_1, z) onto the fourtuple (w_0, w_∞, w_1, w)? respecting the order?

Clearly if we map the points z_0, w_0 to 0 and z_1, w_1 to 1 etc ... it is necessary and sufficient that z and w are mapped to the same point, i. e.

$$\frac{(z - z_0)(z_1 - z_\infty)}{(z - z_\infty)(z_1 - z_0)} = \frac{(w - w_0)(w_1 - w_\infty)}{(w - w_\infty)(w_1 - w_0)}$$

Thus it is convenient to introduce the symbol $(z_0, z_\infty; z_1, z)$ to denote the value of z under the Möbius transformation that sends z_0 to 0, z_1 to 1 etc ... This is called the crossratio of the four points.(*Note that the order is crucial. It is also strictly speaking an element of* $\mathbf{CP}^1$, *but if the four points are distinct we need not worry about the value* ∞ *nor* 0 *and* 1 *for that matter*). If we change the order of the points, this has the effect of composing the Möbius transformation by some other Möbius transformation, and in particular changing the value of the crossratio by some element of **PGL**(2,**C**). In this way we get a representation of the symmetric group of four elements $\mathcal{S}_4$ into **PGL**(2,**C**). Now this representation is not faithful. The crossratio is easily seen to be invariant under the three permutations

$$(0, \infty)(1, z), (0, z)(1, \infty) \text{ and } (0, 1)(z, \infty)$$

they together with the identity form a subgroup H isomorphic with the *Klein vier-gruppe.* On the other hand if we fix z and only permute $0, 1$ and ∞ then we get a faithful representation of $\mathcal{S}_3$ into **PGL**(2,**C**). In fact we can write down the following useful table

permutation	value of crossratio	element of $\mathbf{PGL}(2,\mathbf{C})$
	z	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$(0,\infty)$	$1/z$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$(0,1)$	$1-z$	$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$
$(1,\infty)$	$z/z-1$	$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$
$(0,\infty,1)$	$1/1-z$	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$
$(0,1,\infty)$	$z-1/z$	$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

In fact we have implicitly shown that there is a remarkable surjection $\mathcal{S}_4 \to \mathcal{S}_3$ with kernel H. (*As anyone familiar with elementary group theory knows $\mathcal{S}_4$ is in fact the semidirect product of H (= $\mathbf{Z}_2 \times \mathbf{Z}_2$) with $\mathcal{S}_3$ with H identified as the group of automorphisms of $\mathcal{S}_3$*) We will return to this fact in another context further on.

THE $\jmath$ -INVARIANT

The problem of determining when two fourtuples of four distinct points can be mapped onto each other, with no respect of their internal order, is a harder problem. As we can see it boils down to the problem of deciding when two elements belong to the same order under the representation of the group $\mathcal{S}_3$ defined above.

It may in this context be useful to compute a few special orbits, in fact orbits that do not contain the full set of six elements that one does expect in general. To get those we equate any two "values" in the middle column. A moments reflection reveals to us that it sufficient to assume that one of those is always z hence we have five cases. $z = 1/z, z = 1-z, \ldots$. The patient reader who performs the check will then be rewarded by the following list of three "exceptional" orbits.

$(0,1,\infty), (-1,2,1/2)$ and $(-\rho,-\rho^2)$ $\rho^3 = 1, \rho \neq 1$

The first orbit only occurs if the four elements are not distinct(*cf remark after definition of the crossratio*), while the two others are of more concern to us. Let us from now on refer to them as the harmonic and the anharmonic orbit

There is now a remarkable rational function $\jmath(z)$ given by

$$\jmath(z) = \frac{(z^2-z+1)^3}{z^2(z-1)^2}$$

which is invariant under the orbits and in addition separates them.

To check this property we need only to observe two things. a) the function is invariant under the two involutions $z \mapsto 1/z$ and $z \mapsto 1-z$ b) the level sets of the function contain at most six elements. The first observation requires admittedly some computation and the realization that $\mathcal{S}_3$ is generated by any two of its involutions; the second only involves checking the maximum of the degree of the numerator of $\jmath(z)$ and 1 + the degree of the denominator.(see exercise 38)

The $\jmath$ function as it is commonly called has deeper connections than is apparent, some of which we will explore further on. Let us just remark at this stage that three of its values are "exceptional", namely $0, 1$ and ∞ corresponding to the three orbits above (*in reverse order*)

Harmonic Division

The value -1 of the crossratio is special as we saw above. We say that two points A and B are (harmonic) conjugate with respect to C and D iff the crossratio (C,D;A,B)=-1. Using the invariance of the crossratio we see that it only depends on the pairs (A,B) and (C,D) without regards to their internal order, in fact it depends only on pairs of pairs, without any reference to orders. We say then that two pairs are in harmonic division iff one pair is conjugate with respect to the other.

One may also of course talk about (equi) anharmonic divison related to the other exceptional orbit, but due to different symmetries this is not so natural.

Harmonic divisons occur in many context some of which we will discuss below, a somewhat metaphysical reason for their ubiquity is the fact that over the field $\mathbf{F}_3$ any four points necessarily are harmonically conjugate (in fact in whatever way see exercise 37) so that any universal construction that makes sense over that puny field will necessarily be restricted[1].

Given three points A,B and C on a line L; one may construct the conjugate D of C with respect to A and B by ruler in the following way.

Construction: Choose a point O outside the line L and draw the lines AO and BO, then choose an arbitrary line (not coinciding with any previously drawn line) M through C intersecting AO and BO in A' and B' respectively. Join the lines AA' and BB' intersecting in P. Draw the line OP and let D be its intersection with L.

(For a straightforward albeit tedious verification see exercise 44.)

The harmonic conjugate also appears in connection with polarities of conics. In fact,

Proposition: Let C be a conic associated to a non- degenerate quadratic form, let P be a point not on C and $\hat{\mathrm{P}}$ its polar. Let L be an arbitrary line through P which is not a tangent to C, it will meet C in two points Q and Q'. Then the conjugate of P with respect to Q and Q' is given by the intersection P' with $\hat{\mathrm{P}}$.

Proof: Let the tangents to C through P be given by $X = 0$ and $Y = 0$ and the polar $\hat{\mathrm{P}}$ by $Z = 0$, then the quadric must be of the form $XY + \lambda Z^2$. Now consider an arbitrary line through P, it may be parametrised by $s(0, 0, 1) + t(\mu, \nu, 0)$ where P corresponds to $(1, 0)$ and P' to $(0, 1)$ and Q and Q' to points (s, t) such that $(s/t)^2 = \mu\nu/\lambda$.

(*the reader who is not convinced, or is puzzled by the case of* $\mathbf{F}_2$ *may consult exercise 45 and 46*)

Involutions

The fixed points of a Möbius transformation $z \mapsto \frac{az+b}{cz+d}$ are given by the zeroes of the quadratic $cz^2 + (d - a)z - b$. This may have two distinct roots or two coinciding. In the latter case it is called parabolic, in the former we may distinguish between

[1] I emphasize that this reason should not be taken too literally

elliptic and hyperbolic, although the general such transformation is neither, but may be written as a product of an elliptic with a hyperbolic (see **Ahlfors** *Complex Analysis* (page 86) or see exercise 47)

If we normalize the fixed points to 0 and ∞ then the Möbius transformation is given by $z \mapsto \lambda z$ where λ can be intrinsically characterized as the crossratio $(0, \infty; z, Tz)$ or equivalently the quotient between its two eigenvalues. (*Note however that* λ *is only determined up to* $\lambda, 1/\lambda$ see exercise 50)

An involution is a non-trivial transformation T such that T^2 is the identity (the trivial transformation). This means that for each element p, p and Tp constitute a pair which is interchanged. (p and Tp may of course coincide, we then say that T is ramified at p) We have already encountered involutions in the context of double coverings, and they will pop up in many other contexts in the future. It is now natural to determine when a Möbius transformation is an involution. We have the following proposition

Proposition: The following conditions are equivalent for Möbius trans formations

a) T is an involution

b) T has trace zero

c) for any z not a fixed point the pair (z, Tz) is conjugate with respect to the two fixed points

d) there is a point z such that $Tz \neq z$ but $T^2 z = z$

Proof: Let $w = (az+b)/(cz+d)$ then $czw + dw - az - b = 0$ thus z and w play symmetrical roles (as they should do in the case of an involution) iff $a = -d$ i.e. Tr(T)=0. This shows a) $\Leftrightarrow$ b). The transformation $z \mapsto \lambda z$ is an involution iff $\lambda = -1$ this shows a) $\Leftrightarrow$ c) in fact the conclusion is valid if we only assume that $z = \lambda^2 z$ for a single $z \neq 0$, thus we can replace c) by d).

For an involution T we call the points z and Tz homologous and we observe that an involution never can be parabolic (see exercise 51), and we see by c) that homologous points are conjugate with respect to the fixed points of the involution. And in fact conversely if z and Tz are conjugate with respect to the two fixed points for some z, then T has to be an involution.

The above discussion has tacitly been performed over **C** as the terminology Möbius transformation should strictly speaking refer to the elements of **PGL**(2,**C**). For modifications over other fields see exercises 54 and 55.

Automorphisms of the line can also be viewed externally. More specifically let L and M be two lines meeting in A, and let O be a point outside the lines. Then O sets up a 1-1 correspondence between L and M in the usual way (a perspectivity), i. e. l and m on L and M respectively are associated if l,m and O are collinear. Now using any point P outside L and M and distinct from O, P defines an automorphism on L (or analogous on M) by associating to each l on L the point l' which is the intersection of L with the line formed by P and m (where m on M is associated to l via the above perspectivity). Such a transformation has two distinct fix points given by A and B respectively where B is on L collinear with O and P. Conversely we see that any automorphism with one or two fixpoints on L can be represented in this way by choosing the line M and the points O and P judiciously. (see exercise 56)

Exercises

35. Using the triple transitivity show directly that $\mathbf{PGL}(2,\mathbf{F}_2)$ is isomorphic with $\mathcal{S}_3$ and that $\mathbf{PGL}(2,\mathbf{F}_3)$ is isomorphic with $\mathcal{S}_4$. Furthermore show that $\mathbf{PGL}(2,\mathbf{F}_4)$ is isomorphic with $\mathcal{A}_5$, and somewhat more surprising that $\mathbf{PGL}(2,\mathbf{F}_5)$ is isomorphic with $\mathcal{S}_5$. (*Hint: Consider pairs of complementray triplets among six elements*)

36. Find the stabilizers of the harmonic and anharmonic orbits respectively

37. For small fields there is not much elbow room. In the case of $\mathbf{F}_2$ there is only one orbit, in fact one may not ever find four distinct points! Show the following for finite fields

a) The harmonic orbit always exists except when $\mathrm{char}(\mathbf{F}) = 2$ and it contains three distinct elements except when $q = 3$

b) the anharmonic orbit exists iff $q \equiv 1 \pmod 3$ and it contains two distinct elements

c) Write down the smallest prime p such that there are exactly one orbit of each kind, including the generic orbit.

38. There is of course a slight gap in the reasoning that the $\jmath$ function indeed separates orbits. A priori two small orbits maybe coalesced, prove that this indeed does not happen without having to resort to an actual check by computing the values of the $\jmath$ function.

39. What happens if we simply add all the elements of an orbit under $\mathcal{S}_3$, or multiply them? Assuming of course that we avoid the "bad" orbit $(0, 1,$ and $\infty)$ More generally what are the symmetric functions of the orbits?

40. Show that the two points α and β are conjugate to each other with respect to 0 and ∞ iff $\alpha = -\beta$.

41. Given A and B and ∞ how do you construct the "midpoint" C of A and B? (*Hint: what is the relationship with harmonic conjugates?*)

42. Given four points on a sphere making up a regular tetrahedron what is its $\jmath$-invariant?

43. Show that the crossratio of four points is real iff the four points lie on a circle or a straight line (a circle through ∞) when projected onto the complex plane. Furthermore if z and z^* are two points such that $(z_0, z_1, z_2, z) = \overline{(z_0, z_1, z_2, z^*)}$ they are said to be symmetric with respect to the circle (or straight line) C through z_0, z_1 and z_2. Show that in the case of a *bona fide* circle C, z^* lies on the polar of z, with C considered as a conic in $\mathbf{RP}^2$, and that the line joining z and z^* is perpendicular to the polar.

44. Refering to the construction of the harmonic conjugate, choose coordinates such that L is given by $X_0 = 0$, A by $(0, 0, 1)$, B by $(0, 1, 0)$ and C by $(0, 1, 1)$ say and let O be $(1, 0, \lambda)$ and let M be the line given by $\mu X_0 + X_1 - X_2$ construct the point D and in particular show that it does not depend on the choices of λ and μ. What happens to the above construction in $\mathbf{PF}_2$? Or more generally over a field of characteristics two?

45. Show that any quadric $Q(X, Y, Z)$ may be written uniquely in the form

$$XL(X, Y, Z) + q(Y, Z)$$

where L is a linear form and q a binary quadric. Furthermore $X = 0$ is a tangent iff q is a *square* . Use this to show the normal form of the quadric stated in the proof of the proposition.

46. In the case of $\mathbf{F}_2$ a line has only three points, hence any line through P has to be a tangent! This is in fact true for any field of characteristic two as can be verified e. g. by checking that any line $X - \alpha Y$ is tangent to $XY + \lambda Z^2$.

47. Show that any Möbius transformation T with fixed points a and b may be written

$$\frac{w-a}{w-b} = \lambda \frac{z-a}{z-b}$$

with $w = Tz$. If λ is positive T is called hyperbolic and if $|\lambda| = 1$ T is called elliptic. Show that any transformation T with fixed points a and b can be written uniquely as a product of a hyperbolic and an elliptic with the same fixed points.

48. Is it true that any two Möbius transformations with the same fix points commute? In fact determine the commutator to each Möbius transformation. (*Hint: next exercise*)

49. Observe that a parabolic transformation is non- diagonalizable, while the non-parabolic constitute the diagonalizable. How does hyperbolic and elliptic fit into this point of view ?

50. Show that the function $f(\lambda) = \frac{(\lambda^2+1)}{\lambda}$ is invariant under $\lambda \mapsto 1/\lambda$ and that it separates its orbits. (*cf the $\jmath$-function*). Furthermore show that for each diagonalizable Möbius transformation T, $Tr(T)^2/Nm(T)$ is an invariant among those with given fix points. How does this invariant relate to quotients of eigenvalues?

51. Show that if T is not the identity then T is parabolic iff $Tr(T)^2 = 4Nm(T)$, in particular show that an involution can never be parabolic. Furthermore show that two distinct involutions never commute, or do they?. Finally show that any two involutions are conjugate (in the group theoretical sense) (see however exercise 55)

52. Consider the notion of symmetric points as in exercise 43, show that the map $z \mapsto z^*$ is an involution. Is it a Möbius transformation? Is it an element of $\mathbf{PGL}(3,\mathbf{R})$?

53. Show that the involutions form a two-dimensional submanifold of the group $\mathbf{PGL}(2,\mathbf{C})$, in fact we may identify it with $\mathbf{CP}^2 \setminus$ (a conic). Show that there is a natural unramified double covering of the involutions by associating to each involution its two fixed points. Show that this covering is not trivial and try and determine it!

54. If a field $\mathbf{K}$ is not algebraically closed, the two fixed points of a transformation may not be defined over K but only over a quadratic extension.
a) Show that the fixed point of a parabolic transformation is always defined over $\mathbf{K}$ and hence that it is conjugate to a translation. (i. e. ∞ as the fixed point).
b) Show that any fixed point free involution of $\mathbf{PGL}(2,\mathbf{R})$ can be written in the form $z \mapsto -\frac{1}{z}$
c) Show that any transformation is conjugate to a transformation of type $z \mapsto a + \frac{b}{z}$ and that such transformations are involutions iff $a = 0$.
d) Show that any transformation is the product of two involutions. Is this representation unique?

55. Show that over a field of characteristics two, parabolic transformations and involutions coincide! In particular show that any involution has a fixed point. Is it still true that any two involutions are conjugate?

56. Recalling the notation of the construction.
a) If O and P should switch positions, what happens to the automorphism? Same

question for L and M?
b) How should O and P be situated to get a parabolic transformation?
c) Let C denote the intersection of the line OP with M. Is it true that the crossratios (B,C;O,P) and (A,B;l,l') are the same, or belong to the same orbit under the action of $\mathcal{S}_3$? In particular if l and l' are conjugate with respect to A and B (the case of an involution) is the same true for P and O with respect to B and C?
d) If L,M and O are fixed, show that different positions of P correspond to different automorphisms of L. In fact show that they constitute a subgroup of (Möbius) transformations and determine this subgroup.
e) Show that the locus of points P such that the corresponding transformation is an involution forms a line. Determine that line!

Non-singular Quadrics

Recall that a quadric C is given by the zeroes of a Quadratic form Q. This means the set $\{\ (X,Y,Z) : Q(X,Y,Z) = 0\}$ (*Note that the fact that Q is homogenous* (i.e. all the monomials have the same degree (=2)) *makes this a welldefined subset of* $\mathbf{CP}^2$) The zeroes may be non-existant, as in the case of

$$X^2 + Y^2 + Z^2$$

over the reals. Thus to get a faithful picture we need to consider zeroes not just over the field the form happens to be defined over (the field generated by the coefficients) but over the algebraic closure.

We have then the following very special case of *Hilberts Nullstellensatz*

Theorem. *Let $k = \bar{k}$ be an algebraically closed field (e.g.$\mathbf{C}$), and let Q be a quadratic form that vanishes on the zeroes of the quadratic form Q', then Q and Q' only differ by a scalar*

Furthermore we say that C a conic is non-singular iff the corresponding Quadratic form Q is associated to a non-degenerate symmetric bilinear form.

This does not work so well in characteristic 2 as the standard correspondence between symmetric bilinear forms and quadratic forms completly breaks down (see exercise 57)

We may however consider the alternate correspondence that to each quadratic form Q associates the bilinear form given by

$$(*) \qquad \sum_{i=0}^{2} \frac{\partial Q}{\partial X_i} Y_i$$

(Note: We have the Euler identity

$$2Q(X_0, X_1, X_2) = \sum_{i=0}^{2} \frac{\partial Q}{\partial X_i} X_i$$

which is trivial to prove, thus this bilinear form $(*)$ is not exactly the standard bilinear form associated to Q but differs from it by a factor 2.)

We see then in particular that if $p = (X_0, X_1, X_2)$ is a point on C the tangent to C at p is given by $(*)$. Thus we may reformulate the notion of non-singularity to say that C is non- singular iff it has a well-defined tangent at each of its points. (i. e. the gradient of Q (∇Q) never vanishes on Q). We may also note that if char $\neq 2$ then by the Euleridentity any point where $\nabla Q = 0$ must lie on C.(for peculiarities in the case of char 2 see exercise 58)

If p is a singular point of a conic C, then any line L through p meets the conic C "doubly" at that point. I. e. if we restrict Q to L we get a binary quadric which is a "square" with its zero at p. Thus in particular if C has another point q, the Q must vanish identically on the line $< qp >$. If this line is given by a linear form L_1 we can write $Q = L_1 L_2$. Over an algebraically closed field every singular conic is the union of two lines, if the field $\mathbf{K}$ is not, the lines may only be defined over some quadratic extension of $\mathbf{K}$ (see exercise 59). Conversly if Q splits as the product of two linear forms (in the algebraic closure if necessary) $L_1 L_2$ then as $\nabla Q = L_1 \nabla L_2 + L_2 \nabla L_1$ the gradient will vanish at the intersection of L_1 and L_2.(Which will be a point defined over the original field see exercise 61)

We are now ready to determine the internal "structure" of a non-singular conic

Proposition. *If C is a non-singular conic defined over some field $\mathbf{K}$ and with a point $p = (x_0, x_1, x_2)$ likewise defined over $\mathbf{K}$ then the points of C can be parametrised by three linearly independent binary quadrics*

$$(p_0(x,y), p_1(x,y), p_2(x,y))$$

defined over $\mathbf{K}$. Conversely given three linearly independent binary quadrics $(p_0(x,y), p_1(x,y), p_2(x,y))$ defined over $\mathbf{K}$ there is a quadratic form Q defined over $\mathbf{K}$ such that $Q(p_0, p_1, p_2) \equiv 0$

Proof. The first part is geometric. The lines through a point (a pencil) are parametrised by any line not going through the point as we have seen before. If the point p happens to lie on our conic this sets up a 1-1 correspondence between the pencil of lines through p and the points on the conic by associating to each line (or its intersection with the parametrising line if you so prefer) the residual intersection of the line with the conic. (Note that if p is defined over $\mathbf{K}$ then so is the residual intersection, and it coincides with p iff in the case of the tangentline at p). That the parametrizing binary forms are quadric follows in any number of ways (see exercise 62 or the example below). The second part is pure linear algebra. Consider the monomials $p_i(x,y)p_j(x,y)$ $0 \leq i \leq j \leq 2$ they constitute six binary quartic forms. However the vectorspace of binary quartics is spanned by the basis $x^4, x^3y, \ldots y^4$ of five elements, hence there must be a linear relation

$$\sum_{i,j} A_{ij} p_i p_j = 0$$

which clearly defines the quadratic form Q.

(Note: one may use this later idea to give an *ad hoc* proof of the Nullstellensatz for quadrics)

It may be instructive to see this in practice

Example: Consider the conic given by

$$x^2 + y^2 = z^2 \tag{C}$$

and the point $p = (0,1,1)$ on it. The pencil of lines through it is parametrised by the line at infinity say. Thus any point $(\lambda, \mu, 0)$ defines the line

$$s(0,1,1) - t(\lambda, \mu, 0) \tag{l}$$

If we plug in $(t\lambda, s - t\mu, s)$ into (C) we obtain the binary quadric

$$t(\lambda^2 t - 2s\mu + t\mu^2) \tag{b}$$

Observe that $t = 0$ is always a root of (b) corresponding to the point $(0,1,1) = p$ (of course); while the residual intersection is given by $t = \frac{2s\mu}{\lambda^2+\mu^2}$ or by exploiting the convenience of homogenous coordinates $(s,t) = (\lambda^2 + \mu^2, 2\mu)$ which when we plug it into (l) yields the well-known parametrisation

$$(2\lambda\mu, \lambda^2 - \mu^2, \lambda^2 + \mu^2)$$

This parametrisation maybe exploited in many ways.
One may consider it over **R** in which case we get a rational parametrisation of the circle $x^2 + y^2 = 1$ given by $x = \frac{2t}{1+t^2}$ $y = \frac{1-t^2}{1+t^2}$ which we can use to simplify an indefinite integral

$$\int R(x, \sqrt{1-x^2})dx$$

by the substitution $x = \frac{2t}{1+t^2}$ yielding

$$2\int R(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2})\frac{1-t^2}{(1+t^2)^2}dt$$

One may also note that the parametrisation is defined over **Q** thus it gives in fact a complete solution to the diophantine equation

$$x^2 + y^2 = z^2$$

yielding all Pythagorean triples by substituting integral values for the parameters of (b).(*Note that in the projective setting, as we have remarked before (ex.17), there is no difference between integral solutions and rational solutions*)

The discussion of the example above obviously applies to any quadratic. In particular we see that the solution of a diophantine problem $Q(x_0, x_1, x_2) = 0$ over say **Q** reduces to finding just one solution defined over **Q** (A so called **Q**-rational point). Sometimes a solution can be guessed (as in the example), or be (naturally) given; but in general it is hard to find.

There are however a few methods to exclude rational solutions. As solutions can be be assumed to be integral as well as the coefficients of the quadrics; we can, as one says, reduce MOD p for various p. This means simply that we consider "everything" modulo p. Let us consider the following simple example.

Example Consider the conic

$$3x^2 - y^2 - z^2 = 0$$

assuming an integral solution (x, y, z) with no common factors we may consider the residues $\bar{x}\,\bar{y}\,\bar{z}$ of x, y, z mod 3 yielding

$$\bar{y}^2 + \bar{z}^2 = 0$$

but this is clearly impossible unless $\bar{y} = \bar{z} = 0$ which would imply that $\bar{x} = 0$ as well, which contradicts the assumption that x, y, z have no common factor.

One may fear that for a given equation one may have to run it through an infinite number of test. But the following proposition shows that in fact one only needs to check a finite number of primes

Proposition. . *Any conic over a finite field $\mathbf{F}_q$ has a point $/\mathbf{F}_q$ (defined over $\mathbf{F}_q$)*

Proof. We will assume for simplicity that q is odd. Furthermore we may assume that the point (0,0,1) does not lie on the conic (otherwise we would be done!) Then by completing squares we may write the quadric as $z^2 - \lambda x^2 - \mu y^2$ and look at points $(1, y, z)$ (if necessary we may interchange x and y). Looking at the set of values z^2 and $\lambda + \mu y^2$ for z and $y \in \mathbf{F}_q$ we see that they both contain $\frac{q+1}{2}$ elements (in $\mathbf{F}_q^*$ exactly half of the elements are squares as the map $x \mapsto x^2$ has exactly two elements in its fiber) hence there must be a non-empty intersetion.

One ought to remark that this is a special case of Chevalleys theorem

Theorem(_Chevalley-Waring_**).** *Every homogenous form of degree n in m variables has a non-trivial solution over a finite field if* $m > n$

(For a proof see **Serre** *Cours d'Arithmétique* page 13)
For an elabouration of the proposition see exercise 65 and 66

In this context one should mention the so called Hasse principle (a proof of which can also be found in Serre's book, on page 73ff)

Theorem(_Hasse_**).** *If a quadratic form has solutions over each p-adic field* $\mathbf{Q}_p$ *and over* $\mathbf{R}$ *then it has a solution over* $\mathbf{Q}$

Remark The conditions are obviously necessary, the remarkable thing is that they are sufficient. (This fails for higher degrees). One may also remark that $\mathbf{R}$ is in a natural way the same as $\mathbf{Q}_\infty$ (There are many excellent references for p-adic numbers, one is the book by Serre cited above). Finally although there is *a priori* an infinite number of tests we have to apply, all but a finite number of them are automatically satisfied.(see exercise 67)

We are now ready to look at the classification of conics over some suitable fields

$\mathbf{C}$ *the Complex numbers*. In this case the quadratic forms (and the conics) split into three cases depending on their rank

rank 3 (*the non-singular case*)

$$x^2 + y^2 + z^2 = 0$$

rank 2 (*the "generic" singular case*)

$$x^2 + y^2 = 0$$

rank 1 (*the "super"singular case*)

$$x^2 = 0$$

(This all follows from completing squares)

$\mathbf{R}$ *the Real numbers*). In this case we need to know the signature (the # of positive squares - # of negative squares) in addition to the rank (*Sylvesters law of inertia*) for the case of quadratic forms, and the signature up to sign in the case of conics

rank 3 $|sign| = 3$ (*the invisible case*)

$$x^2 + y^2 + z^2 = 0$$

rank 3 $|sign| = 1$ (*the visible case*)

$$x^2 + y^2 - z^2 = 0$$

rank 2 $|sign| = 2$ (*the "point"-case*)

$$x^2 + y^2 = 0$$

rank 2 $|sign| = 0$ (*the "line"case*)

$$x^2 - y^2 = 0$$

rank 1 $|sign| = 1$ (*the double line case*)

$$x^2 = 0$$

and finally

$\mathbf{F}_q$ (*the finite field case*). In this case the quadrics depend on rank and discriminant (an element of $\mathbf{F}_q^*/\mathbf{F}_q^{*2}$ - the Legendre symbol), while in the non- singular case, the classification of conics do not depend on the latter

rank 3 (*the non-singular case*)

$$xy + z^2 = 0$$

rank 2 (*the "line"case*)

$$x^2 - y^2 = 0$$

rank 2 $\left(\frac{\lambda}{p}\right) = -1$(*the "point"case*)

$$x^2 - \lambda y^2 = 0$$

rank 1 (*the double line case*)

$$x^2 = 0$$

(As this is less standard we might supply an argument. As every conic over a finite field $\mathbf{F}_q$ has a point it has two (in fact many $= q + 1$) we may then design two of its tangents with x and y and the line joining them by z. The equation has now been normalized to $xy + \lambda z^2$, where λ is the discrimiant. The coordinate change $x' = \lambda x, y' = y$ and $z' = \lambda z$ changes the form by a scalar multiple λ which does not change the conic)

Involutions and Non-singular Conics

As we have seen that conics are parametrisable by projective lines, the Möbius transformations may have some nice representations on them. One example is given by the projection from points outside the conic. In fact

Proposition(*Fregier***).** *Given a point P outside a conic C, then for every point O on C the residual intersection O' of C with OP defines an involution on C. And conversely every involution occurs in this way*

proof. That P does define an involution is clear. (The only mote point is to show that this involution is indeed a Möbius transformtaion). Conversely given two homologous points of some involution I on C, they define two lines meeting in a point P. The involution defined by P coincide with I on four points, hence it has to be identical with I.

We call the point P the *Fregier* point of the involution. (Note that this explains exercise 53)

The two fix points of an involution now have a nice geometric interpretation. They simply correspond to the two tangency points of the two tangents to C through its Fregier point P

We can then state the following corollary

corollary. *The two points of intersection of a conic with a line through a point P are conjugate with respect to the two tangency points of the two tangents to C from P*

All of this has to be suitably modified in the case of char 2. (see exercise 71)

Given two involutions S and T then they have exactly one homologous pair in common. This common pair is nothing but the two fix points of their product.(see exercise 72). Geometrically we get it by intersecting C with the line joining the corresponding Fregier points. This also gives a geometric way of factoring an arbitrary Möbius transformation into a product of two involutions, In fact a Möbius transformation A is determined by its two fixed points and a pair (z, Az). The fix points may or may not be defined over the field, but the line F joining them is. Now we may choose any point S on F as the Fregier point of one of the factors, the other factor T is constructed accordingly: Join S with z, and let z' be the residual intersection. Now form the line $z'Az$ and let T be its intersection with F.

Exercises

57 Show that for a field $\mathbf{K}$[2]with char($\mathbf{K}$)=2 we have for each symmetric bilinear form A that

$$A(x,x) = L^2(x) \text{ where } L \text{ is a linear form}$$

and conversely that

$$B(x,y) = Q(x+y) - Q(x) - Q(y)$$

is symmetric for any quadratic form Q and any x, y. But for any Q of form $A(x,x)$ we have that $B = 0$ and for any A of form $B(x,y)$ we have that $L = 0$

58 Let Q be a quadratic form over a field $\mathbf{K}$ with char($\mathbf{K}$)=2
a) Show that there is a point $x \in \mathbf{KP}^2$ such that $\nabla Q(x) = 0$.(*Hint: In char=2, a symmetric form is always skewsymmetric! and the latter have always even rank*)
b) Show that if C is non-singular there is a unique such point x and that x lies on any tangent to C!

59 Assume that $(0,0,1)$ is a singular point of the quadric Q show that

$$Q(ta, tb, s) = t^2 q(a,b)$$

for some binary quadric q and conclude that Q splits into two linear forms iff q does i. e. q has a zero in the field

60 Show the following Taylor formula for a quadratic form Q

$$Q(x + hx') = Q(x) + 2hA(x,x') + h^2 \ldots$$

Where A is the associated symmetric bilinear form. What happens in char 2?

61 If Q is defined over a field $\mathbf{K}$ and splits as $L_1 L_2$ then show that if the coefficients of L_i are not in $\mathbf{K}$ then they belong to a quadratic extension $\hat{\mathbf{K}}$ of $\mathbf{K}$ and the forms are conjugate (under the action of the Galoisgroup Gal($\hat{\mathbf{K}}$:$\mathbf{K}$) (=$\mathbf{Z}_2$)) and in particular that the intersection of L_1 and L_2 is defined over $\mathbf{K}$.

62 If the conic C is parametrised by

$$(*) \qquad (p_0(x,y), p_1(x,y), p_2(x,y))$$

and L is the line $A_0X_0 + A_1X_1 + A_2X_2$ show that the intersection of L with C is given by $(*)$ for the values of (x,y) given by the zeroes of

$$A_0p_0(x,y) + A_1p_1(x,y) + A_2p_2(x,y)$$

in particular deduce that
a) The p_i's are of degree two
If $X_2 = 0$ is the line at infinity then
b) C is a parabola iff p_2 is a square
c) C is a hyperbola iff p_2 splits into linear factors
d) C is a circle iff p_2 is proportional to $x^2 + y^2$

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[2]We actually need to assume that any element of the field is a square to show the first case. This will be automatic in the case of an algebraically closed field, or for a finite field of characteristic 2

63 In the proof of the parametrisation of non-singular conics we tacitly assumed (as we indeed could) that the conic did not contain a line as a component. What would have happened it it had? And we had projected from a non-singular (singular) point. Furthermore if the binary quadrics are linearly dependent what can we say about the quadric that we construct?

64 In the inhomogenous case there is of course a big difference between integral solutions and merely rational. Compare the integral solutions with the rational solutions in the case of
a) $x^2 + y^2 = 1$
b) $x^2 - y^2 = 1$

65 Assume that we have found a nontrivial solution (x_0, x_1, x_2) such that

$$Q(x_0, x_1, x_2) \equiv 0 \mod p^n$$

(i. e. a solution in the ring $\mathbf{Z}_{p^n}$ *NOT* in the field $\mathbf{F}_q$) for a non-singular quadric. Show that by a suitable modification $(x_0, x_1, x_2) + p^n(x_0', x_1', x_2')$ we may write $Q(\check{x}_0, \check{x}_1, \check{x}_2) \equiv 0 \mod p^{n+1}$. (*Hint:see exercise 60 and use the fact that* $A(x, *)$ *is a non-degenerate linear form*). Note this is a special case of Hensels lemma

66 Show that the quadric $x^2 + y^2 + z^2 = 0$ has solutions over all p-adic fields $\mathbf{Q}_p$ for p odd, what about $\mathbf{Q}_2$? Try and find a quadric $ax^2 + by^2 + cz^2 = 0$ with a, b and c positive integers which has solutions for all p-adic fields $\mathbf{Q}_p$

67 Show that if p is an odd prime and p does not divide any of the coefficients of Q then $Q(x_0, x_1, x_2) = 0$ has a solution in $\mathbf{Q}_p$. Give a counterexample for $p = 2$

68 The quadric $5x^2 + 3y^2 - 2z^2 = 0$ satisfies the Hasse principle, Try and find a rational solution!

69 Recall the notion of height in exercise 17.
a) Defining the height of a parametrisation in an obvious way (the maximum height of any coefficient, assuming that they do not contain a common factor) give an upper bound on the height of a parametrisation in terms of the height of the conic and that of the point of projection on it.
b) Given a parametrization of a conic give an upper bound on the number of rational points with bounded height in terms of the height of the parametrization.
c) For a given conic with rational points, consider the minimal height of any of its points. Obviously arbitrarily complicated conics may have points with very small height. But is the reverse true, does the height of a conic in an *effective* way bound the minimal height of a rational point? In other words in order to look for a rational point by trial and error we do have a finite search. (Note for a given height there is only a finite number of conics, the minimal heights of their rational points must then have a maximum; *effectivness* means that we can *a priori* compute this, not just *a posteriori*)

70 Show that the Fregier point is outside a conic iff it has two real fix points and inside iff the fix points are not real (then necessarily complex conjugate)

71 Show that in case of char 2, any point outside the conic and the "singular" point through which all tangents pass, still defines an involution. This involution has, as we know and expect, just one fix point. Conclude also that each point, except one, lies on exactly one tangent

72 Show that if S and T are two involutions then the following are equivalent
a) z is a fix point of ST
b) Sz = Tz

73 Give an algebraic factorization of a Möbius transformation into two involutions (*Hint: see exercise 54*)

74 Show that two involutions may commute even if they have different fix points. (Note that the fix points determine the involution) (cf.exercise 51)

75 Let C be the conic $xy - z^2 = 0$ parametrised by (s^2, t^2, st)
a) Determine the involution I (in terms of s, t) defined by the Fregier point $(0, 0, 1)$ and find its two fix points
b) Determine in the same way the involution S defined by the Fregier point $(1, 1, 0)$
c) Factor I into two involutions by choosing one to be S and find the other one T (cf.the previous exercise)

76 Let C be a conic and P a point outside. Show that the locus of the points Q such that the involution of Q commutes with that of P forms the polar of P with respect to Q

Ellipses, Hyperbolas, Parabolas and circular points at ∞

As we have seen (over **C**) all non-singular conics look the same in the projective setting. In order to get the finer classical classification of the ancients, we need to provide the projective plane with somewhat more structure.

The first thing to do is to single out one particular line L_∞ and honor it with the distinction of being the line at infinity. $\mathbf{P}^2 \setminus L_\infty$ is affine space, described by affine coordiantes.(Dehomogenizing homogenous coordinates by L_∞. We can now distinguish between parabolas and other conics, as the former are those tangent to the line at infinity. In the complex setting we still cannot see the difference between an ellipse and a hyperbola, although in the real setting (when the line L_∞ should be real) the latter can be distinguished by whether they meet or not meet the line at infinity. One should also note that the two asymptotes of a (real) hyperbola are the two tangents at its intersection with the line at infinity

Still we cannot single out circles among ellipses. To get a clue we consider the equation of a circle

$$(x-a)^2 + (y-b)^2 = R^2$$

as we are used to from childhood. Letting $z = 0$ be the line at infinity and homogenize

$$(x-az)^2 + (y-bz)^2 = R^2z^2$$

and then putting $z = 0$ (i.e. intersecting at the line at infinity) we get

$$x^2 + y^2 = 0$$

the solution of which $(1, \pm i, 0)$ is independent of the particular circle. We see that a circle can be characterized by being a conic passing through two particular points. The *circular points at infinity*. Conversely we can postulate two points (which we will single out as the *circular* points,(conjugate over the reals to make it more interesting and relevant) and define the line at infinity to be the line through the two points and a circle to be a conic through the circular points. In this way we can define, on the sly, the metric notions without having to resort to some cumbersome apparatus.

The first notion we can define is orthogonality

Definition(Perpendicularity). *Two lines L and M are said to be perpendicular iff their intersections with the line at infinity are conjugate with respect to the two circular points*

We may define the foci of a (real) conic as follows

Definition(Foci). . *The two foci of a conic are constructed as follows: From each of the two circular points we draw the two tangents to the conic. Those tangents are pairwise complex conjugate and hence they intersect in four points two of which are real (the two foci) and two of which are complex conjugate*

The line joining the two foci will be called the major axis, while the line joining the two conjugate points will be called the minor axis. Note that in the case of a circle, the four tangents coalesce to two, one at each point, and their intersection will be the center of the cirle.(The coincidence of the two foci) In the case of a parabola one of the foci has coalesced with the tangency point, the major axis is clearly the line joining the remaining focus and the intersection at infinity, the minor axis may be, if one so wishes, the line at infinity

There is clearly no absolute scale, we would need to specify a unit circle to calibrate distances; but we may of course define angles between two lines L and M (note the order) by specifying the crossratio of their intersection with respect to the circular points(note that we need to order those as well).(see exercise 83)

The Space of Conics

A general conic can be written in the form

$$A_{0,0}X_{0,0} + A_{1,1}X_{1,1} + A_{2,2}X_{2,2} + A_{0,1}X_{0,1} + A_{0,2}X_{0,2} + A_{1,2}X_{1,2} = 0$$

Thus it is determined by the sixtuple $(A_{0,0}, A_{0,1}, A_{0,2}, A_{1,1}, A_{1,2}, A_{2,2})$ (which is determined up to a scalar multiple). Thus the space of conics form in a natural way a projective space $\mathbf{P}^5$. (One may compare this to the dual projective plane, where each point corresponds to a line, here each point corresponds to a conic)

A more "functorial way" would be to do the following; To each (3-dimensional) vector space V we can associate S^2V the vectorspace of symmetric bilinear forms on V. The space of conics associated to $\mathbf{P}(V)$ is then simply $\mathbf{P}(S^2V)$. Choosing a basis for V means that S^2V consists of all symmetric (3x3) matrices.*Note that in char 2, we have to be a bit more careful.*

That the dimension of the space of conics is five we can use to prove the following handy fact

Proposition. *Through any five points there is a conic passing through them*

Proof. We need only to solve five linear equations

$$\sum_{ij} A_{ij} x_i^{(\nu)} x_j^{(\nu)} = 0$$

with ν ranging from $1 \ldots 5$ corresponding to the five points

$$(x_0^{(\nu)}, x_1^{(\nu 0}, x_2^{(\nu)})$$

. Note sometimes we may have more than one solution if the linear equations happen to be dependent (see exercise 88)

In the following we will assume tacitly that we are working over an algebraically closed field (e.g. $\mathbf{C}$ the field *par excellence* although much of this goes over for any field, the discrepancies will be discuseed at the end and in the exercises)

The singular conics form a hypersurface, in fact they are characterized by the determinantal condition

$$\det \begin{vmatrix} 2A_{0,0} & A_{0,1} & A_{0,2} \\ A_{0,1} & 2A_{1,1} & A_{1,2} \\ A_{0,2} & A_{1,2} & 2A_{2,2} \end{vmatrix} = 0$$

thus they define a cubic hypersurface.
(*This can also be seen geometrically*)

And the double lines ("supersingular" conics) are characterized by matrices having rank 1, hence they are given by the vanishing of the six 2x2 minors.(In the symmetric case we can obviously throw out 3 of the nine minors as redundancies)

$$4A_{1,1}A_{2,2} - A_{1,2}^2, 2A_{0,1}A_{2,2} - A_{0,2}A_{1,2}...$$

all of which we do not bother to write down. (see exercise 90)

The dimension (4) and the degree (3) of the determinental hypersurface is easy to determine. The corresponding questions for the locus of "super-singular" conics is tougher, but can be answered in an elementary but *ad hoc* way in our case

We have not yet defined the notion of dimension in a strict formal way (and may very well never come around to doing it) but the intuitive notion should be clear. Dimension is simply the number of *"independent"* parameters to describe an object. Clearly $\mathbf{P}^n$ has dimension n in particular $\mathbf{P}^5$ has dimension 5. To give one "condition" (i.e. an equation) means to cut down the dimension by one. Naively we are "solving" one parameter in terms of the other. More precisely given a polynomial equation $F(z_1, z_2 \dots z_{n+1}) = 0$ we write it as a polynomial in say z_{n+1} with "variable" coefficients

$$f_0(z_1, \dots z_n)z_{n+1}^k + f_1(z_1, \dots z_n)z_{n+1}^{k-1} \dots f_k(z_1, \dots z_n) = 0$$

the solution (which we can of course not write down explicitly for high k) will then depend ("independently") on the remaining variables. Not also that it is very important that we work over an algebraically closed field ($\mathbf{C}$ as the examples of sums of squares illustrates over $\mathbf{R}$. The discriminating Calculus student recognises this of course as a special case of the implicit function theorem. This theorem informs us that in a small neighbourhood of a *non-singular* point the zeroes of a hypersurface are given as the graph of a function. (We need to choose the dependent parameter z_{n+1} with care in fact we need $\partial f/\partial z_{n+1}$). In particular it informs us that the zeroset forms a (complex) manifold at the non-singular points.

Now this discussion is not confined to just the case of one equation, but works with any number of equations, but for simplicity we will concentrate on the easiest case.

Thus we have defined dimension "*locally*" and with *transcendental* methods, as the diemnsion of the small neighbourhoods of a manifold. Such an approach is of course stillborn when it comes to finite fields, and hence algebraically one needs a more global (and necessarily less intuitive) definition, to which we *may* retrun later.

In the digression above we touched upon the notion of "*non-singularity* which boiled down to beaing able to choose a good direction. In other words that (in the case of hypersurfaces) the *gradient* does not vanish. This is a purely algebraic concept, which we have already encountered in our discussion of conics. We say ...

Definition (of *non-singularity***).** *A point p on a hypersurface $F = 0$ is said to be non-singular iff* $\nabla F_{|p} \neq 0$

To emphasize the algebraic and above all geometric aspect of *non-singularity* we prove the following proposition

Proposition. *Let p be a point on a hypersurface $F = 0$ and let L ($\lambda p + \mu q$ for some $q \neq p$) be a line through p. The restriction $F_{|L}$ of F to L will be a binary form that vanishes at p ($\lambda = 0$). We say that q lies on the tangentplane Π or that L is a*

tangent to $F = 0$ at p iff $F_{|L}$ has a double zero at p (divisible by λ^2). Furthermore we say that p is a singular point of $F = 0$ if $F_{|L}$ has a double zero at p regardless of L i.e. that the tangentplane "degenerates" to filling up the whole space.
The tangentplane of $F = 0$ at p is given by the linear form $(\nabla F_{|p}$ and p is non-singular iff this linear form does not "degenerate" (i.e. the gradient of F does not vanish at p)

Proof. This all follows from the Taylor formula

$$F(\lambda x_0 + \mu y_0, \ldots, \lambda x_n + \mu y_n) = \mu^n F(y_0, \ldots y_n) + \mu^{n-1}\lambda \sum_{i=0}^{n} \frac{\partial F}{\partial x_i} y_i + \ldots$$

which the reader should have little trouble establishing

(Note: We may talk about the multiplicity of a singularity as the minimum multiplicity of a line at the point, see exercise 93). We may also talk about the multiplicity of a line L at a point p, or its order of tangency, as the multiplicity of the zero at p. Thus at a non-singular point p the lines tangent at p trace out a hyperplane, the tangentplane, while at a singular point p we may talk about the tangent cone, as the totaility of lines with higher order contact than the multiplicity of p ,see also exercise 93)

We are now ready to look at our stratification of $\mathbf{P}^5$ (the space of conics) in more detail. First we may determine the singular locus of the determinental cubic hypersurface of singular conics. We have the following little lemma

Lemma. . *Let $|(X_{ij})|$ be the determinant of a square matrix whose entries should be thought of as independent variables, then its gradient is given by the corresponding (ij) minors M_{ij}*

in particular

Corollary. *In the space $\mathbf{P}^{n^2-1}$ of nxn matrices, the singular ones form a hypersurface of degree n and its singular locus consists of matrices of rank at most $n-2$*

Proof. We need only observe that if we collect all monomials in the determinental expression containing a certain X_{ij} it can be written as $X_{ij}M_{ij}$ (with the appropriate sign on M_{ij}.

(For a slight modification for symmetric matrices that we actually need, see exercise 95)

We can now state in fact

Proposition. . *The singular locus of the hypersurface of singular conics consists of "supersingular" conics*

A geometric interpretation of this will follow in the next section

As we discussed above, the notion of dimension is an intuitive one, made somewhat more precise by the notion of locally solving for the appropriate number of parameters. This can however be made global (!) in our case. We can parametrise the determinental hypersurface by the product $\mathbf{P}^2 \times \mathbf{P}^2$ as follows

For each pair of points (x_0, x_1, x_2) and (y_0, y_1, y_2) in the dual projective space, we associate the product

$$(x_0y_0, x_1y_1, x_2y_2, x_0y_1 + x_1y_0, x_0y_2 + x_2y_2, x_1y_2 + x_2y_1)$$

of linear forms considered as a quadratic form.
It is now easy to see that this gives a surjection onto the determinental hypersurface. $(\mathcal{D})$ (Over **C** any singular quadric splits into two linear forms) and that it is 2:1 except on the diagonal where it is 1:1. Furthermore the singular locus $(\mathcal{S})$ is exactly the image of the diagonal (a "supersingular" conic is the product of two identical linear forms)

Thus we have another verification that our hypersurface of singular conics is in fact 4-dimensional (it depends (globally even) on four independent parameters, pairs of lines in $\mathbf{P}^2$) But also the much less obvious fact that the singular locus is 2-dimensional. (It is parametrised by lines)

A finer invariant of a "variety" (i.e. anything defined by equations) than its dimension is given by its degree. The degree of a hypersurface is simply given by the degree of the equation. This definition does not work for the subtler cases of many equations, thus we need a geometrical reformulation of degree, which is given below

Lemma. *A hypersurface $F = 0$ is of degree n iff each line intersects it in n points counting multiplicities*

This maybe too obvious to merit a formal argument, simply restrict the equation to a line. (Note that this show how crucial it is to work over **C** when the number of zeroes actually determines the degree of the polynomial (and their positions incidentally the polynomial itself). This leads to the following definition.

Definition(of degree). *Let V be a variety (i.e. defined by equations in some projective space). Then the degree of V is given by the number of intersections with a linear variety (i.e. defined by linear equations) of complementary dimension*

Comment: If V is living in $\mathbf{P}^n$ and of dimension d then the complementary dimension is $n - d$. This definition has two weaknesses. First it may not be so easy to count points because points maybe multiple. In the hypersurface case this is not so hard as the points are coded by the restriction of the hypersurface (i.e. by a binary form) and their multiplicities may be read off from the polynomial. Second it is not so clear that this number actually is independent of the position of the linear space. Those difficulties we will postpone and use this definition provisonally for the moment.

Returning to the parametrization of the singular locus $\mathcal{S}$ we see that it is given explicitly by

$$(x_0, x_1, x_2) \mapsto (x_0^2, x_1^2, x_2^2, 2x_0x_1, 2x_0x_2, 2x_1x_2)$$

in particular we see that the six components are given by a basis of monomials for the quadratic forms in three variables (cf exercise 102 and 62). If we now take a hyperplane

$$\sum_{ij} C_{ij}A_{ij} = 0$$

its intersection with $\mathcal{S}$ will be on one hand a curve in $\mathbf{P}^5$ on the other hand it will correspond to points (x_0, x_1, x_2) in $\mathbf{p}^2$ such that

$$C_{00}x_0^2 + C_{11}x_1^2 + C_{22}x_2^2 + 2(C_{01}x_0x_1 + C_{02}x_0x_2 + C_{12}x_1x_2) = 0$$

that is points on a quadric. (Note: Each hyperplane of $\mathbf{P}^5$ reconstructs our conic by its intersection with $\mathcal{S}$). Now if we take two hyperplane, they will intersect in a linear space of complementary dimension (unless they are dependent of course) and the intersection with $\mathcal{S}$ will correspond to the intersection of two conics in $\mathbf{P}^2$. We now need the following lemma

Lemma. *Two conics intersect in four points (counted with the appropriate multiplicities)*

Proof. Let C be one conic parametrised by binary quadrics

$$(p_0(x, y), p_1(x, y), p_2(x, y))$$

plugging in the parametrization into the quadratic equation of the other conic C' we get a binary quartic whose four roots correspond to the intersection points.

Note:If C is singular, the lemma is even easier, as C splits up into two lines. One should observe that this is a very special case of **Bezouts** theorem that gives the number of intersection points for any two curves in the plane. The *ad hoc* proof given here does not work so well in general as it is hard to parametrise a curve

From the above lemma we conclude that the degree of $\mathcal{S}$ is indeed four. (Something that would have been hard to see just from the equations).

In the real case (or in the finite case for that matter) one observes that the smooth points of the determinantal hypersurface $\mathcal{D}$ splits up into two components corresponding to those singular quadrics that split over the field and those who split over a quadratic extension. This splitting into components is not "algebraic" in the sense that it does in no way correspond to a splitting of the determinental equation into two factors (see exercise 103). Furthermore in the real case the non-singular conics come in two types, and the determinental hypersurface actually separates them (see exercise 104)

Exercises

77 A circle has two given points on it (the circular points). Show that two points on the circle are conjugate with respect to the two circular points (after all the circle is a $\mathbf{P}^1$) iff the line joing them goes through the center. (A diameter-such points are of course refered to as anti-podal)

78 Show that the diameter of a circle is always perpendicular to a tangent at any of its two intersection points with the circle

79 Show that the polar of a point p with respect to a circle is always perpendicular to the diameter through p

80 Show that the center of a circle is always inside (see Duality and Conics)

81 If p and q are two antipodal points on a circle and r a third point, show that the two lines pr and qr are perpendicular

82 Show that the two axi of an ellipse (or a hyperbola) are perpendicular

83 Show that the cross ratio between the circular points $(1, \pm i)$ at infinity and the pair $(\cos t, \sin t)$ and $(\cos u, \sin u)$ only depends on $(u - t)$ hence justifying the definition of angle. Also give the expression for the crossratio in terms of $\tan(u - t)$(Note it is only in the perpendicular case we need not to specify orderings)

84 Consider the two points $(\cos t, \sin t)$ and $(\cos u, \sin u)$ on a circle, compute the crossratio with respect to the two circular points using some identification of the circle with $\mathbf{P}^1$. Does it change? Does it only depend on (u- t)?

85 Using exercise 83 define the notion of a bisector and use this to prove the following optical property of conics
Any ray emenating from one focus of a conic is reflected to the other focus. In particular the rays parallell to the major axis of a parabola all are reflected through its focus

86 The polar to a focus of a conic is called the *directrix* of the focus. Find the equation of the directrix of the focus in the left halfplane of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\text{where } a \geq b \geq 0$$

and for a point on the ellipse compare the distance to the focus and its directrix

87 Show that the space of all binary quadrics form a $\mathbf{P}^2$. Furthermore show that the discriminant hypersurface of singular quadrics is now given by a conic. If we identify the conic with $\mathbf{P}^1$ show that there is a natural 1:1 correspondence between pairs of points on the conic and lines in $\mathbf{P}^2$, or using the conic again, between points of $\mathbf{P}^2$ and binary forms (pairs of points) on the conic. In particular show how we can identify the conic with the double points on the conic. This is nothing but an orgy in tautologies that however maybe amusing and even instructive

88 Show that if at least four points of the five lie on a line there is an infinitude of conics passing through the five points. (All incidentally singular) On the other hand if at most three points lie on a line there is a unique conic through the five points

89 Show that through any three points there is a circle passing through. Defining and using the notion of "midpoint" normal to two points, construct the center of the circle

90 Show that the six quadratic forms that consititute the different 2x2 minors of a 3x3 symmetric matrix are linearly independent. However what can be said about their gradients as linear forms (in six homogenous) variables for fixed values (i.e. fixed conics).(Does this depend on whether the conic is singular or "supersingular"?)

91 Is it possible to find 4x2 matrix of linear forms such that its six 2x2 minors are identical with those for a symmetric 3x3 matrix?

92 Prove the Euler identity

$$nF = \sum_i \partial F/\partial x_i$$

for a homogenous form of degree n. In particular show that if the gradient vanishes at a point p then p necessarily lies on the hypersurface $F = 0$ except in finite characteristics when we have to exclude a few primes p, depending on n (*Hint: prove the formula for monomials*)

93 Let F(x,y,z) be a homogenous form of degree n in three variables; show that we can write F as follows

$$f_0(x, y)z^n + f_1(x, y)z^{n-1} + \ldots f_n(x, y)$$

where $f_k(x, y)$ is a binary form of degree k
In other words we have written the dehomogenous form F(x,y,1) as a sum of homogenous forms
a) Show that the point $(0, 0, 1)$ satisfies $F = 0$ iff $f_0 = 0$
b) Assuming a) show that $(0, 0, 1)$ is a singular point of $F = 0$ iff $f_1 = 0$
c) Show that the point $(0, 0, 1)$ is a point of multiplicity $\geq m$ iff $f_k = 0$ for $k \leq m$
d) write down the equation of the tangent cone for a point of multiplicity m

94 Given a homogenous form $F(x, y, z, w)$ of degree n in four variables and assume that $p = (0, 0, 0, 1)$ is a point on the corresponding hypersurface such that $z = 0$ is a tangentplane. Show that F can be written on the form

$$zw^{n-1} + f_2(x, y, z)w^{n-2} + \ldots$$

Conclude that p is a singular point on the intersection of $F = 0$ with its tangentplane at p

95 Compute the gradient of the determinant of a symmetric matrix (*Hint: the chainrule*)

96 For each (n-1)x(n-1) minor of a nxn matrix (X_{ij}) compute its gradient. For a fixed (but arbitrary) matrix those can be considered as linear forms on an appropriate projective space. Compute its rank (i.e. number of linearly independent forms) and observe under what circumstances it will drop.
You may profitably restrict this analysis to n=4 (or even to n=3)

97 The quadratic form

$$3x^2 + 5xy - 7xz + 2y^2 - 5yz + 2z^2$$

is singular, write it as a factor of two linear forms

98 Given a singular quadratic form over **R** how can you determine whether it splits into two linear forms each defined over **R** or merely splits into two complex conjugate forms?

99 The quadratic form

$$x^2 - xy + 3xz + y^2 + yz + z^2$$

splits over $\mathbf{F}_5$ find the corresponding linear forms (over $\mathbf{F}_{25}$ if necessary)(Does it split over any other prime?)

100 Given a singular quadratic form over $\mathbf{F}_q$ how can you determine whether it splits over $\mathbf{F}_q$ or merely over a quadratic extension $\mathbf{F}_{q^2}$?

101 Is it possible to find a quadratic form that splits into two linear forms only for the primes 3 and 5 say? if so give an explicit example of such a form!

102 Show that a basis for the binary quadrics gives a map of $\mathbf{P}^1$ into $\mathbf{P}^2$ (cf exercises 87 and 62)

103 If a cubic equation splits into factors, one factor needs to be linear. Give an example of six singular (non- splittable over $\mathbf{R}$) conics that do not lie on a hyperplane; and similarly for six singular splittable conics

104 Show that only one component of $\mathcal{D}$ is parametrised by $\mathbf{RP}^2 x \mathbf{RP}^2$. Show that this component meets every hyperplane, while the other component may miss real hyperplanes.Where in this picture does the invisible conics fit?

105 Compute the number of non-singular conics over a field $\mathbf{F}_q$.(*Hint: Compute the number of points on* $\mathbf{P}^5$ *and subtract the number of points on* $\mathcal{D}$ *which can be computed on each component separately*)

Pencils of Conics

Definition. *By a pencil of conics is meant a line in the space* $\mathbf{P}(S^2(\mathbf{C}^3))$ *of conics*

Thus a pencil of conics are given by the conics

$$\lambda Q_0(X,Y,Z) + \mu Q_1(X,Y,Z)$$

We note that the (four, according to the lemma of the previous chapter) intersection points of the conics Q_0 and Q_1 are common to all the conics in the pencil. Conversely any point lying on all conics must be an intersection point of any two distinct members of the pencil. Those points are called the base points of the pencil. Conversely

Lemma. *Any four points determine a pencil of conics*

In fact four points impose four linear conditions and hence determine a line in the five-dimensional space of conics.

Given a pencil of conics, then to every point outside the base points, there is a unique conic of the pencil passing through the point. Thus the entire projective plane is sweeped out by the elements of the pencil. Thus except for the base points, one may think of the elements of a pencil as level curves of a function. (A rational function given by the quotient of two quadratic polynomials) In the real case this can sometimes be the exact case. See below. The generic pencil has four distinct base points. Let us denote them by p_0 p_1 p_2 and p_3. To those four points there are three singular members associated, namely to each partition of the four base points into two disjoint pairs. Those three singular members corresponds to the intersection of the line with the discriminental hypersurface, which we have seen is in fact of degree three.

Given a conic C in a generic pencil we have two sets of four points on $\mathbf{P}^1$. Namely on one hand we have C (isomorphic to $\mathbf{P}^1$) together with the four base points of the pencil, on the other hand we have the pencil itself with the three singular members and C (now considered as a point on the parameterspace ($\mathbf{P}^1$)). It is now a natural question whether those four points determine the same $\jmath$-invariant. In fact this is so and we can explicitly determine a map from the conic C to the parameter space $\mathbf{P}^1$ which carries one set of four points onto the other.

Construction. . *For each point P on the conic C we associate the conic* $\varphi(P)$ *of the pencil which is tangent to the line* Pp_0 *at* p_0*. In this way the three base points* p_1 p_2 *and* p_3 *correspond to the three singular members, while* $\varphi(p_0)$ *corresponds to C.(see exercise 108)*

The association of three points (the singular members, or if you prefer the singular points of the singular members as points in the plane) to any four points is yet another instance of the remarkable surjection between $\mathcal{S}_4$ and $\mathcal{S}_3$. In fact any permutation of the four points induces a permtation of the three points. Algebraically this is related to the fact that a fourth degree equation can be reduced to a cubic equation. (This can also be illustrated geometrically,see exercise 109)

Now two conics may intersect not transversally but be tangent, this means that they have three base points of which one is counted with multiplicity two. Through three points we may only draw two singular conics, thus the pencil intersetcs the discriminant hypersurface in only two points, one which has to be counted with multiplicity two, geometrically corresponding to a point of tangency. If the three points are given by $2p_0\ p_1\ p_2$ and the two singular members by $(p_0p_1)(p_0p_2)$ and $(p_1p_2)T_0$ (where T_0 is the common tangent to all the conics in the pencil) one of those has to correspond to the double zero. Which one can be seen by a simple geometrical argument, which is actually more visual than rigorous as we have not introduced any metric, if we let p_0 split up into two very close points $p\prime_0\ p_0$ then we will have three singular members, two of which are very close, and which "in the limit" converge to $(p_0p_1)(p_0p_2)$.(see exercise 111)

A further degeneration occurs if the two conics in the pencil are flexed. Then we have only two intersection points $3p_0$ and p_1 and only one singular member T_0p_1 (with T_0 the common tangent at p_0 of the conics in the pencil). This singular member is of course a triple root of the discriminant cubic; and geometrically the pencil is "flexed" to the discriminant hypersurface.(see exercise 113 and 114)

Now two conics maybe bitangent. That means that they intersect in the points $2p_0\ 2p_1$. We have now two singular fibers $2(p_0p_1)$ which is a double line and T_0T_1 where T_i is the common tangent at p_i of the conics in the pencil. Now geometrically the pencil intersects the singular locus of the discriminant hypersurface, the double root now corresponds to the double line.

Finally two conics maybe what one calls "hyperflexed", they then intersect in a single point $4p_0$ and there is only one singular fiber $2T_0$ which occurs with multiplicity three in the discriminant. Geometrically the pencil is tangent to the singular locus. (see exercise 115)

We have now presented the classification (over **C**) of all pencils of conics with the generic member smooth. There are five cases, all classified by the intersection behaviour with the discriminant hypersurface, or by the multiplicities of their base points. We note that any two pencils of the same type are projectively equivalent. (see exercise 116)

	describtion	*basepoints*	*discriminant*	*normal forms*
I	*generic*	$(1,1,1,1)$	$(1,1,1)$	$< x^2-y^2, xy-z^2 >$
II	*tangent*	$(2,1,1)$	$(2,1)$	$< x^2-z^2, xy-z^2 >$
III	*flexed*	$(3,1)$	(3)	$< x(z+y)-z^2, xy-z^2 >$
IV	*bitangent*	$(2,2)$	$(2^*,1)$	$< xy+z^2, xy-z^2 >$
V	*hyperflexed*	(4)	(3^*)	$< x(x+y)-z^2, xy-z^2 >$

To this list we ought to add those pencils which only contain singular members (see exercise 117)

Over the reals the cases I,II ... V splits into several subcases. let us consider those one by one

I)

We have three subcases

a) All four base points are real

This gives a faithful picture of the complex case. All the singular fibers are defined over the reals, as well as their components. They are so to speak visible.

b) Two base points are real the other two complex conjugate

Two of the singular fibers are complex conjugate, the remaining is real and visible. Of the three associated points one is real the other two complex conjuagete

c) No base points are real, they come in two complex conjugate pairs. All of the three singular fibers are real, but only one is visible, in the case of the other two only the singular points are real

II

We have two subcases

a) All base points are real

The two singular fibers are both real and visible as well

b) The tangency point is real, the other two points are complex conjugate

The two singular fibers are both real, the simple fiber is visible while the double is invisible. (All what we can see is the tangency point)

III

Everything is real. The base points and the one singular fiber, which of course is visible

IV

We have two subcases

a) All base points are real

Everything is real, the singular fibers are real and visible

b) The two base points are complex conjugate

The two singular fibers are real, but only one - the double line, is visible

We will now comment on case Ic) then the conics of the pencil corresponds to the levelcurves of a welldefined real rational function on $\mathbf{RP}^2$ taking values in $\mathbf{RP}^1$. The two invisible singular fibers will disconnect the circle ($\mathbf{RP}^1$) into two components, the one component containing the visible singular fiber will be singled out. This component corresponds to the visible conics of the pencil, or alternatively to the values of the rationalfunction (given as the quotient of two conics of the pencil). Thus one singular point of an invisible singular fiber corresponds to the "minimum" value and the other to the "maximum" value. The visible singular fiber corresponds to a "critical" value. (for further comments see exercises 118-126)

Given a line L that does not pass through any base points, the intersections of the members of a pencil form a pencil of binary quadrics on L. Such a pencil will in general have two singular members (double points)(the discriminant hypersurface is in this case a conic in $\mathbf{P}^2$) these will correspond to members of the pencil which are tangent to L. A pencil of binary quadrics having only one singular member is necessarily one with a fixed point, this will occur iff L passes through one of the base points.

Exercises

106 Given the two conics $x^2 + y^2 - 2z^2$ and $xy - 2x^2 + yz$ find the conic in the pencil spanned by them that
a) passes through the point $(2, 1, 1)$
b) is tangent to the line $x - 2y + z$ at the base point $(1, 1, 1)$

107 Given the five points $(1, 0, 2), (1, 1, 3), (1, 8, 2), (0, 4, 3)$ and $(5, 1, 2)$find the conic passing through them.(*Hint: Take four of the points, find two singular members passing through them, consider the pencil spanned by them, and force a member of it to pass through the fifth point*)

108 Given an explicit conic C $x^2 + y^2 = z^2$ and an explicit pencil generated by C and C' (xy) write down the correspondence between a point $(t^2 - s^2, 2st, t^2 + s^2)$ on the conic and the pencil $\lambda C + \mu C'$

109 Let $t^4 + pt^2 + qt^+ r = 0$ be a a quartic equation.(Any quartic can easily be reduced to one missing the t^3 term by the traditional trick). Consider the conic C $XZ - Y^2$ parametrised by $(1, t, t^2)$
a) Show that the basepoints of the pencil spanned by the conics C and C' $Z^2 + pY^2 + qYZ + rX^2$ are given by the quartic above.(*Hint: Plugging in the parametrisation of C into the equation of C'*)
b) Show that the singular elements of the pencil C+λC' correspond to a cubic equation in λ. Write down this cubic equation explicitly!

110 Given two non-singular conics C and C'; determine when they can be transformed (by projective transformations) into each other, preserving their four base points

111 The two conics C $XY - Z^2$ and C' $X^2 + 2XY - Z^2$ are tangent at the point $(0, 1, 0)$. Find the two other intersection points and determine the two singular members. Finally proceed as in exercise 109 and verify which singular member corresponds to the double root of the cubic

112 If two smooth conics C and C' are tangent, show that there is always a projective transformation that carries one into the other preserving their three intersection points

113 Verify that the two conics C $XY - Z^2$ and C' $X^2 + XZ - Z^2$ are flexed. Find the singular fiber and verify that it is indeed a triple root of the discriminant

114 Given a (smooth) hypersurface F(x,y,z,w) in $\mathbf{P}^3$. Normalizing such that p $(0, 0, 0, 1)$ lies on the surface and $z = 0$ is the tangentplane (cf exercise 94). The tangent lines through p are then given parametrically $p + t(x, y, 0, w)$ Show that such a line has "contact" three at p iff (preserving the notation of exercise 94) $f_2(x, y) = 0$. Generalize this to hypersurfaces in higher dimensions!

115 Show that the two conics $XY - Z^2$ and $X^2 + XY - Z^2$ are hyperflexed at the point $(0, 1, 0)$. The one singular fiber is given by X^2. Verify directly that the pencil spanned by them is tangent to the singular locus at the "point" X^2.(*Hint: cf exercise 90, compute the gradients of the six minors and show that they all contain the pencil*)

116 Show that there exist a unique projective transformation that carries a given set of four distinct points into another set of four distinct points. (preserving the ordering of the points) (**PGL**(3,**C**) acts four-tuply transitive). Conclude that any generic pencil can be carried into any other generic pencil. This is however not unique, show that it becomes so if we insist on a given ordering of the three singular members. Generalize this to any set of four points (not necessarily distinct)

117 Show that if a pencil contains two double lines x^2 and y^2 say then every member of the pencil is singular. Show that this corresponds geometrically that any secant of the singular locus of the discriminant hypersurface lies in the discriminant. (It fails to fill up the whole space $\mathbf{P}^5$ as one would expect for dimension reasons). Finally show that this is the only type of singular pencil with isolated base points; show that all the oethers have one component in common.

118 Show that a real pencil with some real base point can never contain any invisible conics. Is the converse true? If every conic of the pencil is visible then there has to be real base points? (How does this relate to exercise 104?)

119 Given a real pencil with four real base points. Then every point outside defines a conic in the pencil. Some of those will correspond to ellipses others two hyperbolas, and provided none of the base points lie at the line at infinity; some will correspond to parabolas.
How does that partition correspond to the four base points?

120 Given a real pencil and a real line L not passing through any of the base points. Is it always possible to find an element of the pencil that does not intersect L?(*Hint: Look at pencil of binary quadrics, do they always contain positive definite quadrics?*)

121 Given two members Q_0 and Q_1 of a pencil, and look at the function

$$\frac{Q_1}{Q_0} : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

locally. Compute the Jacobian of this map and determine when it is zero and the corresponding values for the function. How do these correspond to the singular members of the pencil spanned by Q_0 and Q_1?

122 Show that if two members of a pencil are circles, so are all members (except of course the singular)

123 Let C_0 and C_1 denote two disjoint circles not enclosed in each other. The pencil they span corresponds to Ic). What is the relationship between the two singular points and the one component of the visible singular fiber which is not the line at infinity? How can the associated function be interpreted?

124 Concentric circles correspond to a pencil of type IVb) where the singular point of the invisible singular fiber corresponds to the common center. If two circles are not concentric there will be a real line (not the line at infinity and hence visible to us) associated. If they intersect, the line is obvious, if not less so. How can the line be constructed?

125 If two ellipses meet in two points, there are two lines associated (the components of the visible singular fiber) one is obvious, what about the other? Find this line explicitly if the two conics are $x^2 + y^2 - z^2$ and $4x^2 - 12yz + 12y^2 - 5z^2$

126 If we have a pencil of type IVb) the rational function given by the quotient of any two of its members will be welldefined. There will however be just one critical point (the singular point of the invisible singular member). What possible values can this function assume?

127 Given two disjoint real conics, not enclosed in each other, and let L be a line outside them both. Those two conics split up the parameter space $\mathbf{RP}^1$ into two components; show that each component contains a member tangent to L! Visualize this by expanding levelcurves!

128 Find the two conics passing through the four points (1,0),(0,1)(-1,0) and (0,-1) and tangent to the line $x + y + z = 0$ (*Hint: find the pencil passing through the four points and restrict it to the line*)

129 Find the circle flexed to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point $(a \cos t, b \sin t)$ and determine its center. For what points on the ellipse are the flexed circles actually hyperflexed?

130 Preserve the setting of the previous exercise. At each point P on the ellipse we can correspond the residual intersection point with the circle flexed at P. (Actually the circle of curvature at P) .This defines a map from $\mathbf{P}^1$ to itself. Determine this by making a rational parametrisation of the ellipse. (Given the trigonometric parametrisation, what is the map?)

131 Find the circle tangent to the unit circle at the point $(1/\sqrt{2}, 1/\sqrt{2})$ and passing through the point (2,4)

132 Find the conic hyperflexed to the unit circle at the point (0,1) and passing through the point (0,-2)

133 Consider two conics defined over the rationals but with no rational intersection points. The three associated singular fibers may not be defined over $\mathbf{Q}$ bt will they be defined over some quadratic extension of $\mathbf{Q}$?

134 Compute the number of lines in $\mathbf{F}_q\mathbf{P}^5$ over a finite field with q elements. This number corresponds to the number of different pencils of conics. Partition this number into those corresponding to types I,II ...V.(*Hint: Those types splits up into many subtypes over a finite field*)

Quadric Surfaces

By a quadric surface is meant a quadratic hypersurface in $\mathbb{P}^3$. As in the case of conics we may to each point p associate the polar to p given by the gradient at p. If p lies on the quadric the polar gives the tangentplane and the quadric is non-singular iff the gradient never vanishes. (for more details see exercises 135-137)

Over $\mathbb{C}$ the quadrics are classified by the dimension of their singular locus (which is always a linear subspace) or equivalently by the rank of the associated bilinear form. Thus

equation	rank	*description*
$x^2 + y^2 + z^2 + w^2$	4	*non-singular*
$x^2 + y^2 + z^2$	3	*cone over non-singular quadric*
$x^2 + y^2$	2	*two planes*
x^2	1	*double plane*

Note that all singular quadrics are cones, the more singular the lower the dimension of the quadric over which it is a cone

In the real case we get a slightly more involved classification as we need the absolute value of the index as an additional invariant. (Sylvesters law of inertia). We can then present the following table

equation	rank	index	*geometric description*
$x^2 + y^2 + z^2 + w^2$	4	4	*invisible*
$x^2 + y^2 + z^2 - w^2$	4	2	*"ellipsoid"*
$x^2 + y^2 - z^2 - w^2$	4	0	*"one-sheeted hyperboloid"*
$x^2 + y^2 + z^2$	3	3	*"point"*
$x^2 + y^2 - z^2$	3	1	*"cone"*
$x^2 + y^2$	2	2	*"line"*
$x^2 - y^2$	2	0	*two planes*
x^2	1	1	*double plane*

The real classification can be refined if we introduce the analogy of circular points and the ensuing plane at infinity

We will speak about the *circle* as an invisible conic defined over the reals.The plane defined by it will be denoted the plane at infinity, and we will define a sphere to be a quadric intersecting the plane at infinity along the *circle* .

The case (4,2) will then split up into three cases depending on whether
the quadric intersects the plane at infinity
two-sheeted hyperboloid
the quadric is tangent to the plane at infinity
(then necessarily (see below) intersecting in just one point)
paraboloid
the quadric is disjoint from the plane at infinity
ellipsoid

While the case (4,0) will split up into two
the quadric is tangent to the plane at infinity
(the necessarily the intersection consists of two lines)
the hyperbolic paraboloid ?
the quadric is not tangent

one-sheeted hyperboloid

We will now present the following important lemma

Lemma. *Let S be a hypersurface and let T be the tangentplane at a (non-singular) point p then the intersection of S with T is singular at p*

This implies the following over $\mathbb{C}$ for a quadric

Corollary. *Through each point p on a non-singular quadric we may find two distinct lines!*

Proof. Letting p=(1,0,0, ...) and letting x_1=0 be the equation of the tangentplane at p we may write the hypersurface as follows

$$x_1 x_0^{n-1} + f_2(x_1, \dots)x_0^{n-2} + f_3(x_1, \dots)x_0^{n-3} \dots$$

Setting x_1=0 we see that the restriction has no linear terms when dehomogenized at p, which is equivalent to being singular

For the corollary we need only spell this out

Proof.

$$xy + f_2(y, z, w)$$

and observe that the binary quadric $f_2(y, z, w)$ is a perfect square iff the quadric is singular

Note that the two lines (in the notation above) through p are given by y=0 and $f_2(0, z, w)$=0
Now there is “no monodromy” there are actually two families of lines on a non-singular quadric

This can be seen as follows. Fix a point p_0 and denote the two lines on the quadric Q through p by L_1 and L_2 and let Π be the plane spanned by them (the tangentplane to Q at p_0). For each point p a line L through p will intersect Π in one point, this point cannot be p_0 (see exercise 138)if it would be we both lines through p would intersect one of the lines L_i and hence we would have three lines in a plane which is impossible on a quadric, thus exactly one of the lines through p intersects L_1 and the other intersects L_2, this gives the desired decomposition of the lines on a quadric into two families. Furthermore we observe that lines in the same family are always mutually skew, while two lines from different families always intersect. The first statement is clear, as lines in one family are characterized by meeting a given line, and intersection would imply three coplanar lines, for the second statement we consider the intersection of the two planes spanned by L_i and the respective intersecting line; these planes intersect in a line that intersects the quadric in two points, one is p_0 the other must be the sought for intersection point!

We have in fact proved the following proposition

Proposition. *Every non-singular quadric over* $\mathbb{C}$ *is given by a product* $\mathbb{P}^1 \times \mathbb{P}^1$

T. o each point p we can associate two lines intersecting L_1 and L_2 in p_1 and p_2 respectively, conversely given a pair (p_1,p_2) on $L_1 \times L_2$ we choose the residual lines L’ (at p_1 to L_1) and L” (at p_2 to L_2) they intersect in a point p.

A normal form for non-singular quadrics which makes the existence of lines particularly obvious is given by

$$XY - ZW$$

And the two families of lines will be given by

$$\alpha_0 X = \alpha_1 Z \quad \alpha_1 Y = \alpha_0 W$$
$$\text{and}$$
$$\alpha_0 X = \alpha_1 W \quad \alpha_1 Y = \alpha_0 Z$$

The form $XY - ZW$ is determinental i.e. given by the determinant

$$\begin{vmatrix} X & Z \\ Y & W \end{vmatrix}$$

Thus if we consider the projective space $\mathbf{P}(\mathbf{M}_{2,2})$ of non-zero 2×2 matrices, the determinental condition gives all singular matrices M to which we may uniquely associate the pair (kerM,ImM) of one-dimensional subspaces, and conversely to a pair (K,I) of one-dimensional subspaces we may associate a unique matrix (up to homotie) M with K=kerM and I=ImM. A pair of one-dimensional subspaces of a two-dimensional space is of course just a point of $\mathbb{P}^1 \times \mathbb{P}^1$. This gives another proof of the above proposition.

Finally we may consider the Segre embedding. A point $(x_0, x_1; y_0, y_1)$ of $\mathbb{P}^1 \times \mathbb{P}^1$ maybe considered as given by "bi-homogenous" coordinates. More precisley we say that

$$(x_0, x_1; y_0, y_1) \backsim (x_0', x_1'; y_0', y_1')$$

iff there are λ and μ non-zero such that

$$x_0 = \lambda x_0'$$
$$x_1 = \lambda x_1'$$
$$y_0 = \mu y_0'$$
$$y_1 = \mu y_1'$$

With this convention the following map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ given by

$$(x_0, x_1; y_0, y_1) \mapsto (x_0 y_0, x_0 y_1, x_1 y_0, x_1 y_1)$$

is welldefined. Note also that the image satisfies

$$(x_0 y_0)(x_1 y_1) = (x_0 y_1)(x_1 y_0)$$

i.e. $XW = YZ$ with the obvious notational convention.

In the real case a tangent plane may or may not intersect the quadric in two real lines. This does not depend on the point but on the quadric (see exercise 140). The quadrics of type (4,0) will in fact be the real quadrics which maybe identified with $\mathbb{RP}^1 \times \mathbb{RP}^1$, while the quadrics of type (4,2) will have no real lines, the tangent planes then intersect the quadrics in just one point (the visible singular points of an "invisible" singular conic).

Quadrics as double coverings

Given a quadric $\mathbb{P}^1 \times \mathbb{P}^1$ we have a natural involution given by $(x, y) \mapsto (y, x)$ and with the diagonal as fixed point locus. The natural question is what is the quotient of $\mathbb{P}^1 \times \mathbb{P}^1$ by this involution. To explain this it is natural to look at a conic in the plane. Pairs of ordered points on the conic form of course a $\mathbb{P}^1 \times \mathbb{P}^1$, while the quotient consists of pairs of unordered points. But to each ordered pair of points we can associate an un-ordered, simply by considering the line through the two points. The lines form a dual $\hat{\mathbb{P}}^2$, which may howevere be canonically identified with $\mathbb{P}^2$ via the conic itself. Thus we have identified the quotient to be the projective plane.(cf exercise 149)

Conversely a double covering of $\mathbb{P}^2$ branched along a conic is a quadric. The geometry of a quadric can also be gleaned from this picture. The inverse images of lines will be conics, and those conics will split up into pair of lines exactly when the lines are tangent to the conic. Algebraically we are considering an equation

$$w^2 = q(x, y, z)$$

with q a ternary quadric defining the branch locus, conversely any representation of a quadric as above defines a double covering onto the plane w=0 by projection from the point (0,0,0,1). Given a line (x(s,t),y(s,t).z(s,t)) with x,y and z considered as linear forms in s and t, the inverse image is then a conic given by $w^2 = r(s, t)$ The line is tangent iff the binary quadric r becomes a perfect square (i.e. has a double root) u^2 say in which case the conic $w^2 = u^2$ splits into two lines

This point of view is illuminating in the rational case. Viewing a quadric means that we look at it from some vantage point p and see it projected. The image will then have a contour (the ramification locus) and this contour will embrace the extent of the quadric, one part of which is visible, while the other part is hidden.

Considering the real equation

$$w^2 = q(x, y, z)$$

the form $q(x, y, z)$ defines a conic (the contour), in the real plane a conic disconnects the plane in an inside and an outside. (The inside is incidentally topologically a disc, while the outside is a Möbius strip cf exercise 34) On the other hand the covering is only defined where q is positive, this may be either on the inside or the outside. If it is positive on the outside the covering is defined over the tangents and we get a real quadric of type (4,0) all of whose lines (or rather half of each line) we can see as the tangents of the contour, topologically this is of course a torus $(S^1 \times S^1)$; on the other hand if it is positive on the inside there are no lines, and topologically we are looking at a sphere (S^2)

Hyperplanesections and Möbius transformations

An intersection of a non-singular quadric surface Q and a plane H (a hyperplane section) is obviously a conic and the conic is smooth iff the plane is not tangent to the quadric. In particular iff the plane is the polar of a point outside the quadric. If we use the structure of the quadric as a product of two lines as exhibited by the Segre embedding above, we can be more specific. Then every smooth hyperplane section defines a correspondence between two intersecting lines (say L and L') as follows. To each point p on L consider the unique line M skew to L' passing through p. M intersects the plane H in a unique point q and through q we may find a unique line M' on Q skew to L. M' will then intersect L' in a point p'. If we choose a distinguished hyperplane Δ we will then define the 'diagonal' and identify L and L'. Then we can identify each hyperplane with a Möbius transformation, and conversely each Möbius transformation considered as a graph on $\mathbb{P}^1 \times \mathbb{P}^1$ defines a hyperplane as follows

If Γ is the graph in $\mathbb{P}^1 \times \mathbb{P}^1$ it is given in terms of bihomogenous co-ordinates $(x_0, x_1; y_0, y_1)$ as

$$\frac{y_1}{y_0} = \frac{ax_1 + bx_0}{cx_1 + dx_0}$$

or as the zeroes of the bihomogenous form

$$bx_0y_0 + ax_1y_0 - dx_0y_1 - cx_1y_1$$

of bidegree (1,1)(i.e. by considering the form in terms of x and y respectively it will be homogenous in both cases of degree 1; or simply speaking it will be bilinear!). The space of bilinear forms is spanned by the four mixed monomials x_iy_j which also form the basis X,Y,Z and W of the Segre embedding. Thus the bilinear form is nothing but the restriction of the hyperplane

$$bX - dY + aZ - cW = 0$$

to the quadric $XW - YZ = 0$ Thus we see that the complement of a non-singular quadric is a homogenous space for the group **PGL**(2,$\mathbb{C}$) and by chosing a distinguished point δ we get an identification (see exercise 154)

Furthermore the number of fixed points of a Möbius transformation is then geometrically given by the intersection of the quadric with the hyperplanes Δ and H (defined by the transformation). The intersection of the two planes is a line which will intersect the quadric in two points, which may coincide of course. Looking at it on Δ we see a conic C (the intersection of Δ with Q) and H is given by a line, thus the hyperbolic H corresponds to planes intersecting Δ along a tangent to C (cf exercise 155)

Line correspondences

Given two skewlines L and L' and an isomorphism $\phi : \text{L} \to \text{L'}$ we may to each pair of points $(p, \phi(p))$ associate the line M(p) passing through the two points, as p runs through L, the lines M(p) trace out a surface S(ϕ). This surface turns out to be a non-singular quadric surface.(Cf exercise 159)

Geometrically this can be seen heuristically as follows. Let Π be a plane containing one of the lines L, then the intersection of S(ϕ) with Π consists of L and another line spanned by the pair $(p, \phi(p))$ where p is the intersection of L' and Π, thus of degree 2. The weakness of such an argument is that either of the lines may have hidden multiplicities which would yank up the degree.

Algebraically we may choose co-ordinates such that L is given parametrically by $(x, y, 0, 0)$ and L' by $(0, 0, z, w)$ and $\phi(x, y)$ by $z = ax + by$ and $w = cx + dy$. Then we consider lines parametrically given as

$$(sx, sy, atx + bty, ctx + dty)$$

thus

$$\frac{aX + bY}{cX + dY} = \frac{Z}{W}$$

giving the quadratic relation

$$aXZ + bYZ - cXW - dYW = 0$$

Given two skewlines L and L' then to any point p outside the two lines there is a unique line M through p and intersecting both L and L'. To constrcut this line we simply take the intersections of the two planes spanned jointly by p and L and L' respectively.(see exercise 160). Thus given a third skewline L" it defines a so called correspondence between L and L' by to each point p" on L" associating the line through p" and L and L'. Those lines will, as we saw above, sweep out a smooth quadric. Thus we have proved that through any three skew lines there is a unique smooth quadric containing them

Conic correspondences

Given two conics and an isomorphism ϕ between them one may ask when the lines $\langle p, \phi(p)\rangle$ trace out a quadric. Now the two conics may not be arbitrary, any conic on a quadric is given by a hyperplane section and thus any two conics on the same quadric will intersect in two points. Furthermore the lines on a quadric always make up one of the rulings on the product $\mathbb{P}^1 \times \mathbb{P}^1$ thus the isomorphism ϕ cannot be arbitrary but must respect the two intersection points and fix those. Thus we have only one "freedom" left and it is natural to state and prove the following proposition

Proposition. . *Given two conics C and C' intersecting in two points p_1 and p_2 and a transformation ϕ:C $\rightarrow$ C' respecting p_1 and p_2 and mapping p_0 to q_0, then there is a unique quadric Q containing C and C' and the line M=$\langle p_0, q_0\rangle$. Furthermore the quadric Q is swept out by all the lines $\langle p, \phi(p)\rangle$*

Proof. The simplest proof is given by a straightforward algebraic construction of the equation of Q. We may assume that the planes spanned by C and C' are given by x=0 and y=0 respectively, and that the conics themselves are given by the additional equations q_2(y,z,w)=0 and q'_2(x,z,w) respectively. The equation of the quadric must then be of the following two forms simultaneously

$$xL_1(x, y, z, w) + q_2(y, z, w)$$

$$yL'_1(x, y, z, w) + q'_2(x, z, w)$$

where the subscripts gives the degrees of the forms. We get as compatibility condition that q_2(0,z,w) is proportional to q'_2(0,z,w) i.e they define the same two zeroes on the line of intersection given by x=y=0. The two conditions then determine all the monomials except xy and we can write the equation as

$$Q(x, y, z, w) + \lambda xy$$

where λ is to be determined. Choosing a point on the line $\langle p_0, q_0 \rangle$ not on either conic gives an additional condition which will determine λ and hence the quadric Q uniquely. This quadric will by construction contain C and C' and also the line M as it contains three of its points already. Now there will be a correspondence between C and C' given by all the lines of Q skew toM, this correspondence will coincide with ϕ at three points hence be identical with it, thus we see that Q is indeed sweeped out by the lines $\langle p, \phi(p) \rangle$

The proof of the above proposition has a nice corollary. One may ask when two conics C and C' are projections of each other, i.e. when there is a point p such that the lines from p to C also pass through C'. Thus this is equivalent with C and C' being two hyperplane sections of a quadric cone. Hence a necessary condition is that they intersect in two points, this is also sufficient, as we see from the above proof that we need only to choose λ such that Q is singular.(The zero of a quartic equation given by setting the determinant of the associated bilinear form as a linear function of λ equal to zero)

Thus we see that from a special vantage point a correspondence between two conics is given by one on a fixed conic C. The two intersection points then will correspond to the two fixed points. The lines will then correspond to chords $\langle p, \phi(p) \rangle$ of an automorphism ϕ on C, where the chords are tangents in the case of the two fixed points (which may of course coincide). Now the chords define a curve in the dual space, this curve is of degree two. In fact a degenerate case is that of an involution the chords all pass through one point (the Fregier point), now an involution I will have two images common with an arbitrary Möbius transformation T (the fixed points of IT^{-1}), this means that two of the chords will pass through any given Fregier point, in other words a line in the dual (the pencil of lines through a point) intersects the curve in two points. Thus the chords form a dual conic, and its dual will be a conic in the plane of C with the property that it is tangent to all the chords of the automorphism ϕ. The interesting question is what conics may occur in this way. As there is a five dimensional family of conics and only a three dimensional family of Möbius transformations. The answer is given below

Proposition. *Let C be a conic (in the plane) and let ϕ be a Möbius transformation, then the lines $\langle p, \phi(p) \rangle$ constitute the tangents of a conic $\hat{C}$. This conic is bitangent to C and the tangency points are given by the fixed points of ϕ. $\hat{C}$ degenerates to a double line iff ϕ is an involution. Conversely any conic C' which is bitangent to C occurs in this way*

Note that $\hat{C}$ is just the contourline of the quadric when viewed from a special point

Exercises

135 If p is a point and Π a plane through p and Q is a (non-singular) quadric, show that the polar of p to Q intersects Π at the polar of p to the restriction Q_Π of the quadric Q to Π. Conclude that the lines through p tangent to Q are tangent to Q along a conic.

136 Show that given a line L outside a quadric Q there are exactly two tangent planes to Q containing L. Conclude that the tangentplanes to a quadric form a quadric in the dual space.

137 Show that each line L outside a quadric Q defines a line L' according to exercise 130. Show that this correspondence is an involution

138 Show that any line on say a hypersurface must lie in the tangentplane of any of its points. In particular conclude that through a non-singular point p of a surface all the lines through p lie in a plane, and consequently through a non-singular point on a quadric we may draw (at most) two lines

139 Given the quadric $XY - ZW$ show that its lines are given by

$$(\alpha_0 t_0, \alpha_1 t_1, \alpha_1 t_0, \alpha_0 t_1)$$

and

$$(\alpha_0 t_0, \alpha_1 t_1, \alpha_0 t_1, \alpha_1 t_0)$$

parametrically. Find the intersection between the two lines

$$(\alpha_0 t_0, \alpha_1 t_1, \alpha_1 t_0, \alpha_0 t_1)$$

$$(\beta_0 t_0, \beta_1 t_1, \beta_0 t_1, \beta_1 t_0)$$

and find the plane spanned by them

140 Show that if a real quadric contains a line, it will in effect contain two, and in fact through each point there will be two lines.(*Hint: Consider the residual line of the intersection of a plane containing one line*)

141 Show that the singular 2×2 matrices form a quadric of type (4,0)

142 Consider a real quadric as in exercise 139, compute the angle between two intersecting lines.(*Hint: By considering the lines as planes in the 3-dimensional real space defining the corresponding tangentplane, we define this independent of any particular dehomogenization*)

143 Show that a real quadric of type (4,2) is locally convex, i.e. the tangent plane lies on one side of the quadric.
If the quadric does not intersect the plane at infinity, save at most in one point, then it will be globally convex as well

144 Show that in the case of a real quadric of type (4,0) every tangent plane is divided into two parts, show that they are characterized by whether the quadric lies "locally" above or below the tangent plane

145 Let Q be a quadric of type (4.0) and let p be a point on it. We may then consider the planes through p and the curvatures of their intersections with Q (so called sectional curvatures) with the appropriate signs (how does this relate to exercise 144?). We may then consider the maximal and minimal sectional curvatures which will correspond to two (orthogonal) directions on the tangent plane. Show that those directions bisect the two angles formed by the two lines through p

146 Preserving the situation of exercise 145 compute the Gaussian curvature at each point In particular show that it is always negative.*Note that the Gaussian curvature is defined as the product of the minimal and maximal sectional curvature*

147 Given a one-sheeted hyperboloid $x^2+y^2-z^2$ (in affine co-ordinates) find the locus of maximal Gaussian curvature (the absolute value) and the locus where the two lines form the most acute angle. Compare! What is the relationship (if any?) between the angle of the two lines through a point and the Gaussian curvature at the point?

148 Two pairs of lines, each pair consisting of a line from each family define a quadrilateral. As in exercise 147 compute the area A(ϵ) of such a quadrilateral whose sides are of equal length ϵ and compute the limit

$$\lim_{\epsilon \to 0} \frac{A(\epsilon)}{\epsilon^2}$$

at various points of the hyperboloid

149 Show that the space of all on-ordered points can be identified with the projective space of all binary quadrics. Use this to give a slightly different argument that the quotient of $\mathbb{P}^1 \times \mathbb{P}^1$ by the natural involution is $\mathbb{P}^2$

150 Show that the equation $w^2 = x^2 + y^2 - z^2$ defines a real quadric with real lines, by checking that the right hand side is positive outside the circle

151 Considering the one-sheeted hyperboloid as in the previous exercise, compute for each point p outside the circle the angle formed by the two tangents through p a) on the plane b) in the space and finally c) the angle between the plane w=0 and those planes formed by any of the two pairs of intersecting lines

152 Show that there are two types of non-singular quadrics over a finite field $\mathbf{F}_q$ depending on whether the double covering of the quadric is defined over the outside or the inside. Compute the number of points in each case

153 Consider the non-singular quadric as parametrised by all singular 2×2 matrices, the diagonal will then form a distinguished hyperplane section, characterize those matrices and the hyperplane they span!

154 By considering the Möbius transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a point (a, b, c, d) in $\mathbb{P}^3$ determine the polar of the point $(1, 0, 0, 1)$ with respect to the quadric $ac-bd = 0$ and compare with exercise 153

155 Consider a distinguished hyperplane Δ and C its intersection with the quadric Q. Show that to each line L in Δ there is a pencil of hyperplanes H with L as base, and that if L is non-tangent all but three of those planes correspond to Möbius transformations with the two intersection points of L with C as fixed points. Try to characterize the unique involution in this pencil! What is the situation for L tangent?(cf exercise 53)

156 The spheres in Euclidean $\mathbb{R}^3$ are characterized by meeting the plane at infinity in the invisible conic $x^2 + y^2 + z^2 = 0$. Then every real plane defines two circular points by its intersection with the conic at infinity. Henceshow that a real conic is a circle iff it intersects the conic at infinity at two points.

157 Given a real quadric surface which is not a sphere, show that it determines two real lines at infinity such that the pencil of real planes with those as base locus cuts the quadric in circles. Show furthermore that the centers of those circles form a line, what is the relationship of this line (axis) to the base locus of the pencil (cf exercise 137). Note that such a pencil of planes are parallell. Does this mean that any quadric surface is a surface of revolution?

158 Let S be a sphere, and consider the involution as in exercise 137, show that this involution is defined on all lines and has no fixed points and determine the involution on the lines tangent to S. If S is the unit sphere determine the involution explicitly, in particular write it down using Plücker coordinates (cf exercise 14)

159 Show that if a singular quadric contains two skew lines it must be the union of two distinct planes. Furthermore show that such a quadric does not conatin any connecting lines between the two skew lines except of course the line of singular points. Show though that this surface is naturally the surface S(ϕ) if ϕ corresponds to a degenerate transformation, that is a hyperplane section degenerating into two lines

160 Show that if p is outside two skewlines L and L' the projection of p onto a plane Π maps the two skewlines to two intersecting lines intersecting in a unique point p'. Show that the line ¡p,p'¿ is the unique line through p intersecting both L and L'

161 Find the equation of the unique quadric that contains the three skew lines

$$x = y = 0; z = w = 0 \text{ and } x = 3z, y = 2w$$

162 Is it possible to find a unique smooth quadric containing four lines making up a quadrilateral with opposite sides skew?

163 Consider the one-sheeted hyperboloid given by $x^2 + y^2 = z^2 + w^2$, and two hyperplane sections given by z=w and z=-w. Show that those are hyperplane sections of the cylinder $x^2+y^2 = 2z^2$ and hence that they coincide from the vantage point of (0,0,0,1). The lines of the hyperboloid then define a Möbius transformation on the projection- the circle C $x^2 + y^2 = 2z^2$, determine the transformation and show that the dual of the chords defined by the lines is a circle concentric with C and determine it explicitly

164 Let C and C' be two circles parametrised by

$$(\sin t, \cos t, 1) and (\sin t', \cos t', -1)$$

. Join them by lines according to a phase shift θ i.e. $t' = t + \theta$. Find the equation of the quadric, depending on θ of course, and in particular find its "waist". Finally determine for which θ the quadric is singular

165 Let C and C' be given by

$$(a_0s^2 + b_0st, b_1st, b_2st, b_3st + a_3t^2)$$

and

$$(c_0s^2 + d_0st, d_1st, d_2st, d_3st + c_3t^2)$$

Show that C and C' has two points in common corresponding to (s,t)=(0,1) and (s,t)=(1,0) respectively. Show that the correspondence given by the parametrisation by (s,t) defines a quadric, find the equation of the quadric and find a point p from which C and C' coincide and determine the corresponding Möbius transformation ϕ and the corresponding contour

Birational Geometry of Quadric Surfaces

Given a point P on a quadric Q we may project from this point onto a plane, as we did in the case of conics. As a line intersects a quadric in two points keeping one point P fixed we expect there to be a 1-1 correspondence between the points of the quadric (the residual intersections) with the points of the plane, in analogy with the case of conics. However the situation is more complicated, unremediably so.

First a line lying in the quadric may go through P (and in the case of $\mathbb{C}$ this will always be so, in which case all the points of the line are mapped onto the same point on the plane. (We say that the line is "blown down"), secondly there is no canonical way of mapping the point P, in fact there is no unique way of choosing a line tangent to Q at P, the tangent lines sweep out a whole plane, and the projection is indeterminate at P. The solution to this is to replace P by a whole line, each point of which corresponds to a possible direction and hence a possible value at P. We refer to this process as "blowing up"

On the level of equations we may assume that P=(0,0,0,1) and the plane of projection to be w=0, and write the dehomogenized conic accordingly

$$L(x,y,z) + Q(x,y,z) \tag{1}$$

where L is a linear form and Q a quadratic form. The lines through P are then given parametrically by (sx, sy, sz, t) and the residual intersection points by

$$tL(x,y,z) + sQ(x,y,z) = 0$$

Note that if both L and Q are zero the entire line lies in Q and if L=0 then so is s and hence the residual is always P (whose image is the entire line defined by L).

On the other hand we may start with the plane and two points P_1 and P_2. Any other point P determines two lines, one PP_1 through P_1 and one PP_2 through P_2. The lines through a point form a pencil thus are parametrised by $\mathbb{P}^1$; hence we have to each point P of $\mathbb{P}^2$ associated a point on $\mathbb{P}^1 \times \mathbb{P}^1$. Now this is not an isomorphism because this construction goes wrong if P happens to lie on the line P_1P_2. If P coincides with one of the P_i's say P_1 then the line PP_1 is not well defined (thus P_1 is "blown up") and if P lies on the line P_1P_2 distinct from either point P_i then it is always mapped to the same point (P_1P_2,P_1P_2). The line P_1P_2 is "blown down".

The observant reader may have noticed that the two constructions are inverses of each other. Given a point T on a quadric Q distinct from the center of projection P through which the lines L_1 and L_2 pass, the two lines through T will meet each one of the lines L and be skew to the other. They will then be projected to lines through the image of T passing through the image points P_1 and P_2 respectively of the lines L. The image of P will "blow up" and constitute the line through P_1 and P_2, which will simply be the intersection with the tangentplane at P (spanned by L_1 and L_2) with the plane of projection.

Projecting from a point also yields a parametrisation of the quadric with quadrics. In fact looking at the line

$$(0,0,0,1) + t(x,y,z,0)$$

plugging it into the expression (1),we obtain

$$tL(x,y,z) + t^2Q(x,y,z)$$

with the common root $t = 0$ and the residual root $t = -\dfrac{L(x,y,z)}{Q(x,y,z)}$ from which we get the parametrisation

$$(x,y,z) \mapsto (xL(x,y,z), yL(x,y,z), zL(x,y,z), -Q(x,y,z))$$

by ternary quadrics.

Note that this 4-dimensional (a so called web) of conics has two base points, namely at the intersection of the line $L = 0$ with the conic $Q = 0$, those are the two points to which the two lines through P are projected to.("blown down"), furthermore points on the line $L = 0$ are all mapped to ("blown down" to) the point P $((0,0,0,1))$

It is now time to make more precise the notions of "blowing up" and "blowing down" which we have refered to loosely.

Blowing Ups and Downs

Blowing up means replacing a point with all its directions. As an example we may think of the union $B_0\mathbb{C}^2$ of all lines through the origin of $\mathbb{C}^2$. The lines through the origin are of course parametrised by $\mathbb{C}P^1$ and we have a map

$$\pi : \mathbb{C}^2 \to \mathbb{C}P^1$$

which to each point p of $\mathbb{C}^2$ associates the line through p and the origin. This map is not well-defined at the origin, not on the space $\mathbb{C}^2$ but on the space $B_0\mathbb{C}^2$, where we intuitively have replaced the origin by the collection of the origin plus a line. We simply then let π of such a point be the corresponding line.

More formally what we have done is to consider the closure of the graph of π in the space $\mathbb{C}^2 \times \mathbb{C}P^1$. That closure is simply our sought after space $B_0\mathbb{C}^2$ and the extension of the map π is just the projection onto the second factor.

One notices that the projection onto the second factor will have as fibers exactly the lines through the origin, while projecting onto the first factor one gets a 1-1 correspondence except over the origin when we will have a whole line. This line is refered to as the exceptional divisor.

The picture is of a spiral staircase. The lines through the origin of $\mathbb{C}^2$ are "lifted up"

This construction can be localised allowing us to blow up any point on any complex manifold of complex dimension two.

Letting x, y be local coordinates of a neighbourhood O around a point p corresponding to $(0,0)$, we will define two charts U and U' using local coordinates u, v and u', v' respectively. The two charts will be glued together to form a manifold B which will map to O.

In fact considering the map $\pi : B \to O$ we will define the transitionfunctions implicitly as follows

$$\pi(u,v) = (uv, v)$$
$$\pi(u',v') = (u', u'v')$$

From which follows the transition functions

$$u' = uv \qquad u = 1/v'$$
$$v' = 1/u \qquad v = u'v'$$

Note that $x/y = u$ and $y/x = v'$. In fact the image of $(u,0)$ and $(0,v')$ is the origin $(0,0)$ (p), and the inverse image of the origin is a projective line $\mathbb{C}P^1$ where the intersections with U and U' give the standard charts of the Riemannsphere. This projective line is usually denoted by E and refered to as the exceptional divisor; it does parametrise all the directions through p (0,0). On the other hand outside the origin π has a unique point in its inverse image.(see exercise 175)

Topologically the inverse image of O is a tubular neighbourhood of the exceptional divisor E in B.(see exercise 176). If E would be a $\mathbf{P}^1$ in a surface, with a tubular neighbourhood isomorphic to B, then the process can be inverted and we can replace B by O. In this way we define a blow down (✠it will actually turn out there there is a simple numerical criterion for this, the "selfintersection" of E ($=\mathbf{P}^1$) should be -1)

The definitions, although implicitly discussed for $\mathbb{C}$, make equal sense over $\mathbb{R}$ of course. Then letting O be a (small) disc, B turns out to be a Möbius strip! The boundary of O will correspond to the (not so immediate) connected boundary of the strip B, and the exceptional divisor E will correspond to the centerline of the strip. (Cutting along the center line does not disconnect the strip, as the complement will be equal to the punctured disc O*

) Considering a real Quadric with no lines and projecting from a point we will get a 1-1 correspondence between the Quadric blown up at P and the real projective plane. As the Quadric is topologically a sphere, we get another illustration of the fact that the real projective plane is glueing the boundary of a Möbius strip to that of a disc. (Removing a disc from a sphere yields another disc). And conversely considering a real projective plane and two complex conjugate points, the quadrics through them will have no real base points, but the line joining them will be real and blown down, giving a sphere. The quadric with lines will correspond to two real blow ups and a real blow down, and yield a torus ($S^1 \times S^1$)

Cremona transformations

Mapping $\mathbb{P}^2$ birationally onto a Quadric and then projecting the Quadric back to $\mathbb{P}^2$ yields a birational map of the plane onto itself, blowing up three points and blowing down the three lines determined by the points.

We are now considering a 3-dimensional linear space (a net) of conics with three base points. The conics through the base points are the inverse images of lines, while the images of lines, not passing through any of the base points are conics

By identifying the image with the original plane we get a birational automorphism, and by choosing the coordinates nicely, the involutive character of the automorphism becomes evident

Namely consider the map

$$(x, y, z) \mapsto (\frac{1}{x}, \frac{1}{y}, \frac{1}{z})$$

associated to the net spanned by the conics yz, zx and xy This map blows up the vertices $(1,0,0),(0,1,0)$ and $(0,0,1)$ and blows down the lines given by $x=0, y=0$ and $z=0$

To get the image of a curve $F(x,y,z)=0$ we simply substitute for x, y and z the conics yz, zx and xy and factor out any factors xyz

One can prove that any birational map of the plane is a composition of Cremona transformations and linear transformations

Exercises

166 Let Q be a quadric defined over the reals. Writing it in the form

$$wL(x,y,z)+Q(x,y,z)$$

with L a linear form and Q a quadratic form; show that there are two real lines through (0,0,0,1) iff L and Q have real intersection. Show that this property is independent of the choice of real coordinates

167 Given the representation of a quadric as the blow up of two points P_1 and P_2 in the plane and the blow down of the line between them. Show that the two rulings of lines of the quadric corresponds to the lines through either point.

168 Show that the birational isomorphism between $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ becomes an isomorphism if restricted to $\mathbb{P}^2\backslash$(line through the two points of reference). What part of $\mathbb{P}^1 \times \mathbb{P}^1$ is avoided?

169 Find all solutions to the diophantine equation

$$x^2 + y^2 = z^2 + w^2$$

expressing numbers as sums of two squares in two different ways, by giving a quadratic parametrisation of the corresponding quadric, using some obvious solution

170 Given four ternary quadratic forms

$$Q_0(x,y,z), Q_1(x,y,z), Q_2(x,y,z), Q_3(x,y,z)$$

show that the ten ternary *quartic* forms $Q_0^2, Q_0Q_1 \ldots$ formed by all possible quadratic monomials in $Q_0, Q_1 \ldots$ are linearly independent unless the corresponding system (web) of conics have two base points

171 The web of all circles (conics passing through the two circular poins at infinity) will map onto a quadric. Show by making no calculations that this quadric must have index ± 2. By choosing a basis of degenerate circles $z^2, x^2+y^2, (x+z)^2+y^2$ and $(x-z)^2+y^2$ say, find an explicit equation for the quadric, and in particular find the equation for a tangent plane corresponding to the point (a,b,a)

172 The linear space of all bihomogenous forms of bidegree $(1,1)$ in bihomogenous coordinates $(x_0,x_1;y_0,y_1)$ (i.e. bilinear forms on a two dimensional vectorspace) is of dimension four. Show that they give an embedding of a quadric Q into $\mathbf{P}^3$. Furthermore show that the linear subspace of such form passing through a fixed point P of the quadric Q, defines a map onto $\mathbf{P}^2$ which is the same as projection from P onto a plane. Thus this gives the inverse map of the parametrisation of Q with ternary quadrics

173 What happens if we project a singular quadric (a cone) from a point different from the vertix? Consider the web of all conics of the form

$$xL(x,y,z)+y^2$$

determine the image in $\mathbf{P}^3$ parametrised by those

174 Given the chart $(u,v) \mapsto (u,1,0,0)+v(0,0,1,u)$ of a quadric $xz-yw$. Consider the projection π from the point $(1,0,0,0)$ onto the plane Π given by $x=0$, which projects the line $z=w=0$ onto the point $(1,0,0)$ on Π, compare this projection with the local presentation of a blowdown

175 Show that if (x, y) is not the origin, then there is a unique point in its fiber. Find the local coordinates for this point (on either chart) from the local presentation of the blow-up

176 Let U be an open subset of a local blowup B, show that the image of U under π is open iff the intersection of U with E is either empty or the whole of E

177 Given the map $\pi : B \to O$ show that the map

$$\pi_* : \mathbf{T}_p(B) \to \mathbf{T}_{\pi(p)}(O)$$

between tangent spaces is an isomorphism iff p does not lie on the exceptional divisor E. If p would lie on E, determine the kernel of π_*

178 Show that E is the only compact subvariety of B except for finite sets of points.(*Hint: Consider the image of such varities under* π *in the open subset* O)

179 Given a metric on a real B, i.e. a Möbius strip, by e.g. considering B as a closed subset of O$\times S^1$ (a solid torus) show that π contracts the lengths of small linesegments, depending on their distance to E, and the angle they make with the "perpendiculars" to E. In fact show that the "perpendiculars" to E can be thought of the inverse image of lines through the origin (the image of E) in O. Assuming that B is made of some glass that absorbs light, say proportionally to the *cosine* of the angle lights hit it, and that the light rays travel in "concentric" circles through the solid torus above, compute the intensity of light falling on a perpendicular circular cut (O)of the solid torus (O$\times S^1$) as a function of its distance from the center.(cf with exercise 177)

180 Given a trivial (i.e. untwisted) strip $S^1 \times I$ where I is a (short) interval. Such a strip will be the tubular neighbourhood of any of its "circular" fibers. Make a real blow up of a point P on it, show that the tubular neighbour of the "circular" fiber F through P now becomes a Möbius strip, and hence F becomes exceptional and can be blown down. Show that the resulting surface is once more an untwisted strip. This process is called an elementary transformation. Show that if an elementary transformation is performed at a point P on a torus, then the resulting space consists of two Möbius strips glued along their boundaries, yielding a Klein bottle

181 Defining the topological sum $U\#V$ of two real surfaces by puncturing by a "small" disc both surfaces and joing along the edges, we get a semigroup structure on the real two dimensional surfaces. The neutral element S is given by the sphere, and P the projective plane and T the torus are two generators. Letting $T_n = T\#\ldots\#T$ n times with $T_0 = S$ we can ask

a) Show the identity $P\#P\#P = T\#P$ In fact this is the only relation

b) Assuming a) show that any surface can be written in either of three ways

(i)T_n

(ii)$T_n\#P$

(iii)$T_n\#P\#P$

c) Compute the Euler characteristics (see exercise 11) of the cases (i)-(iii) and show that the first case corresponds to orientable surfaces, the second and third to non-orientable

d) To every non-orientable surface we can find a unramified double cover which is orientable.(Associoate to each point the two possible local orientations). Find the orientable double covers of the cases (ii) and (iii)

182 Consider the net of circles passing through the origin. This defines a Cremona transformation on the real projective plane blowing up the origin and blowing down the line at infinity. Relate this to inversion in circles

183 Given a Cremona transformation centered at the points

$$(1, 0, 0), (0, 1, 0) \text{ and } (0, 0, 1)$$

find the image of
(a) a line $Ax + By + Cz = 0$
(b) a conic $x^2 + y^2 + z^2$ passing through none of the base points
(c) a conic $(x + y + z)^2 - 4(xy + yz + zx)$ tangent to the three sides
(d) a conic $x^2 + y^2 - (x + y)z$ passing through one of the base points
(e) a conic $x^2 + xy + yz + zx$ passing through two of the base points
(f) a conic $Axy + Byz + Czx$ passing through all basepoints

184 Find the image of the quadric $x^2 + y^2 + z^2$ under a Cremona transformation with base points at (1,1,1) and the two circular points at infinity

185 By projecting a Quadric to the plane and then going back again, one gets a birational automorphism of a quadric. Show that this is given by an elementary transformation (cf exercise 180)

Plane cubics

By a plane cubic is meant the zeroes of a cubic form in three homogenous variables. As there are ten monomials of degree three in three variables (see exercise 186) we have a nine dimensional family of cubics. As the dimension of $\mathbf{PGL}(3,\mathbb{C})$ is eight dimensional, we see that no orbit can be dense, nor that there maybe only a finite number of orbits. This is in contrast to the case of conics. Thus the case of plane cubics presents the first case of continous moduli in their description

A cubic C is said to be non-singular iff its gradient never vanishes. Unlike the case of conics this is hard to check in practice. One may though write down a (horrible) expression in terms of the coefficients of a cubic which will vanish iff the cubic is singular (cf exercise 193). Thus one may decide that a cubic is singular without finding its singular point!

Classification of Singular cubics

While there are only two types of singular conics there is a whole slew of singular cubics.

Recall that a cubic may be written in the inhomogenous form

$$A_0 + A_1(x,y) + A_2(x,y) + A_3(x,y) \tag{1}$$

where A_i is a binary form of degree i. The point P(=(0,0,1)) lies on the cubic iff $A_0 = 0$ and in that case it is a smooth point iff $A_1 \neq 0$. If on the other hand $A_1 = 0$ then P is a singular point and it will be called a double point iff $A_2 \neq 0$. We see now in passing that a cubic with a triple point must consist of three concurrent lines (the equation given by $A_3 = 0$).(Which may be all distinct, or one double, or all three coinciding into a triple). Thus we may concentrate on the case of double points.

If A_2 has two distinct roots, we say that P is an ordinary double point, or a node. Such a singularity consists of two distinct branches, tangent respectively to the two lines through P given by $A_2 = 0$, those are called nodal tangents (see exercise 194). Now all binary forms split into linear factors, if A_2 and A_3 have a factor say L in common, then L becomes a linear component of the cubic through P. Thus among cubics with a node at P we have three cases, either it consists of three lines, not concurrent, two of which pass through P (the case of A_2 and A_3 having two common factors), or of a line and a conic, meeting transversally at P (the case of A_2 and A_3 having just one linear factor in common) or the cubic is what one says, irreducible (of one piece) (the case of no common linear factors), because any reducible cubic must contain a line.(If you partition three in at least two summands one has to be one!). The last case is important and called a nodal cubic.

If A_2 has a double root then it is a square of a linear form L, and L defines what one calls a cuspidal tangent.(see exercise 194). As before L can be a factor of A_3. If it is a square factor then the cubic consists of a double line (L) and a simple line, if it is a simple factor of L, then the cubic consists of a line (L) and a conic tangent to the line at P. In case L is not a factor of A_3 then the cubic is irreducible and we have the case of a cusp, and the singular cubic is refered to as a cuspidal cubic

Flexes and the Hessian

A new phenomena for cubics is the existence of flexes. A flex is by definition a smooth point such that the tangent has excess contact, i.e. contact of order three. If the tangentline is parametrised then the restriction of the cubic has a triple root at the tangency point. (see exercise 198)

Writing down the Taylorformula for a homogenous form F we have

$$F(x+t\zeta, y+t\eta, z+t\xi) = F(x,y,z) + t(\zeta\frac{\partial F}{\partial x} + \eta\frac{\partial F}{\partial y} + \xi\frac{\partial F}{\partial z}) +$$
$$\frac{1}{2}t^2(\zeta^2\frac{\partial^2 F}{\partial x^2} + \eta^2\frac{\partial^2 F}{\partial y^2} + \xi^2\frac{\partial^2 F}{\partial z^2} + 2\zeta\eta\frac{\partial^2 F}{\partial x\partial y} + 2\zeta\xi\frac{\partial^2 F}{\partial x\partial z} + 2\eta\xi\frac{\partial^2 F}{\partial y\partial z} + \dots$$

Assuming that (x,y,z) is a point on the curve given by $F=0$ we see that the t terms disappear iff (ζ,η,ξ) lie on the line defined by the gradient, furthermore if this is true then the t^2 terms disappear (i.e the tangent is a flexed tangent) iff the quadratic form (in (ζ,η,ξ))splits.

In fact if it vanishes for some (ζ,η,ξ) it will do so for all points on the tangent line (defined by the gradient), thus a flex point forces the form to split; conversely if the form is split means that we can find nontrivial (ζ,η,ξ) such that

$$\begin{array}{ccccccc}
\zeta\frac{\partial^2 F}{\partial x^2} & + & \eta\frac{\partial^2 F}{\partial x\partial y} & + & \xi\frac{\partial^2 F}{\partial x\partial z} & = 0 \\
\zeta\frac{\partial^2 F}{\partial y\partial x} & + & \eta\frac{\partial^2 F}{\partial y^2} & + & \xi\frac{\partial^2 F}{\partial y\partial z} & = 0 \\
\zeta\frac{\partial^2 F}{\partial z\partial x} & + & \eta\frac{\partial^2 F}{\partial z\partial y} & + & \xi\frac{\partial^2 F}{\partial z^2} & = 0
\end{array}$$

If we now multiply the rows by x, y and z respectively and add using Eulers identity for homogenous forms we obtain

$$\zeta\frac{\partial F}{\partial x} + \eta\frac{\partial F}{\partial y} + \xi\frac{\partial F}{\partial z} = 0$$

Thus we have proved that the flexes are given as the intersection of the curve $F=0$ with the curve given by the determinant

$$\begin{vmatrix}
\frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x\partial y} & \frac{\partial^2 F}{\partial x\partial z} \\
\frac{\partial^2 F}{\partial y\partial x} & \frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y\partial z} \\
\frac{\partial^2 F}{\partial z\partial x} & \frac{\partial^2 F}{\partial z\partial y} & \frac{\partial^2 F}{\partial z^2}
\end{vmatrix} = 0$$

This form is called the Hessian of F, and we note that when F is a cubic then the Hessian is as well. As we have seen two conics intersect each other in four (not necessarily distinct) points, and we then expect two cubics to intersect in nine points. As we cannot parametrise (at least not algebraically and rationally, see below on the section on Weierstraß $\wp$-function) we cannot prove this in the same elementary *ad hoc* way.(For a discussion of an elabourate *ad hoc* proof see exercise 201). Let us at this stage just wave our hands, and refer to the principle that this number should be unpertubed if the cubics are deformed ("wiggled") in which case it would be obvious if one of the cubics consists of lines say.(cf exercise 202). We have thus established the existence of at least one flex on a non-singular cubic. (The intersection maybe of multiplicity nine! (although that will never happen as we will see below) or may be completly swallowed up by the singular point in the singular case, the cases when the intersection is infinite, which of course can happen, all refer to reducible cubics)

Weierstrass Normal Form

The fact that every cubic has a flex allows us to write down a standard and useful normal form attributed to Weierstraß.

Assuming that the flex is given by $(0,1,0)$ and the flexed tangent by the line at infinity $z=0$ we can write the cubic in the form

$$zQ(x,y,z) = x^3$$

where $Q(x,y,z)$ is a quadric.(Restricting the cubic to $z=0$ we get a triple root at $(0,1)$) Reshuffling terms and completing squares we may write it as

$$zQ_0(x,y,z) = q(x,z)$$

where $q(x,z)$ is a binary cubic with x^3 as leading term, and $Q_0(x,y,z)$ is a perfect square $L(x,y,z)^2$. Replacing y by $L(x,y,z)$ does not change the coordinates of our flex, nor does replacing x by a linear form in x,z cause any harm. The latter switch allows us to write $q(x,z)$ without any x^2 term. Dehomogenising we have the Weierstraß form

$$y^2 = x^3 + px + q$$

Factoring the cubic x^3+px+q into $(x-e_1)(x-e_2)(x-e_3)$ We observe the following

Lemma. *A cubic in Weierstraß form is non-singular iff the three roots* e_1, e_2, e_3 *are distinct. The condition for this is that the discriminant* $4p^3 + 27q^2$ *does not vanish. If two roots coincide we get a singularity at the corresponding* x*-value. This is a node iff the multiple root has multiplicity two, and a cusp iff the multiple root has multiplicity three, i.e. all three roots coincide. The condition for the latter is that both* p *and* q *vanish*

For a proof and further ramifications see exercise 204

We note that the Weierstraß mormal form is symmetric with respect to the x-axis, under the reflection $(x,y) \mapsto (x,-y)$. This symmetry is induced from the projection from the flex at infinity $(0,1,0)$, the corresponding vertical lines are tangent to the cubic exactly at the points corresponding to $x = e_1, e_2$ and e_3 the significance of which we will return to

The normal form of the cubic is also handy in getting the real picture. Assuming at first that all three roots are real, and say $e_1 < e_2 < e_3$ then the righthand side becomes positive iff $e_1 \leq x \leq e_2$ or $x \geq e_3$, the first finite interval determines a "small" oval, while the unbounded interval determines an "unbounded" portion of the curve. If only one of the roots is real, then only the "unbounded" portion appears. For further details see exercises 207,208 and 209

Nodal and Cuspidal cubics

Both the nodal and cuspidal cubic can be parametrised by $\mathbb{P}^1$. In fact we consider the lines through the singular points, they intersect the cubic in one additional point (which may coincide with the singular points) setting up the correspondence. (cf exercises 190,191 and 210)

The standard form of the cuspidal cubic is given by

$$y^2 = x^3$$

in inhomogenous coordinates with a cusp at the origin, with its standard parametrisation given by

$$(s,t) \mapsto (st^2, t^3, s^3)$$

in homogenous coordinates.

The singular point will correspond to $t = 0$, with cuspidal tangent given by $y = 0$, while it will have exactly one flex corresponding to $s = 0$ and flexed tangent given by $z = 0$ (see exercise 212)

There really is not an equally canonical form for the nodal cubic, but the following form has its advantages as we will see

$$(x + y + z)^3 = 27xyz$$

with its node at $(1, 1, 1)$ and with a (inhomogenous) parametrisation given by

$$t \mapsto ((t-1)^3, (t-\rho)^3, (t-\rho^2)^3)$$

The singular point will correspond to either $t = 0$ or $t = \infty$ with complex conjugate nodal tangents $x + \rho y + \rho^2 z = 0$ and $x + \rho^2 y + \rho z = 0$ (cf exercise 215). The three flexes correspond to $t = 1, \rho$ or ρ^2 and are all real $(0, 1, -1), (-1, 0, 1)$ and $(1, -1, 0)$ respectively. (See exercise 216 for further remarks)

Topologically the cuspidal cubic is homeomorphic to the Riemannsphere.(A 1-1 map between two compact Hausdorff spaces is always a homeomorphism) while the nodal cubic is given by the identification of two distinct points on the sphere.

The non-singular points of a cuspidal cubic form $\mathbb{C}$, while the non-singular points of a nodal cubic are isomoorphic with $\mathbb{C}^*$. Both of these are groups under addition and multiplication respectively.

The Group Law

The intrinsic groupstructure on the nodal and cuspidal cubics are remarkably enough also external and geometrical. In fact if ϑ denotes the standard parametrisation of a standard cuspidal cubic, then the following two statements are equivalent

(1) $t_1 + t_2 + t_3 = 0$

(2) $\vartheta(t_1), \vartheta(t_2), \vartheta(t_3)$ lie on a line

The proof is easy (see exercise 218)Note that the flex corresponds to the *zero* of the group.

Using similarly the above parametrisation (say ϕ) for the nodal cubic we can set up in a completly analogous way the two equivalent statements

(1')$t_1 t_2 t_3 = 1$

(2')$\phi(t_1), \phi(t_2), \phi(t_3)$ lie on a line

The verification is straightforward, but necessarily a little bit more involved in this case (see exercise 219). Note that one of the flexes corresponds to the *unit* element of the group

Now we are motivated to try and use the same construction on a non-singular cubic, whose internal structure is still a mystery to us. Letting a flex be *zero* , like the one at infinity in the Weierstraß model, we simply postulate that three points of the cubic add up to zero iff they lie on a line.

Using the picture of the Weierstraß model this becomes rather explicit. Given two points P and Q on the cubic, we construct their sum P+Q accordingly: Connect P and Q with a line, this will intersect the cubic in a residual point R. (this will be -(P+Q) by definition) Now join R with the flex at infinity, this will be a "vertical" line intersecting the cubic in -R which is our sought for point (P+Q). Note -R is just the reflection of R with respect to the x-axis. Thus we have incidentally interpreted the reflection or the symmetry $(x, y) \mapsto (x, -y)$ as the involution R $\mapsto$ -R.

We can also be quite explicit, in terms of coordinates, to write down this construction.(See exercise 220)

Now there is actually some work involved to prove that this construction actually yields a group structure, notably to show that the addition so defined is associativ. We will not do that, but will get this for free below using a wide diversion into analysis. But as a first step in trying to get some grip on the underlying groupstructure we would like to determine the topology of the complex cubic curve. (The topology of the real curve we have already seen, it is either one or two circles)

The involution R$\mapsto$-R defines the cubic as a double cover of the line $x = 0$ or the Riemannsphere. This double cover will be branched at four points e_1, e_2, e_3 and ∞. Making two slits joining two disjoint pair of branch points, we get the topological double cover by taking two such slitted sphers and glueing them along the slits. Note that each slit has two edges, and the glueing is done with no "crossings". What we actually are constructing is the Riemann surface corresponding to the multivalued function

$$y = \sqrt{(x - e_1)(x - e_2)(x - e_3)}$$

with its four branch points. The resulting surface is a torus (see exercises 223 and 224, the book *Topics in Complex Function theory (I)* (pp 22-29) by **Siegel** is a good and exhaustive reference).

The postion of the four branchpoints is not determined by the cubic, it obviously depends on the particular coordinates we choose, but any such four points are equivalent under a Möbius transformation. Thus to a non-singular cubic we have a $\jmath$-invariant associated.(✠In fact as we will see, through any point on a non-singular cubic we can draw four tangents, tangent at residual points. Thus we will have four points on the Riemann sphere determined. As such tangents can never coincide (see exercise 224 for a preliminary observation) those four points will determine the same $\jmath$ invariant as it can never be allowed to take infinite values). The $\jmath$ invariant constitute the continous invariant to which we refered in the beginning of this chapter. Its value is straightforward to compute in terms of p and q yielding

$$\jmath = \frac{27p^3}{4p^3 + 27q^2}$$

THE WEIERSTRASS℘-FUNCTION

Now there is a remarkable function (Weierstraß ℘-function) defined as follows

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*} \frac{1}{(z-\omega)^2} - \frac{1}{w^2}$$

Once you convince yourself that the sum on the right converges, you see that the function ℘(z) is doubly periodic i.e it satisfies

$$\wp(z+\omega) = \wp(z) \quad \text{for any } \omega \in \Lambda$$

One says that $\wp(z)$ is a so called Elliptic function. In virtue of its double periodity it will be a meromorphic function that naturally lives on the quotient space $\mathbb{C}/\Lambda$ which is a so called complex torus (see exercise 225), as it is also a group, the quotient of two additive groups, it seems natural to assume that this is indeed the group structure defined above geometrically.

One may also take the derivative of the ℘-function, yielding

$$\wp'_\Lambda(z) = -2\sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^3}$$

and between those two functions there is a remarkable polynomial identity

(2) $$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2(\Lambda)\wp_\Lambda(z) - g_3$$

for a judicious choice of the constants $g_2(\Lambda)$ and $g_3(\Lambda)$.

The "trick" is to consider the difference, and note that it is an analytic function which is doubly periodic (hence constant by Liouville or equivalently the maximum principle) vanishing at the origin.(see exercise 228)

Thus we have the "transcendental" parametrization

$$\Theta : \mathbb{C}/\Lambda \to E_\Lambda$$

of the elliptic curve

(3) $$y^2 = 4x^3 - g_2x - g_3$$

by

$$z \mapsto (\wp_\Lambda(z), \wp'_\Lambda(z), 1)$$

What we need to show is that

(i) Every non-singular elliptic curve occurs as (3)

(ii) The canonical addition on $\mathbb{C}/\Lambda$ agrees with the geometric addition

For the first part we make the trivial rearrangement

$$(y/2)^2 = x^3 + (-g_2/4)x + (-g_3/4)$$

showing that we need for any choice (p, q) with $4p^3 + 27q^2 \neq 0$ find a lattice Λ such that $p = -g_2/4$ and $q = -g_3/4$. To do so we need to make a short detour into Modular forms

Introducing the Eisenstein forms

$$E_n(\Lambda) = \sum_{\omega \in \Lambda^*} \frac{1}{\omega^{2n}}$$

we note that the Laurent expansion of $\wp_\Lambda(z)$ is given by

$$\frac{1}{z^2} + \sum_{n \geq 1} E_{n+1}(\Lambda) z^{2n}$$

and that

$$g_2(\Lambda) = 60E_2(\Lambda)$$
$$g_3(\Lambda) = 140E_3(\Lambda)$$

(cf exercise 228)

The Eisensteins forms have the homogenity property that

$$E_n(t\Lambda) = t^{2n} E_n(\Lambda)$$

We can then consider them as homogenous functions on the space of all lattices. We say that two lattices Λ and Λ' are equivalent iff

$$\Lambda = t\Lambda' \text{ for some } t \neq 0$$

(note the analogy with homogenous coordinates!)

Two equivalent lattices Λ and Λ' give rise to isomorphic manifolds $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ by the homothety $z \mapsto tz$, what is less obvious (but not less important) is that two tori are isomorphic iff the lattices are homotheties of each other (see exercise 229). We are thus interested in the space of all lattices up to homothety. As a first step we can normalize the basis (ω_1, ω_2) to $(1, \tau)$ with τ an element of the upper halfplane $\mathcal{H}$. Now a basis is not uniquely determined by a lattice, the group $\mathbf{SL}(2,\mathbb{Z})$ acts naturally on any basis, which descends to an action on $\mathcal{H}$ as follows. The basis $(1, \tau)$ is transformed into the basis $(c\tau + d, a\tau + b)$ and normalizing we get the action

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

Note that by slight abuse of notation considering the Eisenstein forms as functions on $\mathcal{H}$ by $E_n(\tau) = E_n(1, \tau)$ we find that they turn out to be modular forms of weight n satisfying

$$E_n(\frac{a\tau + b}{c\tau + d}) = \frac{1}{(c\tau + d)^{2n}} E_n(\tau)$$

We are thus interested in the quotient

$$\mathcal{M}_1 = \mathcal{H}/\mathbf{SL}(2, \mathbb{Z})$$

The remarkable thing is that we can write down a function $J(\tau)$ (or equivalently $J(\Lambda)$ which classifies the orbits under this action. The function can be expressed in terms of the Eisenstein forms as follows

$$J(\Lambda) = \frac{g_2^3(\Lambda)}{\Delta(\Lambda)}$$

where

$$\Delta(\Lambda) = g_2^3(\Lambda) - 27g_3^2(\Lambda)$$

is a discriminant which vanish iff the lattice Λ degenerates to rank 1

The function J identifies $\mathcal{M}_1$ with $\mathbb{C}$, and (up to a multiplicative constant) coincides with the $\jmath$-invariant! (For further discussions of this see exercise 230) This settles (i), as to (ii) we need only to show that the determinant

$$\begin{vmatrix} \wp(z_1) & \wp(z_2) & \wp(z_3) \\ \wp'(z_1) & \wp'(z_2) & \wp'(z_3) \\ 1 & 1 & 1 \end{vmatrix}$$

vanishes whenever $z_1 + z_2 + z_3 = 0$. This is left as an exercise (232).

We have thus established a non-singular cubic as a complex torus $\mathbb{C}/\Lambda$ via the Weierstraßfunction, and interpreted the canonical group structure geometrically. Λ is a sublattice of $\frac{1}{n}\Lambda$, and the subgroup $\Lambda/\frac{1}{n}\Lambda$ is finite of order n^2 and corresponds exactly to the points of order n on the elliptic curve. In particular if $n = 2$ we have four points, the three non-zero points are called primitive two torsion points. In the Weierstraß normal form those corresponds to the intersection of the cubic with the y-axis (i.e. to the three roots e_1, e_2 and e_3 pf the cubic $x^3 + px + q$, or the three zeroes of $\wp'(z)$). $\wp(z)$ is an even function and the map

$$\wp(z) : \mathbb{C}/\Lambda \rightarrow \mathbb{C}P^1$$

defines a double covering of the Riemann sphere $\mathbb{C}P^1$ branched exactly over the three points $\wp(e_1), \wp(e_2)$ and $\wp(e_3)$

We also observe that the three-torsion points will correspond to the flexes, and that there will be hence nine distinct flexes. As the sum of any two three torsion points as well as the inverse of one is a three torsion point, we see that the line through any two flexes must meet the cubic in a third flex. The flexes of the cubic then form a remarkable system of nine points (impossible over the reals see exercise 233) which is a faithful representation of the finite affine space $\mathbb{F}_3^2$. We will discuss this at greater length in the context of the Hesse normal form

Two special cubics

The connection between the lattice Λ and the cubic $y^2 = x^3 + px + q$ and its coefficients is rather subtle and indirect.

It is hard to compute the Eisenstein forms for a given lattice (and hence get the numbers p and q), and conversely given the equation it is hard to find a corresponding lattice Λ.(see exercise 236). There are however two special cubics for which we can go back and forth explicitly

If Λ is generated by $< 1, \rho >$ (ρ a primitive cube root of unity) we have a “hexagonal” lattice invariant under multiplication by ρ, thus computing the Eisenstein form of weight two we have

$$E_2(\Lambda) = E_2(\rho\Lambda) = \rho E_2(\Lambda)$$

from which we conclude that $E_2 = 0$ and hence $p = 0$ and $\jmath = 0$. This curve is a double cover of $\mathbb{C}P^1$ branched at an equianharmonic set of four points. The curve is commonly known as the Fermat cubic (because it can be put in the form

$x^3+y^3+z^3=0$) and its Weierstraß normal form has an additional symmetry given by

$$(x,y)\mapsto(\rho x,y)$$

Thus there is a cyclic group of order six which fixes a flex and leaves the curve invariant. Conversely any elliptic curve of the Weierstraß form $y^2=x^3+q$ is equivalent to the Fermat cubic

If Λ is generated by $<1,i>$ (i.e. a "square" lattice) which is invariant under multiplication by i. Computing the Eisenstein form of weight three we get

$$E_3(\Lambda)=E_3(i\Lambda)=-E_3(\Lambda)$$

concluding that $E_3=0$ and hence $q=0$ and $\jmath=1$. This curve is a double cover of $\mathbb{C}P^1$ branched over a harmonic set of four points. This curve has no commonly accepted '*nick name*, I prefer to call it the Gauß cubic, as the lattice is given by the Gaussian integers, others prefer to refer to itas the "leminiscate" (see exercise 246). The Weierstraß form of a Gauß cubic has also an additional symmetry given by

$$(x,y)\mapsto(-x,iy)$$

Thus in this case there is a cyclic group of order four which fixes a flex and leaves the cubic invariant. Conversely any elliptic curve of the Weierstraß form $y^2=x^3+px$ is equivalent to the Gauß cubic

Those two cubics are special, but very distinguished, examples of a more general concept, so called elliptic curves with complex multiplication (or of CM-type)

Isogenies

As we have seen (exercise 229) any map between two elliptic curves respecting their *zeroes* is a homomorphism. Such a map is either trivial (i.e. the image is *zero* or surjective. In the latter case we say that it is an *isogeny*. Given two arbitrary elliptic curves (E $=\mathbb{C}/\Lambda$ and E' $=\mathbb{C}/\Lambda$) we expect no interesting map between them, the condition that there exists an isogeny (ι : E $\rightarrow$ E') is that there exists some t such that $t\Lambda\subseteq\Lambda'$ If that is the case the quotient $\frac{\Lambda'}{t\Lambda}$ is a finite subgroup (the kernel of the isogeny) and if N is its order (the index of $t\Lambda$ in Λ', and also by definition the degree of the isogeny) then Λ' is a sublattice of $\frac{1}{N}t\Lambda$ and we get an isogeny ι^* : E' $\rightarrow$ E' also of degree N. ι^* is called the dual isogeny to ι. We thus see that the notion of an isogeny is symmetric, in fact an equivalence relationship (see exercise 250)

Given a point P of order N translation by P defines a map Θ_P (not a homomorphism!) of order N, the quotient by this map gives an isogeny of degree N, and all isogenies occur in this way (provided of course their kernels are cyclic, cf exercise 252). The point P is not determined by the isogeny (except if for trivial reasons see below) but the subgroup generated by it is.(cf exercise 255)

The isogenous curve is seldom isomorphic with the original curve, unless of course the isogeny is given by multiplication by an integer (cf exercise 251). Thus the ring $End(E)$ of an elliptic curve is in most cases just $\mathbb{Z}$. However for certain lattices Λ it turns out that the endomorphism ring is strictly bigger, or what is equivalent, there exists cyclic isogenies in the endomorphism ring. Examples of such curves are the Fermat and Gauß cubic, but as the notion has little geometric content, we will relegade it to some exercises (see exercises 259 and 260)

A particular important case are isogenies of degree two. They correspond to the primitive two-torsion points (hence there are three of them) via the involutions given by their respective translations.

Given a two-torsion point ϵ we have that the translation $x \mapsto x + \epsilon$ commutes with the involution $x \mapsto -x$ as $-\epsilon = \epsilon$. Thus the involution descends down to the covered Riemann sphere permuting the four branch points (see exercise 256 for more details). This involution on the Riemann sphere will have two fixed points and the quotient will be another Riemann sphere with four distinguished points on it, namely the two images of the four branch points on the original sphere, and the two branch points of the covering. The associated elliptic curve will be the image of the associated isogeny.

Hesse Normal Form

Translations are usually not induced by linear transformations of the ambient projective plane, unless they are effected by elements of order three (see exercise 261). We are now going to study a normal form of cubics in which those translations are prominent and conspicious

By the Hesse normal form we will mean a cubic of form

$$x^3 + y^3 + z^3 = 3\lambda xyz$$

A cubic of this form will be denoted by H_λ

As we will see every *non-singular* cubic may be written in Hesse form (see exercise 262). The Hesse form has many symmetries. The most obvious is the one under $\mathcal{S}_3$ acting by permutation of the coordinates. Setting

$$I(x, y, z) = (y, x, z)$$
$$T(x, y, z) = (y, z, x)$$

With $I^2 = T^3 = 1$ and $ITI = T^2$ we establish $\mathcal{S}_3$ as the semidirect product $\mathbb{Z}_2 \ltimes \mathbb{Z}_3$, adding the somewhat less obvious symmetry

$$S(x, y, z) = (x, \rho y, \rho^2 z)$$

satisfying $ST = TS$ and $ISI = S^2$, we find a big symmetry group G given as $\mathbb{Z}_2 \ltimes \mathbb{F}_3^2$. We will presently interpret the group G intrinsically with respect to the cubics of the Hesse pencil

The Hesse form is invariant under Hessians, and the Hessian of H_λ is $H_{\lambda'}$ with

$$\lambda' = \frac{8 - 2\lambda^3}{6\lambda^2}$$

(see exercise 263) Our next task is determine which members of the Hesse pencil are singular. One case is obvious $\lambda = \infty$ corresponding to the triangle $xyz = 0$, three other cases occur for those values λ such that $\lambda^3 = 1$ as those are exactly those for which $\lambda = \lambda'$. Are there others? As a singular cubic can at most contain three singular points, we may simply make a catalogue of all the points whose orbit under G is at most three (cf exercise 264)

If H_λ is a Hesse cubic not of the above type, its intersection with the Hessian is given by

$$xyz = 0$$
$$x^3 + y^3 + z^3 = 0$$

Thus we can give a table of all the flexes (which will be common for all cubics H_λ in the Hesse pencil) and an explicit identification of this set with $\mathbb{F}_3^2$

$(0,0)$	$(1,-1,0)$	$(1,0)$	$(-1,0,1)$	$(-1,0)$	$(0,1,-1)$
$(0,1)$	$(1,-\rho,0)$	$(1,1)$	$(-\rho,0,1)$	$(-1,1)$	$(0,1,-\rho)$
$(0,-1)$	$(1,-\rho^2,0)$	$(1,-1)$	$(-\rho^2,0,1)$	$(-1,-1)$	$(0,1,\rho^2)$

There are twelve lines (cf exercise 267) they come in four packages of three parallel lines, each corresponding to a point at infinity (at the projective completion of $\mathbb{F}_3^2$) or a singular fiber of the Hesse pencil.

Now we can identify the action of the group G. The normal subgroup (generated by S and T) isomorphic to $\mathbb{F}_3^2$ corresponds to translations by flexes (see exercise 268) while the involution I acts as $z \mapsto -z$ (provided $(1,-1,0)$ is *zero*)

We are also interested in the points of order two, keeping as usual $(1,-1,0)$ as *zero* those correspond to points invariant under I, which can be listed as $(1,-1,0)$ (of course) and points of type $(1,1,\alpha)$ which are solutions to the equation

$$\alpha^3 - 3\lambda\alpha + 2 = 0$$

Thus we can compute the $\jmath$-invariant of a Hesse cubic H_λ as follows

$$\jmath(\lambda) = -\frac{\lambda^3(8+\lambda^3)^3}{64(1-\lambda^3)^3}$$

Polars to Cubics

Let P be a point in the plane. We are interested in finding the tangents to a cubic C which passes through P.

If P is given by (ζ, η, ξ) and (x, y, z) a point on the cubic then its tangent clearly passes through P iff

$$\zeta\frac{\partial C}{\partial x} + \eta\frac{\partial C}{\partial y} + \xi\frac{\partial C}{\partial z} = 0$$

For fixed (ζ, η, ξ) we have defined a conic, the polar conic to P with respect to C, and whose intersection with C gives exactly the points whose tangents pass through P

Thus we observe that through a point outside C we can expect to draw six tangents, and furthermore the corresponding six tangency points lie on a conic, which is somewhat remarkable

In analogy with the case of polars to conics we have the following

Proposition. *. A point P lies on a cubic C iff it lies on its polar Q. The conic Q is then tangent to C at P, and it splits into two lines iff P is a flex, in that case one of the components is the flexed tangent, and the other is the line that joins the three primitive two-torsion points (if P is considered as the zero)*

Proof. We may assume that P is given by (0,0,1) and that the cubic C is given by (cf (1))

$$A_0z^3 + A_1(x,y)z^2 + A_2(x,y)z + A_3(x,y)$$

then the polar Q is simply given by $\partial C/\partial z$, hence

$$3A_0z^2 + 2A_1(x,y)z + A_2(x,y)$$

Thus we see that the condition that P lies on C or Q in both cases is given by $A_0 = 0$. If that is the case we see that the common tangent is given by $A_1(x,y) = 0$. Now (cf exercise 198) P is a flex iff A_1 is a factor of A_2, thus if P is a flex Q splits with one component being the flexed tangent and the other joining the tangency points to the residual tangents passing through P. (cf exercise 274) will be the primitive two-torsion points. Finally if the polar splits one component has to be tangent to the cubic, hence $A_1(x,y) = 0$ has to be a component.

Furthermore the polar Q to a point P is tangent to the cubic iff P lies on a flexed tangent. The polars to points on a flexed tangent form a pencil of conics. This pencil has a double base point exactly at the flex, and it contains two degenerate members. One being of multiplicity one, the polar to the flex, and one being of multiplicity two, with its singular point on the flex. In fact we have

Proposition. *Given a flex F, its polar is a singular conic, and the polar of that conics singular point F' is singular at F. Furthermore F' lies on the Hessian of the cubic, and if F is taken as zero it is a point of order two*

For a proof see exercise 276 In case the cubic has singular points, the polars will all go through those, as is easily checked (exercise 278)

The dual cubic

The tangents to a cubic form a curve in the dual plane. The degree of that curve is sometimes refered to as the class of the cubic, and equal to the number of tangents to the cubic through a given point. As the tangents are given by the intersection of the polar with the cubic we get six (for a topological proof see exercise 279). If the cubic is singular, the count will not be correct as their will be false tangents (lines through the singular point), those will have to be subtracted. We see then that the dual of a nodal cubic will be four, and that of a cuspidal cubic three, (exercise 278).

The case of the cuspidal cubic $y^2z = x^3$ can also be checked by direct computation. The tangent to the point (t^2s, t^3, s^3) is given by $(-3ts^2, 2s^3, t^3)$ thus the dual cuspidal cubic is given by $4x^3 = 27y^2z$ another cuspidal cubic. The cuspidal tangent of the original cubic (corresponding to $t = 0$) will now correspond to the flex of the dual cubic, while the flexed tangent of the original cubic will correspond to the cusp of the dual

This global calculation is really local, thus we see that the dual of a non-singula cubic will be a sextic with nine cusps, while the dual of a nodal cubic will be a quartic with three cusps and a bitangent corresponding to the node. This particular quartic is known as the Steiner quartic

Exercises

186 Let Q be a homogenous form of degree N in k homogenous variables

$$x_1, x_2, \ldots x_k$$

Show that we can write Q in a unique way as

$$x_k Q_1 + Q_2$$

where Q_1 is a form of degree $N-1$ in k variables and Q_2 a form of degree N in $k-1$ variables $x_1, x_2, \ldots x_{k-1}$
Denote by $B(N,k)$ the number of linearly independent forms of degree N and with k variables use the above to conclude the following recursive formula

$$B(N,k) = B(N-1,k) + B(N,k-1)$$

and compare this with binomial coefficients and write down an explicit formula for $B(N,k)$ in terms of a binomial coefficient

187 Show that any cubic written as the sum of two monomials must either be reducible or cuspidal. Can every cubic be written as the sum of three monomials with the proper choice of coordinates?

188 Show that any cubic defined over the reals must have real points,in fact show that any real line has to intersect a real cubic. Is this still true if we restrict to $\mathbb{R}^2$?

189 Compute the dimension of rational cubics (i.e. cubics which can be parametrised by binary forms like e.g smooth conics).(*Hint: Show that a rational cubic must be parametrised by cubic forms*)

190 Given a cubic parametrised by cubic binary forms,(p(s,t),q(s,t),r(s,t)) show directly that either
(i) There are two distinct points on $\mathbb{P}^1$ mapped to the same point
or
(ii) Some combination of (p_t,q_t,r_t) and (p_s,q_s,r_s) vanish at some point

191 Determine for which λ the cubics $(x+y+z)^3 = \lambda xyz$ are singular, and find a parametrisation in those cases

192 Show that there exists a polynomial in the coefficients of two binary quartics (this will of course work for any degree) which vanishes iff the two forms have a common zero. Such a polynomial is refered to as the resultant.(*Hint: If P and Q are two quartics with a common root then one may find two cubics p and q such that Pp=Qq. Treating the coefficients of p and q as unknown show that we have too many equations and not enough unknowns*)

193 If C is a cubic, show that the question of whether C is singular i.e. whether the three conics $\frac{\partial C}{\partial x}, \frac{\partial C}{\partial y}, \frac{\partial C}{\partial z}$ have common base point can be reduced to whether two quartics have a common root
Use this to establish the existence of a polynomial in the coefficients of the cubic, which vanishes iff the cubic is singular. If you are a bit more careful determine the degree of such a polynomial, and if you are really courageous try and determine it explicitly!(*Hint: exercise 192*)

194. Show that a nodal tangent or a cuspidal tangent intersects a cubic in just the singular point, unless of course it is a component of the cubic

195 Show that over the reals there are two types of nodes, one in which the nodal tangents are real, one for which they are complex conjugate. Show that in the latter case the singularity consists of an isolated point, while in the former it has two real branches. On the other hand show that a cusp defined over the reals never can consist of an isolated point

196 Write down all the different types of singular cubics (8) and show that they are all projectively equivalent within their type. Also make a partial order in which one type dominates another if it can be "specialised" to it. Note this is not a total order, as neither a conic with a non-tangent line nor a cuspidal cubic can be "specialised" to each other. Finally try and compute the dimension of each type, as "subvarities" of the big 9-dimensional space of cubics. Note that with the exception of one type (which?) those subvarities are not "closed" but their closures contain types of lower order in the domination ordering. In this way we can make precise the notion of "specialising". Note finally that those types are simply orbits under the induced action of $\mathbf{PGL}(3,\mathbb{C})$ on $\mathbb{P}^9$ and hence are smooth, but their closures are not!

197 Extend the classification of singula cubics to the affine case by considering the position of the line at infinity visavi the cubics. Furthermore refine the classification by considering the real case, and the position of the two circular points at infinity!

198 Show that, conserving the notation of (1), P is a flex iff $A_0 = 0$ (of course) and A_1 is a factor of A_2

199 Let P be a point through which two flexed tangents to a cubic can be drawn, by choosing coordinates such that P is given by (0,0,1) the two flexed tangents by $x = 0$ and $y = 0$ and the flexes by (0,1,0) and (1,0,0) respectively, and such that an additional tangent from P is given by $x = y$ and tangent at (1,1,1), find the equation of the cubic (up to a parameter)

200 Observe that the Hessian of a linear form is always identically zero, and that of a conic constant and equal to zero iff the conic is singular. How does this tally with inflexion points?

201 Given two cubics they form a pencil, the intersection of any two members of this pencil give the same intersection points (the base points), thus we will be done if we can assume that one cubic is singular, as it then can be parametrised by cubic binary forms which can be plugged into the cubic equation of an arbitrary member of the pencil. (If the singular cubic turns out to be reducible, the argument is even simpler). Prove that any pencil oc cubics has a singular member!(cf exercise 193)

202 Find the Hessian of the Fermat cubic $x^3 + y^3 + z^3 = 0$ and determine its flexes explicitly. How many are real?

203 Show that any real (non-singular) cubic has an odd number of real flexes. In particular observe that any real cubic can be put in real Weierstraß form.

204 Prove the claims of the lemma! In particular show that a cubic $x^3 + px + q$ has a multiple root iff it has a common root with its derivative $3x^2 + p$, and use this to get the expression for the discriminant. Furhermore the roots of the derivative corresponds to the local extrema of the cubic, show that we have three real roots iff the local extrema have opposite signs, and just one real root if they have qual signs. Use this to give a criteria in terms of the sign of the discriminant whether the cubic has one or three real roots.

205 Show the identity

$$4p^3 + 27q^2 = (e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2$$

and compare with exercise 204

206 Show that if two non-singular cubics have the same $\jmath$-invariant, then they belong to the same orbit under $\mathbf{PGL}(2,\mathbb{C})$. By considering the two real cubics

$$y^2 = x^3 + px + q$$
$$y^2 = -(x^3 + px + q)$$

show that this is not true over $\mathbf{PGL}(2,\mathbb{R})$

207 Show that the "unbounded" portion of the real cubic meets the line at infinity at the flex. Furthermore show that any line intersects this component in $\mathbb{R}^2$ except some of those parallel to the y-axis. In particular show that the component cannot have any asymptotes. Finally show that all the flexes lie on the "unbounded" component, and that they correspond to x-values which are roots to the polynomial $2qq'' - (q')^2$, where $q(x) = (x-e_1)(x-e_2)(x-e_3)$. You may also try and prove that this polynomial has exactly one real root (hence there are exactly two additional real flexes to the one at infinity),

208 Show that the bounded component encloses a convex region. Write down the integrals giving the length of its perimeter, its area and rotated volume, in terms of the cubic q. Which of those integrals can you do?

209 Show that if the cubic is nodal we have two cases corresponding to whether $e_1 = e_2 < e_3$ or $e_1 < e_2 = e_3$, one case corresponds to a isolated singular point (the nodal tangents are complex conjugate) the other to a connected curve. Which corresponds to which? What about the number of real flexes in those two cases?

210 Give a parametrisation of the the two nodal cubics $y^2 = x^3 + x^2$ and $y^2 = x^3 - x^2$ defined over the reals, and show that in one case we can identify the real part of the cubic (except the isolated singularity) with a circle ($\mathbb{R}P^1$), while in the other case, it is identified with the figure 8

211 Show that the inhomogenous equation $y = x^3$ defines a cuspidal cubic when it is homogenised. Where is the cusp?

212 Find the flex(es) of a cuspidal cubic, either by computing its Hessian and find the intersections (which is now easy why?) or by finding directly a line $Ax + By + Cz = 0$ which becomes a perfect cube when the parametrisation of the cubic is plugged into it.

213 Exploit the parametrisation of a cuspidal cubic to find the equation of a conic which is tangent to the cubic $y^2 = x^3$ at the points $(1,1,1)$,$(4,8,1)$ and $(12,-8,27)$ (*Hint: Write down the binary form of the conic when restricted to the parametrisation of the cuspidal cubic, and go backwards*)

214 By using a parametrisation of a nodal cubic, show that there are three points at which it is possible to find a non-singular conic intersecting the cubic at just one point - with multilicity six (one more than you expect). Such a conic will be refered to as a super-flexed conic. If the nodal cubic is parametrised as in exercise 219, compute the equation of the superflexed conics, how many are real?

215 Show that the nodal cubic $(x+y+z)^3 = 27xyz$ is invariant under the action of $\mathcal{S}_3$ via permutation of coordinates; thus conclude that the nodal tangents most belong to orbits with only two elements under the dual action on $\mathbb{P}^{2*}$ and determine them by finding a unique such orbit

216 Prove that the point $((z-1)^3, (z-\rho)^3, (z-\rho^2)^3)$ is real (with $\rho^3 = 1, \rho \neq 1$) iff $|z| = 1$

217 Show that the cuspidal cubic is not diffeomorphic to the Riemannsphere

218 Substituting the parametrisation $(t^2, t^3, 1)$ of a cuspidal cubic into the equation of a line $Ax + By + Cz$ observe that there is no t-term and interpret this intelligently. Furthermore assuming that the determinant

$$\begin{vmatrix} t_1^2 & t_1^3 & 1 \\ t_2^2 & t_2^3 & 1 \\ t_3^2 & t_3^3 & 1 \end{vmatrix} = 0$$

conclude that $t_1 + t_2 + t_3 = 0$

219 Substituting the parametrisation of exercise 216 into the equation $Ax + By + Cz$ of a line we will observe that the constant coefficient will always equal the leading coefficient. make as above an intelligent interpretation of this. Furthermore considering the vanishing of the determinant below

$$\begin{vmatrix} (t_1-1)^3 & (t_1-\rho)^3 & (t_1-\rho^2)^3 \\ (t_2-1)^3 & (t_2-\rho)^3 & (t_2-\rho^2)^3 \\ (t_3-1)^3 & (t_3-\rho)^3 & (t_3-\rho^2)^3 \end{vmatrix}$$

try and conclude that $t_1 t_2 t_3 = 1$

220 Given a cubic in Weierstraß form $y^2 = x^3 + px + q$ and two points (x_1, y_1) and (x_2, y_2) on it, write down the coordinates (x_3, y_3) of their sum, and show that this is given as rational functions with coefficinets defined over the field generated by p and q. In particular write down the coordinates of a “doubling” of a point (x, y). As a comparison do the same thing for the cuspidal cubic and for some nodal cubic (in Weierstraß form)

221 The above construction allows us to produce new rational solutions provided we already have rational solutions. The rational solutions will form a subgroup, the so called Mordell-Weil group, of the group of all points on the cubic. It is a deep theorem that although this group may be infinite, the rank (as a **Z** module) is always finite. This is known as Mordells theorem.
In particular check that $(1, 2, 1)$ is a rational solution to the curve $y^2 = 3x + 1$ and construct others (less obvious)

222 Although there may be an infinite number of rational solutions to a smooth cubic, there is never more than a finite number of integral solutions. A theorem of Siegel. Show that on the cuspidal cubic $y^2 = x^3$ we may find an infinite number of integral solutions, what about nodal cubics?

223 Let S^1 be a group, defined by its identification with the unit circle $|z| = 1$ inside $\mathbb{C}$. Consider the torus $S^1 \times S^1$ as a group, and show that the involution defined by $(z_1, z_2) \mapsto (-z_1, -z_2)$ has exactly four branch points. Find them!

224 Given a surface one may triangulate it and consider the alternating sums of vertices, edges and sides. (cf exercise 11). This alternating sum, as observed by Euler, is independent of the triangulation and denoted the Euler characteristics. Show that the Euler characteristics of a double cover of the sphere at four points is zero, by choosing a triangulation of the sphere (with Euler characteristics 2) such that the four branch points are part of the vertices, and then lifting it up to a triangulation of the double cover

225 Given a lattice Λ we may take two periods ω_1 and ω_2 which generate the lattice (i.e. $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Those span a parallelogram, which is a so called fundamental region for the action of the lattice Λ on the complex plane $\mathbb{C}$. Now opposite sides of the parallelogram are identified, as they differ by translation of a lattice element, a so called period. Use this to argue that the quotient $\mathbb{C}/\Lambda$ is indeed a torus.

226 Considering the two real cubics in exercise 206, show that they are real subgroups of the same complex torus. Explain how their components fit as closed paths in the torus

227 Show that the unbounded component is a real subgroup of the real cubic

228 Compute the Laurent expansions of the Weierstraß function $\wp_\Lambda(z)$ and its derivative $\wp'_\Lambda(z)$ and conclude that the difference between the two sides of (2) has a Laurent expansion with no poles nor any constant terms

229 Any isomorphism $\theta : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ lifts to a map $\Theta : \mathbb{C} \to \mathbb{C}$ on the universal double coverings, respecting the lattices (i.e. $\Theta(\Lambda) \subseteq \Lambda'$). Show that such a map Θ must be of the form $z \mapsto tz$ for some t. Conclude also that any map between two elliptic curves E and E' respecting the *zeroes* must be a homomorphism

230 We say that F is a modular function of weight n if

$$F(\frac{a\tau + b}{c\tau + d}) = \frac{1}{(c\tau + d)^{2n}} F(\tau)$$

. Examples are given by the Eisenstein forms. A modular function is automatically periodic with respect to $\tau \mapsto \tau + 1$ thus we may may expand it in a so called q-expansion, with $q = \exp(2\pi i\tau)$

$$F(\tau) = \sum_{-\infty}^{\infty} a_n q^n$$

and we say that F is a modular form iff it has a removable singularity at $q = 0$ and a parabolic form if in addition it vanishes at 0.

Show that the space M_n of modular forms of weight n is a finite dimensional vectorspace, that $\dim_{\mathbb{C}} M_n = 1$ for $0 \leq n \leq 5$ and write down the generators in each case. Furthermore show that $\dim_{\mathbb{C}} M_6 = 2$ containing (up to a multiplicative constant) a unique parabolic form Δ. Show that multiplication with Δ yields an isomorphism between M_n and the 1-codimensional subspace M^0_{n+6} of M_{n+6} consisting of parabolic forms. Finally observe that the quotient by any two linearly independent forms in the same space M_n yuields a holomorphic function on $\mathcal{H}/\mathbf{SL}(2,\mathbb{C})$

231 Show that there are polynomials $P_n(x, y)$ with rational coefficients. such that $E_n = P_n(E_2, E_3)$ and give a recursive defintion of those polynomials.(*Hint: Use the the Laurent expansion of the identity (2)*)

232 Show that the function

$$\wp'(z) - \alpha\wp(z) - \beta$$

has three zeroes adding up to a period (cf exercise 239). Use this to show that for suitable choice of α and β we may assume that the zeros are x, y and $-(x + y)$.

Conclude that for $x + y + z = 0 \mod (\Lambda)$ the determinant

$$\begin{vmatrix} \wp(x) & \wp(y) & \wp(z) \\ \wp'(x) & \wp'(y) & \wp'(z) \\ 1 & 1 & 1 \end{vmatrix}$$

vanishes

233 Show that if a finite collection of points in the real projective plane has the property that any line joining two points contains a third, then they all lie on a line. In particular show that a cubic may at most have three flexes.

234 Given the parametrisation of a smooth cubic by $\mathbb{C}/\Lambda$ (Λ generated by $< 1, \tau >$ say) show that the "unbounded" oval corresponds to the image of the real line. While the "small" oval, corresponds to the image of the line $\mathbb{R} + \frac{1}{2}\tau$ which will be real iff the two two-torsion points $\frac{1}{2}\tau$ and $\frac{1+\tau}{2}$ are real.

235 Show that one may find a super-flexed conic (see exercise 214) at a point P iff P is of order 6 but not of order 3. Show that if P is of order 3, the super-flexed conic degenerates into a double line. In particular conclude that a cuspidal cubic has no super-flexed conics

236 The identity (2) is really an example of a non-linear differential equation, showing that the "inverse" function Ψ is a primitive of the function

$$\frac{1}{\sqrt{x^3 + px + q}}$$

Now a primitive does not exist on the torus, as we may find closed paths Γ such that the integrals

$$\int_\Gamma \frac{dx}{\sqrt{x^3 + p + q}}$$

do not vanish. The values of such integrals on closed paths of the elliptic curve are called the periods of the integral, the periods form a lattice Λ and the elliptic curve is recaptured via that lattice.

By chosing suitable "slit"paths between the branch points downstairs on the Riemann sphere, give an integral expression for τ

237 The presentation of $\wp(z)$ in terms of partial fractions is not a very efficient way of computing the function, nor is the Laurent expansion, giving the extra difficulty of approximating the Eisenstein forms

a) Give an error estimate of the approximation

$$\wp_{(N)}(z) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda \\ |\omega| < N}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$

b) Give an error estimate of the Laurent approximation

$$\wp_{<N>}(z) = \frac{1}{z^2} + \sum_{n=1}^{N} (2n+1) E_{n+1} z^{2n}$$

A more efficient way is outlined in exercise 238 below

238 Define the theta function $\theta(z,\tau)$ accordingly

$$\theta(z,\tau) = \sum_{n=-\infty}^{\infty} \exp(\pi(n+\frac{1}{2})^2\tau + 2(n+\frac{1}{2})z)$$

Due to the exponential factors in the summand this converges very fast. This function is not periodic, but it has the following quasi-periodic behaviour

$$\theta(z+1,\tau) = -\theta(z,\tau)$$
$$\theta(z+\tau,\tau) = -1\exp(-\pi i(\tau+2z))\theta(z,\tau)$$

from which we conclude that the function

$$\Theta(z) = \frac{d^2 log\theta(z,\tau)}{dz^2}$$

is doubly periodic.
Show that Θ coincides with the Weierstraß $\wp$-function up to an additive constant
c) Give an error estimate by approximating the theta function by

$$\theta_N = \sum_{n=-N}^{N} \exp(\pi(n+\frac{1}{2})^2\tau + 2(n+\frac{1}{2})z)$$

239 A function doubly periodic with respect to a lattice Λ is called elliptic. Show that for en elliptic function E the integral

$$\frac{1}{2\pi i}\int_{\partial P} Edz$$

where P is a period parallelogram. vanishes. Conclude
a)The sum of the residues for an elliptic function equal zero. Hence an elliptic function cannot have just one pole simple at that
b)The number of zeroes of an elliptic function equal the number of poles (each counted with the appropriate multiplicity). This number is called the order of the elliptic function. Note that no elliptic function has order 1(*Hint: Consider instead of E the elliptic function* $\frac{E'}{E}$)
c) Show that the sum of the distinct zeroes and poles inside a fundamental parallellogram of an elliptic function is an element of the lattice Λ(*Hint: Consider* $\frac{zE'}{E}$)

240 Denote by $\wp_\tau(z)$ the Weierstraß function associated to the lattice $< 1,\tau >$. Show that we have

$$\lim_{\tau\to\infty} \wp_\tau(z) = \frac{\pi^2}{\sin^2 \pi z} - \frac{\pi^2}{3}$$

241 Check that the point $(0,1,-1)$ is a flex on the Fermat cubic $x^3+y^3+z^3=0$ and write down all the linear transformations that fixes the flex and leaves the cubic invariant

242 Show that an elliptic curve is a Fermat cubic iff three flexed tangents go through a point. And if that is the case, the flexed tangents come in three triplets each consisting of concurrent lines

243 Find all complex numbers t such that $t\Lambda = \Lambda$ for the "hexagonal" lattice, they form a finite subgroup of $\mathbb{C}^*$ determine it!

244 Show that the automorphisms of a Fermat cubic permutes the primitive two-torsion points simply transitively (cyclically). What are its orbits under its action on the nine three-torsion points?

245 What is the condition that three three-torsion points lie on a line, or that the corresponding flexed tangents are concurrent in the case of a Fermat cubic?

246 Show that the Riemann surface associated to the function

$$y = \sqrt{1 - x^4}$$

is given by the Gauß cubic. Furthermore if

$$x^4 + y^4 + 2x^2y^2 + y^2 - x^2 = 0$$

gives the equation of a lemniscate (the locus of points whose product of the distances to two fixed points is constant) the lines through the origin (in this case) give a parametrisation via the length $r = x^2 + y^2$ of the radi vectors of the curve
a) Show that the origin is a node of the curve, what are its nodal tangents
b) Are there any other singularities (complex, at infinity?)
c) Show that

$$2x^2 = r^2 + r^4$$
$$2y^2 = r^2 - r^4$$

constitute a parametrisation of the curve
d) Using the parametrisation show that the arclength $s(r)$ (from the origin to the point (x, y)) is given by

$$\int_0^r \frac{dr}{\sqrt{1 - r^4}}$$

e) Show that if

$$r^2 = \frac{4u^2(1 - u^4)}{(1 + u^4)^2}$$

then

$$\int_0^r \frac{dr}{\sqrt{1 - r^4}} = 2\int_0^u \frac{du}{1 - u^4}$$

this is known as a formula for doubling the arclength of a lemniscate, devised by the Italian count Fagnano at the beginning of the 18^{th} century (cf exercise 220)

247 Determine the complex numbers t such that $t\Lambda = \Lambda$ for a "square"lattice, show that they form a finite subgroup of $\mathbb{C}$ and determine it

248 Show that the automorphisms of the Gauß cubic switches two primitive two-torsion elements and fixes a third. Which? How does it act on the flexes?

249 Show that the Gauß cubic is characterized by the existence of a flexed tangent and a super-flexed conic (see exercise 214) which are tangent

250 Show that there are to each degree N only a finite number of isogenies of degree N. How many? Conclude that each equivalence class only contains a countably infinite number of curves

251 Show that the endomorphism $z \mapsto Nz$ is an isogeny of an elliptic curve onto itself. Determine its degree and show that the composition of an isogeny and its dual is always of that type

252 Show that any isogeny can be factored into a cyclic isogeny and one of form $z \mapsto Nz$

253 Are the Fermat cubic and the Gauß cubic isogenous?

254 If E,E' and E" are isogenous curves, with an isogeny of degree N between E and E' and an isogeny of degree M between E and E", show that there exists an isogeny of degree P between E' and E", where $P|MN$

255 Let E_N denote the subgroup of N-torsion points, and let P be a primitive such point, generating a subgroup P_N of E_N. P defines an isogeny of degree N onto an elliptic curve E'. Show that we have a surjection from E_{N^2} to E'_N

256 Given the four branch points normalized to $1, 0, \infty$ and λ say on the Riemann sphere. Assuming that 0 corresponds to the *zero* upstairs, write down the three involutions corresponding to the points $1, \infty$ and λ and find their corresponding fixed points

257 Given a lattice Λ generated by $< 1, \tau >$ and consider the isogenies corresponding to the points $\frac{1}{2}, \frac{\tau}{2}$ and $\frac{1+\tau}{2}$, determine the period parallelograms, or equivalently the "τ" values of the quotient. Use this to compute the $\jmath$-value of the quotient in terms of the original elliptic curve

258 Make a similar calculation of the $\jmath$-invariants of quotients of degree two, but this time using exercise 256

259 Recall that any endomorphism of an elliptic curve is given on the universal cover by $z \mapsto \lambda z$ (cf exercise 229), thus the existence of non-trivial isomorphisms is equivalent to the existence of non-integral λ such that $\lambda\Lambda \subseteq \Lambda$.
a) Show (assuming as we may that $\Lambda =< 1, \tau >$) that this is equivalent to τ satisfying a quadratic equation
b) Show that the endomorphism ring of the elliptic curve will turn out to be the ring of integers of the quadratic field $\mathbb{Q}(\tau)$

260 Determine all elliptic curves who have endomorphisms of degree two (i.e. determine the "τ" values) and compute their $\jmath$-invariants. Use this to compute the Eisenstein forms for such lattices

261 Assume that z_1, z_2, z_3 lie on a line, show that $z_1 + \theta, z_2 + \theta, z_3 + \theta$ do so iff $3\theta = 0$ i.e. a flex

262 Show directly that neither the nodal nor the cuspidal cubic can be written under Hesse form

263 Write down the Hessian of a cubic H_λ in Hesse form and verify that it is also of Hesse form $H_{\lambda'}$

264 Determine all the orbits under the group G containing at most three elements. Use this to confirm that there are no other singular members than those four accounted for by $\lambda = \lambda'$

265 Observe that if $\lambda^3 \neq 1, \infty$ then H_λ has nine distinct flexes, and hence cannot be singular. Why?

266 Write down the equations of the flexed tangents to the flexes of a Hesse pencil. Those equations will depend on λ. Try and find those λ for which we may find three concurrent flexed tangents (cf exercise 242)

267 Show that G acts on the dual projective plane of lines, and that the orbits with three elements divide into four groups, each corresponding to a singular member (cf exercise 264)

268 S and T acts as translations on the subgroup of flexes, identify that action setting $(1, -1, 0)$ as *zero*

269 Given (x_0, y_0, z_0) on cubic H_λ and letting $(1, -1, 0)$ be *zero* use the vanishing of

$$\begin{vmatrix} x & y & z \\ x_0 & y_0 & z_0 \\ 1 & -1 & 0 \end{vmatrix} = 0$$

to determine explicitly the involution $z \mapsto -z$ on the coordinate level

270 Show that points of form $\theta + \epsilon$ with θ flexes andϵ points of order two, form points of order six; and that conversely every point of order six can be written uniquely in such a form. Try and find all the six-torsion points on a Hesse cubic H_λ

271 Show that a Hesse cubic H_λ with λ real has one or three real points of order two depending on the sign of $1 - \lambda^3$. Show that a Hesse cubic has three real points of order two iff its Hessian has just one real point of order two. Will this still be true if the cubic is not under Hesse form?

272 For certain values of λ the group leaving H_λ invariant is even bigger than G. Prove that we have the two following cases
(0) $U(\xi, \eta) = (\xi, \eta - \xi)$ lifts to a linear transformation leaving H_λ invariant for $\lambda = 0$ and $\lambda^3 = -8$ (the case of the Fermat cubic)
(1) $V(\xi, \eta) = (-\eta, \xi)$ lifts and leaves H_λ invariant in the case of $\lambda = 1 \pm \sqrt{3}$ (the case of the Gauß cubic)
And determine explicitly the linear transformations that lift U and V, and how they will transform $H - \lambda$ in general

273 Show that there exists 12 λ values for each $\jmath$-value, except for certain $\jmath$ values. Which?

274 Show that if P is a point on a cubic, we can draw four tangents from P, not counting the tangent at P, unless of course it is a flex. Show that in the latter case the three residual tangents are tangent at three points lying on a line, and those three points are the primitive 2-torsion points, in case P is taken as the zero

275 Show that a polar conic is bitangent to the cubic iff the point lies on the intersection of two flexed tangents. What can we say in the case the polar conic is triply tangent?

276 Letting F=$(1, -1, 0)$ be the *zero* on the Hesse cubic H_λ show that the polar of the point F'=$(\lambda, \lambda, -2)$ has its singular point on F. Show furthermore that it actually is the singular point of the polar of F, and that it lies on the Hessian. Finally show that the tangent at F' of the Hessian passes through F and conclude that F' is a point of order two (on the Hessian, with F=0)

277 Preserving the notation from exercise 276. The polars of the points lying on the flexed tangent of the Hessian $H_{\lambda'}$ to F, form a pencil of conics. Show that those conics are flexed to each other at F' and that their common tangent is given by the flexed tangent of H_λ, and that their simple base point is given by $(3\lambda, 3\lambda, \lambda^3 + 2)$

278 Show that a singular point of a cubic lies on every polar. Show furthermore that if the singular point is a cusp, every polar will be tangent to the cuspidal tangent

279 Let C be a smooth cubic and P a point outside. Projecting from P to a line we get C presented as a triple cover of the Riemann sphere. By chosing a triangulation with vertices at the branch points, compute the Euler characteristic of C in terms of the Euler characteristic of the Riemann sphere and the number of branch points. (cf exercise 224). As two quantities in the "equation" will be known, find the unknown (the number of branch points) and interpret this number

280 Show that the cuspidal tangents of the Steiner quartic all are concurrent

281 Show that the real points of a Steiner quartic are formed by the hypocycloid of a circle of radius r rolling inside a circle of radius 3r

282 Show that the Steiner quartic is given by the image of a conic tangent to the edges of a triangle blown down under a Cremona (cf exercise 183 c)

Nets of Conics

By a net of conics we mean a 2-dimensional linear subspace N of the space $\mathbb{P}^5$ of conics.

A net in analogy with a pencil can be written as all the conics of the form

$$\lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2$$

where $(\lambda_0, \lambda_1, \lambda_2)$ then are homogenbous coordinates for the plane N.

Associated to a net is a discriminant cubic corresponding to the conics in the net which are singular. This is clearly given by the intersection of N with the discriminant hypersurface.

Furthermore the discriminant cubic of a net comes with a canonical double covering, namely to each singular conic we can associate either component. This double cover the "dual" discriminant is a curve that naturally lives in the dual $\mathbb{P}^*$. This is also a cubic, this can be seen as follows. Each point p of $\mathbb{P}^2$ defines a line in the dual space, the number of times this line meets the "dual" discriminant is given by the number of components of singular conics in the net passes through p. The conics in the net passing through p form a pencil (unless of course p is a base point), a pencil has three singular members counted of course with appropriate multiplicities, thus the degree of the "dual" discriminant is also three.

We may also associate to N a curve in $\mathbb{P}^2$ being the locus of singular points of singular members of the net. This curve will be denoted the Jacobian of the net A member of the net is singular iff all its partials vanish at some point. For such a point this means that the system

$$\begin{array}{ccccccc} \lambda_0 \partial Q_0/\partial x & + & \lambda_1 \partial Q_1/\partial x & + & \lambda_2 \partial Q_2/\partial x & = & 0 \\ \lambda_0 \partial Q_0/\partial y & + & \lambda_1 \partial Q_1/\partial y & + & \lambda_2 \partial Q_2/\partial y & = & 0 \\ \lambda_0 \partial Q_0/\partial z & + & \lambda_1 \partial Q_1/\partial z & + & \lambda_2 \partial Q_2/\partial z & = & 0 \end{array}$$

has a nontrivial solution $(\lambda_0, \lambda_1, \lambda_2)$ which translates into the vanishing of the determinant

$$\begin{vmatrix} \partial Q_0/\partial x & \partial Q_1/\partial x & \partial Q_2/\partial x \\ \partial Q_0/\partial y & \partial Q_1/\partial y & \partial Q_2/\partial y \\ \partial Q_0/\partial z & \partial Q_1/\partial z & \partial Q_2/\partial z \end{vmatrix}$$

which gives the equation for the Jacobian and shows that it is a cubic as well.

Given a net N we may canonically associate a map

$$\Theta : \mathbb{P}^2 \to N^*$$

By simply to each point p $\in \mathbb{P}^2$ associating the subpencil of N (i.e. a line in N and hence an element of N*) consisting of the conics passing through p. If p happens to be a basepoint this will not be defined, unless we blow up p. We will from now on assume that the net does not have a basepoint, and later discuss the rather easy modifications when it has

This map is 4:1 as each fiber consists of the basepoints of the corresponding pencil. A point P (P for pencil) of N* is called a branchpoint iff the corresponding fiber does not have four distinct points. And a point p of $\mathbb{P}^2$ is called a ramification point iff it is a multiple point of some fiber

A pencil which has a multiple base point is tangent to the discriminant hence to the discriminant cubic. Thus the locus of branchpoints on N^* is simply the locus of tangents to the discriminant, hence its dual curve. As a dual cubic it is a sextic with nine cusps, the cusps corresponding to the flexed tangents of the discriminant, or equivalently to branchpoints whose fibers contain points of multiplicity three.

At a ramification point p, the corresponding pencil of conics through p has (at most) two singular members. In any case one of those has p as singular point. Thus the locus of ramification points coincide with the Jacobian.This gives a geometric explanation of the determinental form of the Jacobian equation above.

At a ramification point p, all the conics passing through p are tangent (cf exercise 283). Thus if the conics of a net would be plotted the Jacobian would appear darkened as a Moire pattern. If p has multiplicity three in its fiber, then all the conics through p are flexed to each other.

At a point p of the Jacobian there are two distinguished lines, namely the two components of the singular member. If the multiplicity of p is two, there is also a third distinguished line, namely the common tangent to all the conics through p. If p is of multiplicity three that common tangent will coincide with one of the two components. There is also of course a fourth line, the tangent of the Jacobian at p. Its relationship to the other three (two) lines is interesting (see exercises 285,286 and 287) We note the following

Lemma. *If P is a singular point of a singular conic L_1L_2 corresponding to a flex of the discriminant, then the tangent to the Jacobian at P coincides with the component L_i which is the common tangent to all the conics in the pencil determined by P*

For a proof see exercise 287. We only note that such points P are mapped onto cusps of the branchcurve on N^* under the canonically associated map $\Theta : \mathbb{P}^2 \to N^*$, but that they are *not* flexes of the Jacobian curve

Assuming now that the net N does not contain any double lines, we will have a 1-1 correspondence between the points of the discriminant cubic D and those of the Jacobian J. Namely each point of the discriminant determines a singular conic which has a unique singular point; and conversely if p is a singular point of a singular conic, that singular conic is unique (otherwise we would have a pencil spanned by two containing double lines)

There is also a direct correspondence between the points of the Jacobian (the ramification curve) and the branchcurve, assuming of course no double lines, as there can be at most one multiple point in a fiber of Θ

The map Θ maps the conics of N onto lines (4:1) and the inverse of each line of N^* is a conic of the net, in fact the conic that corresponds to the line by duality. This gives another proof that the Jacobian is a cubic, in fact a line L in N^* intersects the branchcurve in six points, those six points corresponds 1:1 with the intersection of their fibers with the Jacobian, hence the Jacobian intersects the inverse image of L in exactly six points, as the inverse image is a conic, we see that the Jacobian has to be of degree three

Furthermore the map Θ maps lines onto conics, except when the lines are components of singular members of the net. (see exercise 288)

Jacobian Nets

Given a cubic C we may consider the net spanned by all the partials

$$\partial C/\partial x, \quad \partial C/\partial y, \quad \partial C/\partial z$$

The particular conics spanning the net depend of course on the choice of coordinates, the net itself is independent of the coordinates. This can be seen by a direct calculation, or better the Jacobian net can be identifed with all the polars of C. Thus presenting a Jacobian net means identifying the net N with the space $\mathbb{P}^2$ on which the conics of the net lives

In particular the two cubics the discriminant and the Jacobian now live in the same space. We have the following remarkable fact

Proposition. *The discriminant and the Jacobian of a Jacobian net coincide. The canonical isomorphism now becomes a canonical involution. This involution defines an isogeny dual to the isogeny given by the "dual" discriminant and its canonical involution described above*

Proof. As we are for the moment considering nets without basepoints, we will restrict ourselves to the case when C is non-singular. Hence we can assume that it is in Hesse normal form.

The fact that a point (x, y, z) is a singular point of a conic in the net, or that a given point $(\zeta_0, \zeta_1, \zeta_2)$ has a polar which is singular can be stated that the system

$$\begin{array}{ccccccc} \zeta_0 2x & + & -\zeta_1\lambda z & + & -\zeta_2\lambda y & = & 0 \\ -\zeta_0\lambda z & + & \zeta_1 2y & + & -\zeta_2\lambda x & = & 0 \\ -\zeta_0\lambda y & + & -\zeta_1\lambda x & + & \zeta_2 2z & = & 0 \end{array}$$

has non-trivial solutions in both $(\zeta_0, \zeta_1, \zeta_2)$ and (x, y, z). Thus it means that both the determinant

$$(1) \qquad \begin{vmatrix} 2x & -\lambda z & -\lambda y \\ -\lambda z & 2y & -\lambda x \\ -\lambda y & -\lambda x & 2z \end{vmatrix}$$

and the determinant

$$(2) \qquad \begin{vmatrix} 2\zeta_0 & -\lambda\zeta_2 & -\lambda\zeta_1 \\ -\lambda\zeta_2 & 2\zeta_1 & -\lambda\zeta_0 \\ -\lambda\zeta_1 & -\lambda\zeta_0 & 2\zeta_2 \end{vmatrix}$$

vanish. Now (1) gives the Jacobian and (2) gives the Discriminant. (Note that it is clear that the Jacobian of a Jacobian net is simply the Hessian of the corresponding cubic, and as we have remarked the Hessian of a cubic in Hesse form is still in Hesse form)

Straightforward computations now yield that both determinants give the same cubic (with the obvious notational change in the case of (2))

$$x^3 + y^3 + z^3 = 3(\frac{8 - 2\lambda^3}{6\lambda^2})xyz$$

Furthermore for each critical value (i.e on the discriminant) $(\zeta_0, \zeta_1, \zeta_2)$ we can associate the point

$$(\text{"i"}) \qquad \left(\begin{vmatrix} -\lambda\zeta_2 & -\lambda\zeta_1 \\ 2\zeta_1 & -\lambda\zeta_0 \end{vmatrix}, \begin{vmatrix} -\lambda\zeta_1 & 2\zeta_0 \\ -\lambda\zeta_0 & -\lambda\zeta_2 \end{vmatrix}, \begin{vmatrix} 2\zeta_0 & -\lambda\zeta_2 \\ -\lambda\zeta_2 & 2\zeta_1 \end{vmatrix} \right)$$

of the Jacobian. And similarly to each point (x, y, z) of the Jacobian we can associate the point

$$(\text{"ii"}) \qquad \left(\begin{vmatrix} \lambda z & -\lambda y \\ 2y & -\lambda x \end{vmatrix}, \begin{vmatrix} -\lambda y & 2x \\ -\lambda z & -\lambda x \end{vmatrix}, \begin{vmatrix} 2x & -\lambda z \\ -\lambda z & 2y \end{vmatrix} \right)$$

of the discriminant. Clearly they are each others inverses. (This does not have to be checked computationally of course, as a singular conic determines uniquely its singular point, which in its turn determines the singular conic).

Those maps can of course be extended to the whole of $\mathbb{P}^2$ and are then given by nets (see exercise 289)

The fact that the involution defines an isogeny dual to the isogeny defined by the natural double cover of the discriminant is deeper.

Given a flex F on the cubic C, it will also be a flex of the discriminant cubic (or if you prefer the Jacobian cubic) and from now on refered to as the *zero* of either cubic involved. Its polar with respect to C will be singular at a point $\tilde{F}$ and consist of two lines LM with L the flexed tangent of the cubic C passing through the flex F and M will be the line joining the three primitive two-torsion points (cf exercise 274).

Considering the pencil Π defined by the flexed tangent to the *discriminant cubic* at F it will have two base points, one of them being $\tilde{F}$ of multiplicity 3, and the other being $\bar{F}$ of multiplicity one. By the previous lemma, the common tangents to the conics of the pencil Π will be L and the point $\bar{F}$ will hence lie on M

Looking at the dual space $\mathbb{P}^{2*}$ the point $\bar{F}$ will correspond to a flexed tangent to the dual discriminant D^* at the flex M (considered now as a point in $\mathbb{P}^{2*}$), while $\tilde{F}$ will correspond to a line through M residually tangent to the point L (on D^*). The canonical involution on the dual discriminant, interchanges L and M, choosing M (being a flex on D^*) as the natural *zero* on the dual discriminant, this involution is given by translation by the point L of order two. There will now be two other points of order two on the cubic D^* namely L' and L'' permuted by translation by L, the corresponding tangents both passing through M will be denoted by G' and G'' will also be considered as two points lying on the line M in $\mathbb{P}^2$. As the points L, L' and L'' lie on a line (in fact the point F considered as a line in $\mathbb{P}^{2*}$) the lines G', G'' and L pass through a common point (in fact F)

The two lines G' and G'' form a singular conic in the net, as they are paired by the canonical involution on D^*, the singular point is of course F and it will be the polar of the point $\tilde{F}$ (cf exercise 276)

We have thus identified the dual isogeny defined by the canonical involution on the dual discriminant with the natural involution defined on the discriminant. In fact the two pairs L, M and L', L'' on the dual cubic correspond to two singular conics, i.e. two points on the discriminant (or Jacobian), which are permuted by the natural involution on that cubic

Singular Nets

A singular discriminant cubic can be obtained in two different ways (and further degenerations thereof). Either there could be base points (typically one base point) or there could be double lines

In the case of base points, there could be one, two or three thereof. (The case of four base points forces the net to be a pencil), which may coincide. There may also be a common line component.

In the case of double lines there could be one, two or three of those, but never more.

In the case of a base point P the map

$$\Theta : \mathbb{P}^2 \to N^*$$

is not defined at P, but the point has to be blown up. The degree of the map is also then reduced by one. Thus a single base points gives a 3:1 map between the blow up $B_P\mathbb{P}^2$ and N^*, while two base points give a double cover of N^*, and finally three base points define a Cremona transformation from $\mathbb{P}^2$ to N^* generically 1:1, as all birational transformations

It is interesting to determine the branch loci in the three cases and the corrsponding relations between the discriminant and the Jacobian and the dual discriminant

Geometrically a net N has a base point P iff it is tangent to the discriminant hypersurface in the space $\mathbb{P}^5$ of all conics. The tangency point $\check{P}$ corresponds to the unique singular conic in the net with its singularity at P (see exercise 294). This point $\check{P}$ will be a singular point of the discriminant cubic, as any pencil containing $\check{P}$ will have its singular point as a base point. This proves the assertion that N is tangent to the discriminant locus.

The nodal tangents at $\check{P}$ will correspond to the two pencils formed by chosing conics in the net tangent to either component of $\check{P}$. We see thus that the singularity at $\check{P}$ has degenerate quadratic term (cf discussion of singular cubics) iff the net has a double line (necessarily passing through P). Thus we may observe that a net which has both a base point and a double line has a cuspidal discriminant (provided there are no other base points or double lines). Geometrically this corresponds to the net N being tangent to the discriminant locus at a singular point, i.e. actually tangent to the singular locus (see exercise 295)

Higher dimensional system of conics

As the space of conics is five-dimensional. any six conics are linearly dependent, and no higher dimensions than five occur. Such a linear system must coincide with the space of conics itself, and nothing more can be said of it.

As to four-dimensional systems, they correspond to hyperplanes, and they are completly determined by their intersections with $\mathcal{S}$ the singular locus of the discriminant hypersurface. $\mathcal{S}$ is a surface, in fact the Veronese surface, identified by $\mathbf{P}^2$ or rather by the dual $\mathbf{P}^{2*}$. Its hyperplanes correspond to conics, thus four-dimensional systems are dueal to 0-dimensional ones, and the classification is already known.

In the same way 3-dimensional systems of conics, so called webs of conics are dual to pencils (1-dimensional families), by associating to each web W the intersections with $\mathcal{S}$ of the hyeprplanes containing W. Thus a web is determined by its four intersection points with $\mathcal{S}$.

Finally we see that to each net N we can associate a dual net N* on the dual projective space, by associating the intersections with $\mathcal{S}$ of hyperplanes containing N.(see exercises 299,300)

We have seen before that the discriminant hypersurface has a natural non-singular double covering given by $\mathbf{P}^{2*} \times \mathbf{P}^{2*}$ ramified at the diagonal, canonically identified with $\mathcal{S}$. The discriminant of a 4-dimensional system F will be doubly covered by the zeroes $Z(B)$ of a bilinear form $B(X,Y)$ with X and Y linear forms on $\mathbf{P}^2$. This bilinear form will in effect be symmetric (see exercise 301). The 3-dimensional space $Z(B)$ will have a projection onto either factors and with fibers $\mathbf{P}^1$. In fact the fiber over a point X will be the line P_X given by $\{Y : B(X,Y) = 0\}$. As a symmetric bilinear form corresponds to a conic C, the fiber will just be the polar to X with respect to the conic C. The conic C will be canonically identified with the intersection of $Z(B)$ with the diagonal Δ (canonically identified with $\mathcal{S}$). Thus we will have two cases.(cf exercise 303)

1) B is non-degenerate, or equivalently the system F intersects $\mathcal{S}$ transversally

or

2) B is degenerate,(of rank 2) or equivalently the system F is tangent to $\mathcal{S}$ at a point L

In this case the fiber of $\pi : Z(B) \to \mathbf{P}^{2*}$ will "blow-up" above L to an entire plane.

3) B is degenerate (of rank 1) or equivalently the system F is tangent to $\mathcal{S}$ along an entire line ℓ

Now away from the points of ℓ the fibers are canonocally identified with ℓ, above ℓ the fibers "blow up" to planes

In the case of a web, we will consider a double covering of the discriminant surface, $Z(B_1, B_2)$ (or Z for simplicity), which is given by the intersection of two symmetric bilinear forms $B_1(X,Y)$ and $B_2(X,Y)$ on $\mathbf{P}^{2*} \times \mathbf{P}^{2*}$. Those bilinear forms will determine a pencil of conics, and the fiber associated to the projection to (either) factors over a point P, will correspond to the intersections of all the polars to P with respect to all the conics in the pencil. In general this will correspond to a point, except when P coincides with a singular point of a singular fiber, when we will have an entire line. The surface Z will be a blow up of $\mathbf{P}^{2*}$ above three points (which may coincide) and it will define a graph of a birational isomorphism between the two factors. In fact in general a Cremona transformation.(see exercise 306,307). The surface Z will be invariant under the natural involution (the forms $B_1.B_2$ are symmetric) and the quotient will be a singular cubic hypersurface in the web, with singular points corresponding to the four base points of the dual pencil, or equivalently the four intersection points with the diagonal.

Finally in the case of a net, we may describe the double covering of the discriminant cubic, as the intersection of three symmetric bilinear forms corresponding to the dual net. Projecting as usual onto a factor $\mathbf{P}^{2*}$ we see that the fiber over a point L is non-empty, iff the polars of L with respect to the dual net all go through a point. By exercise 284 this happens iff L is a point of the Jacobian of the dual net. As clearly the double covering defined by the bilinear intersection in $\mathbf{P}^{2*} \times \mathbf{P}^{2*}$, is none but the canonical one, associating either component of a singular conic, we have shown that the "dual" discriminant is in fact isomorphic with the Jacobian of the dual net, and hence isomorphic with its discriminant.(As the linear forms are symmetric, the natural involution leaves the curve invariant) These ideas can be exploited to show the dualism between the two natural isogenies discussed in the context of Jacobian nets.(See exercise 309)

The birational map between $P(S^2V)$ and $P(S^2V^*)$

Given any conic C we may associate the dual conic C* in the dual projective plane. This presents no problem when we just consider non-singular conics. Then there is a 1-1 correspondence. The problem arises when we consider singular conics, in particular double lines.

The duals of a pencil Π of conics do not form a pencil of conics in the dual space, and conversely. Given a pencil of conics in the dual space, they will (in general) have four base points, and dualizing we get a one dimensional family of conics tangent to four given lines (corresponding to the base points). This is not a linear family, as the reader can easily convince herself of.(see exercise 310). This means that the association between the conics and their duals cannot be extended to a linear one between the complete 5-dimensional spaces.

The dual of a singular conic (not a double line) will correspond to the lines through the singular point. Thus two singular conics with the same singular points will have the same dual, naturally a (double) line. It is now natural to encode the dual, by distinguishing two points on it corresponding to the two components. In this way we are lead to the definition of so called complete conics. A complete conic is either a non-singular conic or a singular conic with two components, or a double conic with two (possibly coinciding) distinguished points.

It is now clear that duality extends to complete conics, in fact in such a way that we have complete duality. The dual of a double line with two distinguished points, is clearly two lines corresponding to the two points (or a double line with a distingusihed point), and the dual of a conic with two component, a double line with distinguished points corresponding to the two components.

Geometrically the space of complete conics can be thought of as a blow up of the space of conics, blown up at the locus $\mathcal{S}$. In fact given a point Q in $\mathbf{P}(S^2V)$, the lines through Q make up a $\mathbf{P}^4$ (parametrising all directions), and the cone over Q a $\mathbb{C}^5$ denoted by D_Q. If Q would lie on $\mathcal{S}$ the tangent directions to $\mathcal{S}$ at Q would form a $\mathbf{P}^1$ and its cone a $\mathbb{C}^2$ denoted by T_Q. If we identify two directions in D_Q iff their difference lie in T_Q we get a space $\mathbf{P}^2$ of "normal directions" to $\mathcal{S}$ at Q. Blowing up $\mathcal{S}$ means "replacing" each point of $\mathcal{S}$ with all its normal directions. The "exceptional divisor" in this case will then be a fibration over $\mathcal{S}$, with each fiber a $\mathbf{P}^2$ corresponding to the normal directions. The fiber above a special double conic Q will be all the complete conics with Q as underlying conic. As the space of two points on the projective line is naturally given by the projective plane (via binary quadrics, cf exercise 87) everything fits nicely. Furthermore a pencil Π of conics through Q determine two points on Q, namely the two basepoints. A pencil corresponds to a direction, and consequently two directions are equivalent, if the corresponding base points are identical (see exercise 312)

A more formal and direct definition of the complete conics would be had if we consider the closure of the graph of the association of dual conics defined on non-singular conics.(see exercise 314).

This closure defines a birational correspondence between the two spaces $\mathbf{P}(S^2\mathbb{C}^3)$ and $\mathbf{P}(S^2\mathbb{C}^{3*}$ By identifying somehow the projective plane with its dual (via some non-singular quadric) we also get a birational correspondence on the space of conics itself. This is a Cremona transformation generalized to higher dimensions. It is given by a six-dimensional family of quadrics in six homogenous variables, given by the 2×2 minors of a symmetric 3×3 determinant (cf exercise 311). It blows up the locus $\mathcal{S}$ of double lines, while it blows down the discriminant hypersurface, in analogy with the Cremona transformation in the plane that blows down the edges of a triangle (a cubic), while blowing up its vertices.(see exercise 315 for further analogies)

The tangent locus to a conic

Given a (non-singular) conic C, we may look at all the conics C' which are tangent to C. Projecting from C, we see that C' is tangent iff the pencil spanned by C and C' is tangent to the discriminant locus. The discriminant locus projects 3:1 onto a hyperplane and will be ramified along a hypersurface of that plane. By considering a net containg C, we see that the degree is six (as there are six tangents to a cubic through a point outside, for an alternative prrof see exercise 317). Thus the conics C' tangent to C form a hypersurface of degree six.(A cone over the ramification locus). In particular given a pencil Π and a conic C, not a member of the pencil, we expect six conics in the pencil to be tangent to C. Conics C' tangent to two conics C and $\hat{\mathrm{C}}$ then form a space of codimension two, the intersection of the two sextic hypersurfaces. And given a net of conics we expect 36 members to be tangent to two given conics.

Continuing reasoning in likewise manner, one may conclude that there are a finite number of conics tangent to five given conics (as there is a finite numbner, in fact just one, conic through five given points (unless in very special position). And even bolder conclude like Steiner did, that there are 7776 such conics.(If you wonder about the number think of 6^5). Now any double line is automatically tangent to any given conic, thus a sextic hypersurface of conics tangent to a given conic always contains $\mathcal{S}$, and thus the intersection of any number, in particular five, always contains $\mathcal{S}$. Thus the number 7776 has little relevance. Yet one may wonder?

The solution to this dilemma is to look at the space of complete conics. Then we consider the proper transforms (i.e. the closures of the inverse images of non-singular conics in the tangent locus to a given conic) of tangent loci. It now turns out that a complete conic, i.e. a double line with two distinguished points is tangent to a conic iff one of its distinguished points lie on the conic. Now it turns out that five proper tangentloci intersect in a finite number of points, and with a mild generality conditions on the five conics. now double line will be tangent to all of them. The problem now is to compute the number.

The number 7776 was computed by Bezout's theorem, generalized to higher dimensions. The argument was that any hypersurface is formally given as $n\mathrm{H}$ where H is a hyperplane, and n is the degree of the hypersurface. In the case of a plane we have that a hyperplane H is a line, and the intersection H^2 is a point. Thus formally $n\mathrm{H}$ and $m\mathrm{H}$ intersect in $mn\mathrm{H}^2$ i.e. in mn points. In the case of the 5-dimensional space of conics, we need to look at higher powers H^5 to get a point. Bezouts theorem can also be worked out for other spaces, e.g the quadric. Then we have two families of lines L_1 and L_2 and every curve is given by a bihomogenous polynomial of bidegree (d_1, d_2) and such curves can be written formally as $d_1L_1+d_2L_2$. To get the number of intersection points we need to look at the three basic intersections $L_1^2 = 0$,$L_2^2 = 0$ as lines in the same family do not meet, and $L_1L_2 = 1$ as lines from opposite families meet in exactly one point. To get the intersections for two curves $d_1L_1 + d_2L_2$ and $e_1L_1 + e_2L_2$ we multiply formally to get

$$d_1e_1L_1^2 + (d_1e_2 + d_2e_1)L_1L_2 + d_2e_2L_2^2$$

which simplifies to the right intersection number

$$d_1e_2 + d_2e_1$$

The space of complete conics needs two generators to describe all the hypersurfaces. One say given by H, the inverse image of a hyperplane under the natural projection onto the space of conics, and E the exceptional divisor. The proper transform of a quadric containing $\mathcal{S}$ will then be given by 2H-E, while the proper transform of the discriminant cubic will be 3H-2E as $\mathcal{S}$ is the locus of double points on the discriminant. The proper transform of the tangent locus will be 6H-2E (see exercise 319), and we are reduced to look at the formal product

$$(6H - 2E)^5 = \sum_{k=0}^{5}(-1)^{k+1}6^k2^{5-k}\binom{5}{k}H^kE^{5-k}$$

and thus we have to compute the intersections

$$H^5, H^4E, H^3E^2, \dots E^5$$

Those computations are unfortunately a little beyond the scope of this "book", we are swimming at deep water as it is, but some of those can be heuristically explained. In fact $H^5 = 1$ is easy, and that $H^4E = H^3E^2 = 0$ should be clear as a line or a plane in general do not meet $\mathcal{S}$ and hence not the exceptional divisor. The remaining three are harder, and just for completeness we give them.

$$H^2E^3 = 4, HE^4 = 18, E^5 = 51$$

The reader can now, with some patience, complete the calculation, yielding the magic number 3264, which has significance. The interested reader can consult **Griffiths** and **Harris** *Principles of Algebraic Geometry* (page 749 ff) for the complete details and the checking that the five hypersurfaces do in fact meet transversally.

Exercises

283 Show that if p is a point *not* on the Jacobian then through every line L through p, there is a unique conic in the net tangent to L at p. What happens if p lies on the Jacobian?

284 Show that if p is a point *not* on the Jacobian then the polars of p with respect to the conics of the net comprise all lines, and hence establish an isomorphism between N and $\mathbb{P}^{2*}$. On the other hand show that if p lies on the Jacobian, the polars all pass through a fixed point p', and in particular given a line L there is an entire pencil of conics in the net such that for each of them L is the polar of p.

285 Given the four lines through a point on the Jacobian, compute the crossratio of the two lines forming the singular conic with respect to the common tangent of conics and the tangent to the Jacobian. In particular determine whether this crossratio is constant(*Hint: Make an explicit calculation by* e.g. *using a Hesse cubic* cf exercises 276,277)

286 Show that if given a singular conic $C_0 = L_1L_2$ the pencil spanned by C_0 and the conic C_1 is tangent to the discriminant cubic at C_1 iff C_1 is tangent to either L_1 or L_2, and that it is tangent at C_0 iff C_1 passes through the singular point of C_0

287 Preserving the notation of exercise 286 conclude that the lines L_1 and L_2 intersect the Jacobian cubic in two additional points each, in addition to the singular point of C_0 (cf exercise 274). In particular observe that neither line L_1 or L_2 is tangent to the Jacobian (cf exercise 285), unless C_0 is a flex of the discriminant

288 Show that a line L is a component of a singular member of a net N, iff the restrictions of three spanning quadrics to L are no longer linearly independent. Use this to get an explicit equation for the "dual"discriminant as a cubic in $\mathbb{P}^{2*}$ for the Jacobian net of the cubic

$$x^3 + y^3 + z^3 = 3\lambda xyz$$

289 Show that the three conics

$$\lambda\zeta_0\zeta_2 + 2\zeta_1^2 \quad \lambda\zeta_1\zeta_2 + 2\zeta_0^2 \quad 4\zeta_0\zeta_1 - \lambda^2\zeta_2^2$$

defined by the minors in (i) define a net of conics with three base points. Find the three base points, and show that the same holds for (ii)

290 Not every translation on a cubic can be induced by a linear transformation, (see exercise 261). Show however that any translation can be induced by a Cremona transformation (for a judicious choice of base points). Interpret the result of exercise 289 in this context.

291 Given a cubic C we can associate another net to it, namely as given by the 2x2 minors of the matrix

$$\begin{pmatrix} \dfrac{\partial^2 C}{\partial x^2} & \dfrac{\partial^2 C}{\partial x \partial y} & \dfrac{\partial^2 C}{\partial x \partial z} \\ \dfrac{\partial^2 C}{\partial y \partial x} & \dfrac{\partial^2 C}{\partial y^2} & \dfrac{\partial^2 C}{\partial y \partial z} \end{pmatrix}$$

Show that this does not depend on the choice of coordinates, and that the net will have three base points.

292 Show that the points inside a small oval of the discriminant cubic in a net corresponds to invisible conics (cf exercise 104)

293 Given a triangle with vertices A,B and C and an inscribed circle O tangent at a on the line BC, at b on the line AC and finally at c on the line AB. Show that the four conics (AB)c,(BC)a,(AC)b and O actually form a net, and that the discriminant cubic is a Fermat cubic (cf exercise 242)

294 Given a point $P = (0,0)$ we can write a polynomial $F(x,y)$ in inhomogenous coordinates

$$f_o + (f_{11}x + f_{12}y) + \dots$$

Thus there will be one condition ($f_0 = 0$) for the curve $F = 0$ to pass through P and two additional conditions ($f_{11} = f_{12} = 0$) for the curve to be singular at P. In our case of P being the base point of a net, show that the net will always contain a conic singular at P

295 Write down the tangent planes to the singular locus (the Veronese surface) of the discriminant algebraically

296 Given the polars of the cubic

$$(x + y + z)^3 = 27xyz$$

Show that they form a net with just one base point at $(1,1,1)$, by finding the three flexes to the discriminant cubic, write down explicitly three conics all passing through a point and being mutually flexed

297 Show that one can find four mutually bitangent conics. Show that all such configurations are projectively equivalent, and that the four conics are linearly dependent. Furthermore write down explicit equations.(*Hint: Show that they span a net with three double lines*)

298 Show that through each point on the discriminant locus we can find exactly two nets meeting the singular locus of the discriminant (the Veronese surface) once each

299 Show that the dual of a dual net is canonically isomoprhic with the net itself

300 Show that if a net N has a non-singular discriminant the same is true for its dual N*. Is it true that its discriminant is the "dual" discriminant of the original net?

301 Given the parametrisation of the discriminant hypersurface $(L_1, L_2) \mapsto L_1L_2$ show that the inverse image of a hyperplane in the space $\mathbf{P}^5$ of conics will correspond to a symmetric bilinear form on the dual space P^{2*}

302 Show that $Z(B)$ is not isomorphic to $\mathbf{P}^1 \times \mathbf{P}^2$, but show that under the natural map $\pi : Z(B) \to \mathbf{P}^2$ we have that $\pi^{-1}(C)$ is isomorphic with $\mathbb{P}^1 \times \mathbb{P}^1$, where C is the associated conic. Is it possible to find a section to the map π?

303 Compute the "polar" to a point in the case of a degenerate symmetric bilinear form B, and show that it "blows up" to the entire plane iff the point is a singular point.

304 Write down basis elements for 4-dimensional linear systems of conics in the three possible cases

305 Given a pencil of Π of conics, we can to each point P associate P' the base point of the corresponding pencil of polars to P. Write down an explicit pencil and determine this association, decide whether or not it will define an involution

306 Given a pencil Π of conics, and consider the lines joing the three associated singular points. Show that points on such a line determine the same pencil of polars with respect to Π.Find the corresponding base point, and conclude that they are blown down.

307 Corresponding to different types of pencils Π of conics, determine the corresponding Cremona transformations. In particular what happens if Π contains double lines?

308 The graph $Z(B_1, B_2)$ of a Cremona transformation contains a natural hexagon of "lines" which splits up into two dijoint parts containing three lines each, blown down to respective factor. What happens to this hexagon under the natural involution?

309 Show that the "dual" discriminant has a natural double covering (in virtue of being a discriminant of a net (the dual net)) and that this double covering is given by the discriminant. Use this to show that the corresponding involution on the discriminant is the same induced by the identification in the case of a Jacobian net.

310 Given four lines (say $X = 0, Y = 0, Z = 0$ and $X + Y + Z = 0$) determine all the conics tangent to the lines. Notice that all double lines are automatically tangent to any line. Furthermore consider the duals of the conics in dual space going through the duals of the four lines and show that they form a conic of conics.

311 Recalling exercise 90, show that the six quadrics given by the 2×2 minors of a symmetric 3×3 determinant define a map of $\mathbf{P}(S^2\mathbb{C}^3)$ onto a $\mathbf{P}^5$. Show that the base locus of this map is given by $\mathcal{S}$ the locus of double lines. Furthermore show that this map is generically 1-1, in particular show that a non-singular symmetric 3×3 matrix is determined by its 2×2 minors. What about singular matrices?

312 Show that the conics in pencils with two fixed base points on a double line Q form a 3-dimensional linear space W, and hence that the pencils themselves correspond to a $\mathbf{P}^2$. Furthermore if T_Q denotes the tangentplane to $\mathcal{S}$ at Q, and U is a plane disjoint from T_Q, the webs containing T_Q will then be in a natural 1-1 correspondnec with the points of U via their inytersectins with U. Show that such webs automatically have two basepoints, and that U naturally parametrises all the normal directions to T_Q.

313 Show that the "exceptional divisor" i.e. complete conics which are double lines, contains a 3-dimensional subvariety which is a so called conic bundle, by considering complete conics whose distinguished points coincide. Is this fibration trivial? Does it have a section?

314 Let L be a double line, and Q an arbitrary conic in the dual space. Furthermore let Γ be the collection of $(C, \check{C})$ with C a non-singular conic and $\check{C}$ its dual. Given C we can look at the pencil $< C, L >$, the duals of its conics are defined except for L, by closing we get a representative L_C for the dual of L. Show that L_C is always a double line, and that it depends on the pencil $< L, C >$, and that $L_C = L_{C'}$ iff the pencils $< L, C >$ and $< L, C' >$ have the same basepoints. Thus identify $\overline{\Gamma}$ with the space of complete conics.

315 Given the inverse image of the discriminant hypersurface in the space of complete conics, it will split up into two components. Show that one consists of the closure of all singular conics which are not double lines, the so called proper transform of the discriminant locus, while the other turns out to be the exceptional divisor. Furthermore show that both of those are non-singular. Finally show that the image of the proper transform of the discriminant locus is blown down to $\mathcal{S}$ in the space of dual conics under taking duals.

316 Define the blowup of the diagonal in the product $\mathbf{P}^2 \times \mathbf{P}^2$, show that the natural involution (switching factors) naturally extends, and show that the quotient is non-singular. Identify both the proper transform of the discriminant locus and the exceptional divisor with this space.

317 Show that every cubic hypersurface can be written under the form

$$X_0^3 + X_0 Q(X_1, \dots, X_n) + C(X_1, \dots, X_n) = 0$$

where Q and C are quadratic and cubic forms respectively. Projecting from the point $(1, 0 \dots)$ to the plane $X_0 = 0$ show that the branchlocus is given by $4Q^3+27C^2$ which is a sextic hypersurface

318 Try and find the six conics in the pencil through the four points $(\pm 1, \pm 1, 1)$ tangent to the unit circle. Will certain conics be counted with multiplicity, and if so why?

319 Let Π be a pencil of conics containing a double line, and let C be a generic conic. Show that there are just four conics in Π additional to the double line which are tangent to C. Conclude that $\mathcal{S}$ consists of double points of the sextic tangent locus associated to C.

320 Consider the net XY, YZ, ZX defining a Cremona ytransformation and find the members which are tangent to the two conics $XY - Z^2$ and $XZ - Y^2$. Would it be possible to find conics tangent to $YZ - X^2$ as well?

321 A curve of bidegree $(1, 1)$ on a quadric, is the intersection with a plane, use this to check that its selfintersection is in fact 2 (1+1) as it should be. Further more a curve of bidegree $(2, 2)$ is given by the intersection with a quadric, and use this to check that its selfintersection is 8 as it ought to be. Finally two quadrics containing a line (L_1) intersects residually in a curve of bidegree $(1, 2)$, use this to ccheck that the intersection with a curve of bidegree $(2, 1)$ is 5 (=8-1-2)

322 Setting H'=2H-E and E'=3H-2E verify that H=2H'-E' and E=3H'-2E'. The significance of this is that the quadrics containing $\mathcal{S}$ give a birational map onto the space of dual conics, and H' is the inverse image of its hyperplane and E' is the exceptional divisor (cf 315). By using the identities H'^5=1, E^5=E'^5 and say H'^4E'=0 (etc if necessary) conclude formally the values of the missing intersections H^2E^3 etc..

Quadrics

The notion of a conic and a quadric hypersurface can readily be generalized to arbitrary dimensions.

Thus we will consider a quadratic form $Q(z_0, \ldots, z_{n+1})$ whose zeroes form a quadratic hypersurface of dimension n. Every Quadric of dimension n will henceforth be denoted by Q^n. As before we can to each point p in $\mathbb{P}^{n+1}$ consider the gradient

$$\partial Q(p)/\partial z_0, \ldots, \partial Q(p)/\partial z_{n+1}$$

and if this is non-zero for each p we say that the Quadric is non-singular, and the corresponding hyperplane

$$\frac{\partial Q(p)}{\partial z_0} z_0 + \ldots, \frac{\partial Q(p)}{\partial z_{n+1}} z_{n+1} = 0$$

defines the polar hyperplane to p.

If p does not belong to the Quadric the polar $\check{p}$ intersects in a non-singular quadric of one lower dimension, and all the tangentplanes of the Quadric along this intersection pass through p. And of course if p lies on the Quadric, the polar is nothing but the tangent hyperplane.

The notion of polars can be extended to linear subspaces. In fact given a linear subspace V its polar $\check{V}$ is defined to be $\bigcap_{p \in V} \check{p}$ and clearly its dimension is complementary to that of V. We should also note that if V is contained in the quadric, it is necessarily contained in its polar. Hence a non-singular n-dimensional quadric can at most contain linear subspaces of dimension $[\frac{n}{2}]$.

By completing squares any Quadric (singular or not) can be written as a sum of such, the number of squares gives the rank, and the rank classifies the Quadrics over $\mathbb{C}$. (Over $\mathbb{R}$ there is a slight complication due to the signature (see exercise 323).) Thus we get $n+1$ different types, ranging from the most degenerate the double hyperplane (rank 1) to the non-singular (rank $n+1$). All of this translates into the usual matrix approach, there is a 1:1 correspondence between quadratic forms and symmetric bilinear forms. The rank of the quadratic equals the rank of the corresponding matrix.

Geometrically each singular quadric is a cone over a non-singular quadric of lower dimension, whose dimension is one lower than the rank.

The group $PGL(n+1, \mathbb{C})$ acts on the family of quadrics. The orbits are given by quadrics of fixed rank. In particular the non-singular quadrics are all projectively equivalent. The stabilizer of the natural projective action on (non-singular) quadrics is denoted by $PO(n+1, \mathbb{C})$ and constitute the automorphism group of the quadric. Its dimension is easily computed from the transitive action of the $(n+1)^2 - 1$ dimensional group onto the space of all quadrics of dimension $\frac{(n+1)(n+2)}{2}$, and it is not too hard to see that it acts transitively on the quadric.

There are alternative normal forms for Quadrics which are quite instructive. We will then have to distinguish between the case of odd- or even dimensional Quadrics. In the even-dimensional case we can write a non-singular Quadric (over $\mathbb{C}$ as (cf page 50)

$$X_0 Y_0 + X_1 Y_1 + \cdots + X_n Y_n = 0$$

while in the odd-dimensional case we may write

$$X_0^2 + X_1 Y_1 + \cdots + X_n Y_n = 0$$

Looking at the first we see that every Q^{2n} contains a linear space of dimension n $(X_0 = X_1 = \cdots = X_n = 0)$ in fact we can say much more as we will see.

Quadrics as Double coverings

Writing a Quadric Q as $x_0^2 + Q\prime(x_1, \ldots, x_n)$ we represent it as a double covering, projecting from the point p $(1, 0, \ldots, 0)$ outside the quadric onto the plane $x_0 = 0$. This covering is ramified along the quadric $Q\prime$ of one lower dimension. In this way we can relate properties of the quadric $Q\prime$ to Q and study quadrics inductively, although a more efficient way of doing so will be treated in the next section.

In particular we can relate the linear subspaces of $Q\prime$ to those of Q. In fact a linear space Π in $\mathbb{P}^{n-1}$ will split upstairs iff the restriction of $Q\prime$ is a double hyperplane. Thus in particular we see that the dimension of a linear subspace of Q is at most one more than the maximal dimension of a linear subspace of $Q\prime$.

If we look at a three-dimensional quadric, we see that through each point there will be a $\mathbb{P}^1$ of lines, in fact they will correspond to the lines through a point on the ramification quadric in $\mathbb{P}^3$ lying on its tangentplane.

Birational Representation of Quadrics

Projecting from a point p of a quadric onto a hyperplane we get a $1:1$ map, except that i) the intersection of Q with the tangentspace T_p at p is a singular quadric, a cone over a quadric $Q\prime$ of dimension two less, and the projection blows it down onto $Q\prime$ and ii) the projection is not well defined at p and we have a blow up.

From the point of view of the hyperplane $\mathbb{P}^{n-1}$ we blow up a quadric of codimension two and blow down the hyperplane it spans.

This map can be represented by a linear system $Q_0, Q_1, \ldots, Q_n$ of quadrics vanishing along a fundamental locus given by the intersection of a hyperplane H with a quadric $Q\prime$.(see exercises 335 and 336). The inverse map is simply given by a n-dimensional system of hyperplane section on Q

Algebraically we are writing the quadric Q as

$$x_0 x_1 + Q\prime(x_2, \ldots, x_n)$$

(projecting from $(1, 0, \ldots, 0)$ onto $x_0 = 0$ with fundamental locus given by $x_1 = Q\prime(x_2, \ldots, x_n) = 0$

In this way we get an inductive step that relates a quadric to one of two less dimensions. In particular wee see that the inverse image of a linear space is linear upstairs iff it intersects the hyperplane spanned by the fundamental quadric exactly in the fundamental quadric. Thus the maximal dimensions of linear subspaces on Q are exactly one more than those on $Q\prime$

Applying this to the three dimensional quadric, we see that it is given by blowing up a conic in $\mathbb{P}^3$ and blowing down the hyperplane it spans. Thus the linear spaces of maximal dimensions are the lines that intersect the conic in just one point. In particular through each point we have a family of lines, a $\mathbb{P}^1$, those are in effect exactly those line that are defined by the cone given by the intersection with the tangentspace

In order to look at the totality of all lines on a three- dimensional quadric look at exercise 337

Linear Subspaces on a Quadric

We have the following basic theorem

Theorem. *On a even-dimensional quadric Q^{2m} the maximal dimension of a linear subspace is m and there exists two disjoint system of m-planes, each of dimension $\frac{m(m+1)}{2}$. On n odd-dimensional quadric Q^{2m+1} the maximal dimension of a linear subspace is likewise m, but here we have but one system of m-planes, and it is of dimension $\frac{(m+1)(m+2)}{2}$*

Proof: There is a one-to-one correspondence between the linear spaces of maximal dimension on a quadric Q^n with those of a quadric Q^{n+2} through a point, by taking the cones. Thus the two initial statements are clear by induction. (From the cases of Q^0 and Q^1 respectively). We need to verify the dimension statements. To that purpose we are going to look at a bigger space, namely all linear subspaces of dimension m together with a distinguished point. The dimension of such a space is clearly $2m$ $(2m+1)$ higher than the space of all m planes through a given point, but the latter space is by induction in 1:1 correspondence between the space of all $m-1$ planes in a quadric of dimension two less. Finally we observe that the difference between the space of m-planes (without distinguished point) and the auxiliary one we considered is of course m. hence we are done by induction.$\diamondsuit$

If we look at the even-dimensional case, we can say more. In fact we have

Theorem. *On a quadric Q^{2m} with m odd, two m-planes in the same system do not generally meet, and if they do, they do so in a space of odd dimension. While two m-planes from opposite systems generally meet in a point, and if more in a space of even dimension.*

On a quadric Q^{2m} with m even, two m-planes in the same system generally meet in a point, and if more, in a space of even dimension, while m-planes of opposite system do not generally meet, but if they do, they do so in an odd number of dimensions.

Proof: Almost all of this, that which pertains to intersecting linear spaces, follows straightforwardly by the standard induction used. Remains to show the statements on generic m-planes. Putting the equation of the quadric under form

$$x_0y_+\dots x_my_m = 0$$

we can exhibit two disjoint m-planes (Π_x and Π_y) namely $x_0 = \cdots = x_m = 0$ and $y_0 = \dots y_m = 0$. We need then show that in case m is odd those two linear subspaces belong to different systems, wheras they belong to the same in case m is odd. If $m = 2k+1$ is odd we can make a pairwise grouping of the terms

$$(x_0y_0 + x_1y_1) + (x_2y_2 + x_3y_3) + \cdots + (x_{2k}y_{2k} + x_{2k+1}y_{2k+1})$$

and consider the following system of linear m spaces

$$sx_0 + ty_1 = sx_1 - ty_0 = \cdots = sx_{2k} + ty_{2k+1} = sx_{2k+1} - ty_{2k} = 0$$

All of those are contained in Q a specialization to $s = 0$ and $t = 0$ respectively gives Π_y and Π_x. Conversely if we have two disjoint m planes given as above, the equation of the quadric can be written after possibly rescaling as $\sum x_i y_i$ and they have to belong to the same system.

To consider odd m we simply consider Q^{2m+2}, any two $m+1$ planes passing through a point of the same system meet in an even-dimensional subspace, hence the corresponding m planes of the blow down has to meet in at least a point, and disjoint m-planes can easily be exhibited, hence they need to belong to different families.

Topology of Quadrics

Given a smooth quadric Q^n and T a tangent hyperplane, then $T \cap Q^n$ is a cone over a smooth quadric Q^{n-2} and we can write the disjoint sum $Q = (Q^n \setminus (T \cap Q^n) \cup (T \cap Q^n)$ the first component will be isomorphic with $\mathbb{C}^n$ while the second can be further decomposed by removing the intersection of Q^{n-2} with one of its hyperplanes. In this way we get

Proposition. *An $2m$-dimensional quadric can be decomposed in the direct sum*

$$\mathbb{C}^{2m} \cup \mathbb{C}^{n-1} \cup \ldots \mathbb{C}^m \cup \mathbb{C}^m \cup \mathbb{C}^{m-1} \cup \cdots \cup \{p\}$$

while an $2m+1$-dimensional quadric has a decomposition

$$\mathbb{C}^{2m+1} \cup \mathbb{C}^{n-1} \cup \ldots \mathbb{C}^m \cup \mathbb{C}^{m-1} \cup \cdots \cup \{p\}$$

Proof: By induction. In fact we add strata succesivley by reducing the rank by two. In the even dimensional case the last quadric we get is the cone over the union of two points i.e. two hyperplanes which can be decomposed as $\mathbb{C}^m \cup \mathbb{C}^m \cup \mathbb{P}^m$, while in the odd dimensional case, the last quadric is a cone over a conic i.e. $\mathbb{C}^{m-1} \times \mathbb{C} \cup \mathbb{P}^m$.◇ We can use this to compute the Euler characteristic of a smooth quadric (for a simpler approach see exercise 344). In fact $e(\mathbb{C}) = 1$ hence $e(\mathbb{C}^n) = 1$ as well by the product formula. Adding we simply get $e(Q^{2m}) = 2m + 2$ and $e(Q^{2m+1}) = 2m + 2$

But the very explicit decomposition gives of course much more information. For anybody familiar with cohomology we can state

Proposition. *The non-zero homology groups of an even- and odd-dimensional quadric Q (of dimensions $2m$ and $2m+1$ respectively) are given by*

$$H^{2i}(Q, \mathbb{C}) \cong \mathbb{C} \quad 0 \leq i \leq 2m \quad i \neq m, H^m(Q, \mathbb{C}) \cong \mathbb{C}^2$$

and

$$H^{2i}(Q, \mathbb{C}) \cong \mathbb{C} \quad 0 \leq i \leq 2m+1$$

respectively.

Now as a change of pace we are going to consider real quadrics.

The topological classification of (smooth) real quadrics is rather simple. Recall that a real quadric is classified by its index, hence one of index $n-2k$ can be written under the form

$$X_0^2 + \cdots + X_k^2 - (X_{k+1}^2 + \cdots + X_n^2) = 0$$

A solution hence satisfies

$$X_0^2 + \cdots + X_k^2 = \lambda = X_{k+1}^2 + \cdots + X_n$$

where $\lambda \neq 0$ as we are looking at real solutions. Hence we can normalize $\lambda = 1$ and parametrise all solutions by the product $S^k \times S^{n-k-1}$ On each sphere we have the antipodal involution $s \mapsto -s$ which diagonalizes to an involution $(s,t) \mapsto (-s,-t)$ on the product, whose quotient gives the real quadric

In particular we conclude that if both the dimension and the index of a quadric is even, then its Euler characteristic is equal to two and zero otherwise.

We also see that index $(n-2)$ for an n-dimensional quadric gives a sphere S^n (as S^0 consists of two points).

Exercises

323 Show that the quadrics over $\mathbb{R}$ are classified by rank and index.

324 Show that if a quadric contains the linear space cut out by $L_0, L_1, \ldots, L_k$ then it can be written under the form $L_0M_0 + \cdots + L_kM_k$ where M_i are other linear forms. Conclude that the maximal dimension of a linear subspace on a smooth quadric of dimension n is $[\frac{n}{2}]$

325 Show that the polar of a point to a non-singular quadric intersects it smoothly unless the point lies on it, and in that case the rank goes down only by one

326 Show that on an even-dimensional quadric, any hyperplane containing a linear subspace of maximal dimension has to be tangent. While it is not true for odd-dimensional quadrics.

327 Given a point p_0 on a smooth quadric, consider its tangentplane $T_0 = T_{p_0}$ and a point p_1 on its intersection, distinct from p_0. Consider then the intersection of the quadric with $T_0 \cap T_1$ and choose a point p_2 on it, not lying on the line spanned by p_0 and p_1, and continue. This process will obviously have to stop, show that when it does we have constructed a linear space of maximal dimension containing $p_0, p_1, \ldots$

328 Given an odd-dimensional quadric Q and a linear space L of maximal dimension, show that among all spaces Π of one higher dimension there is a unique space Π_L such that $Q \cap \Pi_L = 2L$

329 Given an odd-dimensional quadric Q^{2m+1} and a linear space L^m of maximal dimension, construct all linear m-planes contained in Q intersecting L in a $(m-1)$-plane. And show that they are in one-to-one correspondence between points on a projective plane. Determine its dimension.

330 Show that $PO(3, \mathbb{C}) \cong PGL(2, \mathbb{C})$. What can be said about the groups $PO(4, \mathbb{C})$ and $PO(5, \mathbb{C})$?

331 For real quadrics we can define the notion of inside and outside, by postulating that for outside points the polar intersects the quadric, but not for inside points. Show that we can associate an orientation to a defining quadratic equation depending on whether it is positive or not on the outside points.

332 Show that over a finite field $\mathbf{F}_q$ the notion of inside and outside (according to 331) does not work unless $n = 2$ but for a defining quadratic equation we can divide the points not on the quadric according to whether the values are quadratic non-residues or not. Show that this divison is independent of the quadratic equation. (But the orientation will of course vary)

333 Show that there are two types of non-singular quadrics over a finite field $\mathbf{F}_q$ and determine the number of points in either case, and try and make a discriminantal characterization

334 Determine the dimension of all transformations that leave the point $(1, 0, ., 0)$ invariant while preserving the quadratic $x_0x_1 + x_2^2 + \ldots, x_n^2$ and conclude that $PO(n+1, \mathbb{C})$ operates transitively.

335 Show that any quadric $Q(x_1, \ldots, x_n)$ which vanish on the conic C given by the two equations $Q\prime(x_2, \ldots, x_n) = x_1 = 0$ can be written on the form $Q\prime + x_1L$ where L is a linear form. Furthermore determine the dimension of the vectorspace of quartics of the form $q_0x_1^2 + q_1Q\prime x_1 + Q^2\prime$ and show that it contains any product of two quadrics that vanish on the conic C.

336 Consider the quadric $x_0x_1 - q(x_2, \ldots, x_n) = 0$ and the projection from

$(1, 0, \dots, 0)$ to $x_0 = 0$ show that the inverse image is given by the system of quadrics $q, x_1^2, x_1x_2, \dots x_1x_n$. (Compare 335)

337 Consider a smooth three-dimensional quadric Q^3. Two points on it are said to be related if they are joined by a line lying in Q^3.

a) Show that two points are related if equivalently

i) They are contained in each others tangentspaces

ii) The intersection of their tangentspaces intersects Q^3 in a double line

Given a line l on Q^3 the planes Π through l are parametrized by $\mathbb{P}^2$ (In fact any plane skew to l may serve as parameterspace).

b) Show that for any Π through l there is a residual line l_π lying on Q^3 and that $l = l_\pi$ iff Π is equal to the intersection of all tangent hyperplanes containing l. Conclude that the lines meeting a given line make up a $\mathbb{P}^2$

For a generic hyperplane, the intersection becomes a smooth two-dimensional quadric Q^2. Let $L(Q)$ is the space of all lines lying on Q^3 and let L_1 and L_2 be the lines of the two rulings of Q^2.

c) Show that we can associate to each point of $L(Q) \setminus (L_1 \cup L_2)$ a unique point of Q^2 with fibres $\mathbb{C}^*$. Compare this with $\mathbb{P}^3$ minus two skew lines, to conclude that $L(Q) \cong \mathbb{P}^3$

338 By considering the three-dimensional quadric in the form $x_0^2 + x_1x_2 - x_3x_4$ find the totality of all its lines explicitly.

339 By considering all the lines on a three-diemsnional quadric as $\mathbb{P}^3$ show that each point is contained in a canonical hyperplane (the hyperplane of all lines meeting the given). Conclude that if two lines meet, the line given as the intersection of their corresponding hyperplanes, is simply the line that parametrises all the lines through a given point.

340 A real three-dimensional non-singular quadric come in two types (excluding the invisible) corresponding to index three and one, determine which has lines and which as not.

341 Give an example of two linear subspaces of maximal dimension that intersect each other in codimension 1. Is it possible to have three subspaces of maximal dimension m intersecting along a subspace of dimension $m - 1$?

342 Assume that a quadric Q^{2m} contains the two disjoint m-planes $x_0 = \dots = x_m = 0$ and $y_0 = \dots = y_m = 0$, show that the quadric can be written as $\sum_i \alpha_i x_i y_i = 0$

343 In $\mathbb{P}^{2m+1}$ consider two disjoint m-planes Π_1 and Π_2 show that for any point p outside the m-planes, there are unique points $p_i \in \Pi_i$ such that p lie on the line spanned by them.

Let Q be a quadric containing the linear spaces Π_i, show that for each point p in Q but not on any of the Π_i there is a unique line L_p containing p and meeting both linear subspaces.

This defines a subset $\Gamma \subset \Pi_1 \times \Pi_2$ and a map $\pi : \tilde{Q} \to \Gamma$ (where $\tilde{Q}$ denotes the quadric minus the Π_i) Taking the closure of the graph we get a map $\pi : Q \to \Gamma$ with fibres the lines L_p.

Determine this set for the case of $m = 1$ and show that we get a natural splitting $\mathbb{P}^1 \times \Gamma$.

Show that it is a hypersurface in general. By choosing suitable equations for Π_i determine the equation for this hypersurface and determine whether we still can factor out a $\mathbb{P}^1$

344 Representing a quadric as a double cover, compute inductively its euler-characteristic using that $e(\mathbb{P}^n) = n+1$ and a formula for the euler characteristic of a double cover in terms of the euler characxteristic of the branch locus

345 Let Q^n be a real quadric of index $(n-k,k)(=n-2k)$, we may then consider the two quadrics $Q_+ = z^2 - Q$ and $Q_- = z^2 + Q$ of index $(k+1, n-k)$ and $(n+1-k, k)$ respectively (cf exercise ?4). If $e_+(n,i)$ and $e_-(n,i)$ denotes the euler characteristics of the inside and outside respectively of a real n-dimensional quadric of index i write down inductive formulas for $e(n,i)$ (the euler characteristic of the quadric itself) and try to determine those in that way.

346 Do the analogous decomposition of a real quadric. Show that it will terminate whenever the intersection with a tangentspace fails to become a cone. (Invisible conic). Compute the real homology of a real quadric and compare with the representation as a quotient of the product of two spheres.

The Quadric Line Complex

We would like to parametrise all the lines in $\mathbb{P}^3$. A line is given either by two distinct points or by the intersection of two hyperplanes. Unfortunately neither choice of points or hyperplanes is canonical. A reformulation of the problem is to look at the linear space $\mathbb{C}^4$ and the space of planes through the origin, A plane is either given by two linearly independent vectors (two distinct points) or by two linearly equivalent linear equations.

However we can to each linear space V associate the space $\bigwedge^2 V$ spanned by the vectors $a \wedge b$. If $\dim V = 4$ then $\dim \bigwedge^2 V = 6$ and consequently we have a space $\mathbb{P}^5$.

Letting $e_i, i = 0, \ldots 4$ be a basis for V and $e_i \wedge e_j, i < j$ a basis for $\bigwedge^2 V$ Writing $a = \sum a_i e_i, b = \sum b_i e_i$ we have by bilinearity $a \wedge b = \sum (a_i b_j - a_j b_i) e_i \wedge e_j$.

In practice we are to each pair of 4-vectors

$$a = (a_0, a_1, a_2, a_3) and b = (b_0, b_1, b_2, b_3)$$

going to associate the 6-vector consisting of the six minors of the 4×2 matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}$$

Those are called the Plücker coordinates of the (plane) line spanned by a and b. It is easy to see directly that up to a scalar the Plücker coordinates do only depend on the (plane) line not on the (vectors) points a and b that span it.

The space of lines in $\mathbb{P}^3$ will be denoted by $\mathbf{G}(2,4)$ and the Plücker coordinates gives a map into $\mathbb{P}^5$.

Now it is not true that any point in $\mathbb{P}^5$ represents a line in $\mathbb{P}^3$ via the Plücker coordinates. In fact a vector v in $\bigwedge^2 V$ represents a line, iff it is decomposable, i.e. $v = a \wedge b$. Now (see exercise 348) a vector $v \in \bigwedge^2 V$ is decomposable iff $v \wedge v = 0$. If we let $v = \sum p_{ij} e_i \wedge e_j, i < j$ this translates into the quadratic condition

$$(1) \qquad p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$$

This quadric is known as the quadric line complex. Now this being a quadric of dimension four, it will contain two systems of planes. Those systems have very nice direct geometrical representations. If we fix one point (say $p = (1,0,0,0)$) and look at all the lines through it (they will have Plücker coordinates $p_{34} = p_{24} = p_{23} = 0$) they will make up a $\mathbb{P}^2$ in $\mathbb{P}^5$. Or if we fix a hyperplane in $\mathbb{P}^3$ (say $X_0 = 0$) and consider all the lines on it (their Plücker coordinates will satisfy $p_{12} = p_{13} = p_{14} = 0$) they will make up another $\mathbb{P}^2$ in $\mathbb{P}^5$.

Two planes in the same system will clearly always intersect in a point (unless they coincide) while two planes in different systems will normally be disjoint (the point not on the plane) or intersect ina line (the lines lying in a hyperplane and passing through a fixed point).

It is clear that both families are parametrised by $\mathbb{P}^3$ Furthermore there is a striking duality. As (cf exercise 347) $\bigwedge^2 V$ and $\bigwedge^2 V^*$ are canonically identified, the space of lines in $\mathbb{P}^3$ and its dual $\mathbb{P}^{*3}$ are the same. Thus there is a natural identification between the lines in $\mathbb{P}^3$ and in its dual. Given a line L in $\mathbb{P}^3$ it determines a pnecil of hyperplanes (those that contain L), hence a line $\check{L}$ in $\mathbb{P}^{*3}$

The plane of all lines L through a point $p \in \mathbb{P}^3$ will then correspond to all lines $\check{L}$ contained in a hyperplane $\check{p}$ (the hyperplanes vanishing on p), while conversely the plane of all lines L contained in a hyperplane H correspond to all lines $\check{L}$ passing through $\check{H}$. Of course both L and $\check{L}$ will have the same Plücker coordinates, and correspond to the same point. It is only the geometric interpretation that will differ as lines of $\mathbb{P}^3$ or of $\mathbb{P}^{*3}$

Given a smooth quadric Q in $\mathbb{P}^3$ we can define an involution on the space of lines, simply by considering the polar to each line. The fixed points of this involution will consist of the lines contained in Q. The involution will permute the two system of planes.

Two points on a quadric line complex will be said to be connected iff they are both contained in a plane lying inside the complex. If so they will be contained in a unique plane from either system. It is easy to see that two points are connected iff the corresponding lines meet. Equivalently, two points are connected iff they are contained in their respective tangentplanes. Thus the intersection of the quadric complex with the tangentspace at a point p (corresponding to a line L in $\mathbb{P}^3$) consists of the points connected to p or equivalently meeting L.(cf 353)

We have in the previous chapter discussed the topology of quadrics, and how they can be built up from simpler entities. This can be given an instructive geometric interpretation. Consider two hyperplanes Π_i meeting in a line L_0. Any line L not meeting L_0 determines a pair of points by its intersections with the planes P_i. Conversely given a point on each plane, not lying on L_0 determine a line. In this way we have parametrised "almost" all the lines in $\mathbb{P}^3$ by $\mathbb{C}^2 \times \mathbb{C}^2$. We can also say that the quadric line complex is birational with $\mathbb{P}^2 \times \mathbb{P}^2$. The lines we have forgotten are exactly those that intersect L_0 namely the intersection of the quadric complex with the tangentspace at L_0. This intersection is a cone over a smooth two dimensional quadric, which can be seen explicitly. (cf exercise 355) The two ruling of lines on a smooth quadric in $\mathbb{P}^3$ will of course show up naturally on the quadric line complex. In fact each quadric determines two skew planes in $\mathbb{P}^5$ whose intersections with the quadric line complex gives the lines.(cf exercise 356)

The Real Quadric

The discussion above works as well for other fields, e.g $\mathbb{R}$. The real quadric linecomplex will hence be a quadric of rank six and index zero. The system of planes will be real as well, as the geometric interpretation will testify.

The real quadric line complex also comes equipped with a canonical double covering. Any basis of a two dimensional real space has an orientation. This double cover is of course induced by the double cover $S^5 \to \mathbb{RP}^5$. Geometrically we are considering directed lines in $\mathbb{RP}^3$. Topologically the directed complex is given by $S^2 \times S^2$ (which incidentally we recall has a complex structure as a complex two dimensional smooth quadric)

If we identify $\mathbb{R}^4$ with $\mathbb{C}^2$ we can identify the complex lines among the real 2-planes. In fact (see exercise 360) those are given by the linear conditions $p_{13} - p_{24} = 0 and p_{14} + p_{23} = 0$ and the inequality $p_{12} + p_{34} \neq 0$ Rewriting the equation (1) using $u_1 = p_{13} - p_{24}, u_2 = p_{13} + p_{24}, v_1 = p_{14} + p_{23}, v_2 = p_{14} - p_{23}, w_1 = p_{12} + p_{34}$ and $w_2 = p_{12} - p_{34}$ we get

(1')
$$(u_1^2 + v_1^2 + w_1^2) - (u_2^2 + v_2^2 + w_2^2) = 0$$

And the conditions for complexity translate into $u_1 = v_1 = 0$ and $w_1 \neq 0$ giving the quadric

$$u_2^2 + v_2^2 + w_2^2 = w_1^2$$

(we see that the condition $w_1 \neq 0$ is a consequence of the two linear conditions) which is clearly a rank four quadric of index two, and topologically equal to the two sphere S^2. It also inherits a complex structure as the complex lines through the origin of $\mathbb{C}^2$ hence $\mathbb{CP}^1$

The canonical double cover is given by the normalization

$$(u_1^2 + v_1^2 + w_1^2) = 1 = (u_2^2 + v_2^2 + w_2^2)$$

Now $\mathbb{RP}^3$ is a group, in fact we can consider the Hamiltonian quaternions $\mathbb{H}$ and the associated real projective space $P(\mathbb{H})$. This space comes equipped with a distinguished point ($P(\mathbb{R})$) the origin (the identity), from now on denoted by **1**. In fact the space can be identified with $\mathbf{S}O(3)$ (see exercise 361). The lines through the origin correspond to planes in $\mathbb{H}$ containing the reals. Any such plane is a subfield of $\mathbb{H}$ (in fact isomorphic with $\mathbb{C}$) (se exercise 362) and define a one dimensional subgroup of $\mathbf{S}O(3)$, conversely any one-dimensional subgroup of $\mathbf{S}O(3)$ is a line through **1**. In general the lines in $\mathbb{RP}^3$ can be thought of as left (or right) cosets of one-dimensional subgroups of $\mathbf{S}O(3)$. The subgroups of $\mathbf{S}O(3)$ or equivalently the lines through a point are clearly classified by $\mathbb{RP}^2$. Fixing a line L corresponding to a subgroup L (using the same notation for economy), we get a whole family of skewlines αL (or equivalently another family by $L\beta$). The family of skewlines is parametrised by S^2 (see exercise 364). Hence we have exhibited $\mathbb{RP}^3$ as a S^1 bundle over S^2. Furthermore the complex lines correspond to the translates of the unique complex line through **1** (cf exercise 365)

Exercises

347 Let V be a 4-dimensional vectorspace, show that $\bigwedge^3 V \cong V^*$ via the natural map $V \ni v \mapsto v^* \wedge v \in \bigwedge^2 V \cong \mathbb{C}$ for each $v^* \in \bigwedge^3 V$. Show also that $\bigwedge^2 V \cong (\bigwedge^2 v)^*$ via the linear amps $\bigwedge^2 V \ni v :\mapsto a \wedge v \in \bigwedge^4 V \cong \mathbb{C}$. Furthermore show that $\bigwedge^2 V$ maps naturally into $\hom(V, V^*) \cong V \otimes V$ via $V \ni v :\mapsto a \wedge v \in \bigwedge^3 V$.

348 An element $v \in \bigwedge^2 V$ is said to be decomposable iff we can write $v = a \wedge b$. Show:
a) If $\dim W = 3$ show that any element in $\bigwedge^2 W$ is decomposable.
b) While in case $\dim V = 4$ an element v in $\bigwedge^2 V$ is either of the form
i) $a \wedge b$ (decomposable)
ii) $a \wedge b = c \wedge d, a, b, c, d$ linearly idependent (indecomposable)
c) Given $0 \neq v \in \bigwedge^2 V$ then
i) v decomposable $\Leftrightarrow \dim \ker x \mapsto v \wedge x = 2$
ii)v indecomposable $\Leftrightarrow \ker x \mapsto v \wedge x = (0)$
d) Show that $v \in \bigwedge^2 V$ is decomposable iff $v \wedge v = 0$

349 Show that two decomposable elements $a \wedge b$ and $c \wedge d$ wedge to zero iff a, b, c, d are linearly dependent.

350 Show that the pencil $\lambda a \wedge b + \mu c \wedge d$ consists of decomposable elements iff a, b, c, d are linearly dependent.

351 Show that the nets of decomposable elements are either given by elements $a \wedge x$ (where $x \in V$) or by $a \wedge b$ (where $a, b \in W$ a 3 dimensional subspace of V)

352 Given a decomposable element $a \wedge b \in \bigwedge^2 V$ ($\dim V = 4$) it determines both a kernel $K(a \wedge b)$ inside V and an image $I(a \wedge b)$ inside $\bigwedge 2V \cong V^*$. Show that $K(a \wedge b)$ is canonically identified by the 2-dimensional vectorspace in V with Plücker point $a\ wedgeb \in \bigwedge^2 V$. Give a similar characterization of $I(a \wedge b)$

353 Show that the polar of a given point $p \in P(\bigwedge^2 V)$ is given by the hyperplane $x \in \bigwedge^2 V : p \wedge x = 0$

354 Given a point p on the quadric line complex, show that it is contained in many different planes from either system. In fact the planes containing p form a $\mathbb{P}^1$. More precisely if $\mathbf{P} \cong \mathbb{P}^3$ parametrise all the lines from one system, those containing p will form a line in $\mathbf{P}$.

355 Any line L intersecting L_0 does of course lie on a unqiue plane containing L_0. Thus show that if M_0 is a line skew to L_0, any line L intersecting L_0 determines a point in $M_0 \times L_0$

356 Given a quadric $x_0 x_1 - x_3 x_4$ determine explicitly the Plücker coordinates of the lines on it, and hence the two skewplanes in $\mathbb{P}^5$.

357 Given three skewlines, show that they determine a plane in $\mathbb{P}^5$, and hence a conic. Show that this conic sweeps out the unique quadric containing the three lines.

358 Given the lines of one ruling of a quadric, they determine a plane Π as in 356, show that the plane defined by the other ruling, is simply the polar $\check{\Pi}$ to Π with respect to the quadric line complex

359 Given a sphere (a real quadric with no lines) in $\mathbb{R}P^3$ show that it defines a fixed point free involution on the real line complex by considering the polars. Determine this involution explicitly in terms of Plücker coordinates.

360 Given an identification $\mathbb{R}^4 \cong \mathbb{C}^2$ a complex line is given by an equation $Az + Bw = 0$. Setting $A = a + ib, z = x + iy \dots$ translate this into two real conditions and compute the Plücker coordinates. (cf exercise 14)

361 For each α consider the linear mapping $x \mapsto \alpha x \alpha^{-1}$. Show that it preserves the pure quaternions (i.e. x such that $x^* = -x$ where $x \mapsto x^*$ denotes conjugation) and the quadratic form $< \alpha, \beta >= \alpha\beta^* + \beta\alpha^*$. Hence conclude that $P(\mathbb{H}) \cong \mathbf{S}O(3)$.

362 Show that if α is a non-real element of $\mathbb{H}$ then α satisfies an irreducible quadratic equation (over $\mathbb{R}$).*Hint:* $\mathbb{R}(\alpha)$ *is a field*

363 A one-dimensional subgroup is often refered to as a 1-parameter subgroup. As such it can be parametrised by $\mathbb{R}$ via the exponential map. Given a non-real element α of $\mathbb{H}$ or equivalently, a non-trivial element of $\mathbf{S}O(3)$, show that it determines a unique line in $\mathbb{H}_-$ (the three dimensional space of pure quaternions) via its eigenspace; and determine explicitly a parametrisation (the exponential map) of the one-parameter subgroup that contains α

364 $\mathbf{S}O(3)$ operates on S^2 in a natural way, show that the different stabilizers are identical with the set of one-parameter subgroups. In particular show that the translates of a given line through the origin form a S^2 and hence that the real quadric line complex can be thought of as a S^2 bundle over $\mathbb{RP}^2$

365 Fixing a line L through the origin, means giving $\mathbb{H}$ the structure of a complex vectorspace. (By identifying $L \cong \mathbb{C}$). We may then ask what planes (lines in $\mathbb{RP}^3$) are complex.

a)Show that each element in $\mathbb{H}$ is contained in a unique complex line, and that each hyperplane contains a unique complex line.

b)Show that the complex lines coincide exactly with the left (or right depending on whether the complex structure on $\mathbb{H}$ is given by left or right multiplication)n cosets of L.

366 Fixing a complex structure as in 365 and a hyperplane $H \cong \mathbb{RP}^2$ in $\mathbb{RP}^3$ we get a map onto S^2 by considering through each point the unique complex line. Show that this map is $1:1$ except along the unique complex line (cf 365 a))contained in H. Conclude that $\mathbb{RP}^2$ is the blow-up of S^2

367 By considering a quaternionic structure on $\mathbb{C}^4$ show that $\mathbb{CP}^3$ is fibered over S^4 by spheres S^2 (actually isomorphic with $\mathbb{CP}^1$)

368 Determine the number of lines in $\mathbb{F}_{\shortparallel}\mathbb{P}^3$ for a finite field $\mathbb{F}_q$

The elliptic quartic

Given two (non-singular) quadrics Q_1 and Q_2 on $\mathbb{P}^3$ we need to find out what its intersection $C = Q_1 \cap Q_2$ is.

Writing $Q_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$ we see that C can be written as a curve of bidegree $(2,2)$ as every line of the ruling of Q_1 intersects Q_2 in two points. Thus it is given by a bihomogenous form $F(x_0, x_1; y_0, y_1)$ of bidegree $(2,2)$. Explicitly we write

$$F = q_{00}(x_0, x_1)y_0^2 + q_{01}(x_0, x_1)y_0y_1 + q_{11}(x_0, x_1)y_1^2$$

where q_{ij} are quadratic forms in x_0, x_1

By dehomogenization we can consider this as a quartic polynomial say in $x = x_1, y = y_1$ by setting $x_0 = y_0 = 1$ defined on $\mathbb{C} \times \mathbb{C}$. Hence we can define tangents and the notion of non-singularity. (see exercise 370) This leads to the notion of transversal intersection. Two quadrics (or more generally any two surfaces) are said to intersect transversally at a point if and only if their tangentplanes are distinct. We then have

Lemma. *The tangentline to an intersection of two surfaces is given by the intersection of their tangent planes*

In particular we have the important corollary

corollary. *Two quadrics meet in a non-singular curve iff they intersect transversally at every point*

We have a projection from C to either of the factors $\cong \mathbb{P}^1$ of Q, this projection is clearly $2:1$ and the fiber over a point (x_0, x_1) is given by the roots of the binary quadric

$$q_{00}y_0^2 + q_{01}y_0y_1 + q_{11}y_1^2$$

Thus we have defined a double cover of C onto $\mathbb{P}^1$ or equivalently an involution on C (with rational quotient). The ramification of this covering is given by the vanishing of the discriminant of the quadric, i.e. by

$$4q_{00}(x_0, x_1)q_{11}(x_0, x_1) - q_{01}^2(x_0, x_1) = 0$$

which is a quartic. Thus we have four ramification points and C is an elliptic curve!

A Curve C given by the transversal intersection of two quadrics is refered to as the *elliptic quartic* . One may show that a curve in $\mathbb{P}^3$ of degree four is either plane, rational or the intersection of two quadrics.(see exercise 373)

We can also exhibit C as a (smooth) cubic. For that purpose we pick an arbitrary point p on C and project onto a plane Π. To check the degree of the projection C' we take an arbitrary line L in Π and consider the plane P spanned by p and L, it will intersect C in four points, namely the intersection of the two conics C_1 and C_2 formed by the intersection of P with Q_1 and Q_2 respectively. Clearly the intersection of C' with L are given by the projections of the three residual intersection points of C with P. (For a computational form see exercise 371 and 372).

Generic Pencils of Quadrics

Given two quadrics Q_1, Q_2 they will span a pencil $\lambda Q_1 + \mu Q_2$, the baselocus of this pencil will clearly be C. The singular members of this pencil will be cones, and they will correspond to the vanishing of a 4×4 determinant, and hence we will expect four singular members.

We thus have two sets of four points on $\mathbb{P}^1$

The four branch points of C considered as a double covering of $\mathbb{P}^1$

The four points of the pencil corresponding to the singular members

One may hazard to guess that those two sets of four points define the same $\jmath$-invariant. In fact we can set up a $1:1$ correspondence between the points. A smooth quadric has two sets of rulings, while a cone has only one. Thus we can associate another canonical elliptic curve to a pencil of quadrics, namely the double covering defined by the rulings of the quadrics of the pencil. This defines a $2:1$ map ramified exactly at the singular members.

Proposition. *The two canonically associated elliptic curves to a pencil of quadrics are indeed isomorphic*

Proof: Pick a point p_0 on C and for each point p of C consider the line $L_p =< p, p_0 >$, there will now be a unique quadric Q_p in the pencil, containing the line L_p. This mapping is $2:1$ as Q_p will have two rulings and also contain the line $L_{p'}$ through p_0. Clearly it will be ramified exactly when Q_p is singular (and have only one ruling).$\diamondsuit$

The elliptic curve associated to a pencil of quadrics is a projective invariant of the pencil. In fact there is a projective transformation of $\mathbb{P}^3$ transforming the quadrics in the pencil Λ into those that constitute the pencil Λ' iff the associated elliptic curves have the same $\jmath$-invariants. (In contrast to the case of pencil of conics, when any two generic pencils (i.e. with four distinct basepoints) are projectively equivalent). The above proposition takes care of one direction, the converse is deeper. (see exercise 374)

We should also note (in analogy to the case of pencils of conics) that given an elliptic quartic C, there is a unique pencil of quadrics containing it, and hence having it as a baselocus (see exercise 373). We may construct this pencil canonically by considering rational involutions on C. For each such involution ι we may consider the surface Λ_ι traced out by the lines $< x, \iota(x) >$ where x runs through the points of C. Knowing that C is indeed the intersection of two quadrics, it is straightforward to see that Λ_ι must be a quadric and a member of the pencil spanned by any two quadrics cutting out C. It is somewhat harder to prove this directly. (see exercise 376)

Given a generic point outside p the quartic C the projection of C onto a plane will yield a binodal quartic (cf exercise 372). The two nodes corresponds to the two lines through p of the unique quadric in the pencil passing through p. (For further degeneracies see exercise 382). If however the point p will be chosen as one of the four vertices of the cones in the pencil, the projection will degenerate into a double cover of a conic.

The four vertices of the singular quadrics are canonically associated to a elliptic quartic, and they determine a so called *coordinate tetrahedron* Using the coordinates corresponding to the faces of the tetrahedron, the equations defining the elliptic quartic gets to be particularly simple (see exercise 383)

The Geometry of Elliptic Quartics

By restricting a pencil λ of quadrics to a plane Π we get a pencil Λ_π of conics, the base points of which are given by the intersection of Π with the baselocus C of Λ, and the degenerate members corresponds to quadrics tangent to Π.

In general we will have four basepoints, if less, it means that the plane is in special position visavi C. Two points coalescing means that the plane Π is tangent to C i.e. containing a tangentline to C, while four points coming together two and two of course means that the plane is bitangent to C. Similarly three points coinciding means that the plane gives the best fit to C at the point in question, it is the so called *osculating plane* . The extreme case is that four points come together, this corresponds to the osculating plane having gratioutous contact, and such planes play similar rôles to the elliptic quartic as flexed tangents do to the non-singular cubic. For want of better terminology let us refer to such points at which the osculating planes have additional contact, as *hyperosculating* . It is easy to see (cf exercise 384) that the hyperosculating points correspond to the ramification points of C corresponding to the four involutions given by the singular cuadrics of the pencil cutting out C, and the corresponding hyperosculating planes are given by the tangentplanes to the quadric cones along the tangent lines . Thus there are 16 hyperosculating points, and one may guess cleverly (and correctly) that those somehow correspond to the sixteen four-torsion points of the elliptic quartic under some suitable choice of origin. In fact this leads us to take one of the hyperosculating points 0 as an origin, and to postulate (in complete analogy with the cubic case) that four points add up to zero iff they lie in a plane. In this way we define an addition by simply choosing two points a, b on C and consider the plane Π spanned by those and 0. This plane will intersect C in a fourth residual point $-(a+b)$ by definition. To get $a+b$ we then choose the plane containing the tangentline T_0 to C at 0 and $-(a+b)$ and consider its residual intersection with C. To actually show, on first principle, that this defines an associative and commutative binary operation on C is however rather involved.

Degenerate Pencils

As in the case of conics we can discuss various pencil when the baselocus degenerates.

The simplest such example is given by two tangent quadrics. In that case the intersection will no longer be elliptic but a rational quartic with a node at the tangency point. The determinental condition on the quadrics will degenerate into a binary quartic with a double point, and there will be only three degenerate members, one of which is double. The double cone will have its vertex at the node, the common tangency point of the members. The pencil maybe generated by two cones one of which has its vertix on the other.

Degenerating once more we may do it in two ways. Either we allow two tangency points, then the pencil will be generated by two cones havinga line in common, and the quartic baselocus will split into a line and a rational cubic curve (a so called twisted cubic), or we allow higher order contact between the tangent quadrics. The latter alternative can be degenerated even further into the case of only one singular member.

In all the cases discussed, the singular members are still cones. We may of course also consider the case when the singular members degenerate. The simples such case consists of a pencil generated by a reducible quadric (consisting of two planes) and a cone. The baselocus then splits up into two conics meeting transversally in two points. The singular members are then three, a reducible anmd two cones. The cone may however become tangent to the singular line of the reducible quadric, in case the baselocus consists of two conics meeting in one point. The two singular cones have then coalesced to one. Then there may also be higher order contact.

As can be seen the classification becomes rather tedious to compile. There are standard ways (dating from Segre) of book-keeping. One obvious, which we did consider from the beginning, is to keep track of the branchlocus, a simpler one is to keep track of the type of the degenerate fibers with multiplicity, as we did in the case of pencils of conics. Representing a cone with $\bullet$ a reducible quadric with $\times$ and a double plane with $\|$ and adjoining the appropriate multiplicities (adding up to 4) we can present the following cases corresponding to generic member of pencil non-singular

$\bullet \quad \bullet \quad \bullet \quad \bullet$	*the generic case, non-singular intersection*
$\bullet(2) \quad \bullet \quad \bullet$	*nodal intersection*
$\bullet(2) \quad \bullet\,(2)$	*line component of intersection*
$\bullet(3) \quad \bullet$	
$\bullet(4)$	
$\times \quad \bullet \quad \bullet$	*two conics*
$\times \quad \bullet\,(2)$	
$\times(3) \quad \bullet$	
$\times(4)$	
$\times \quad \times$	*intersection fourlines*
$\|(3) \quad \bullet$	*double conic*

The reader is invited to fill out (some of) the gaps of the commentaries.(Exercise 386)

Exercises

369 Show that any bihomogenous form F of bidegree $(2,2)$ may be written in the form

$$\sum_{0\leq i,j,k,l\leq 1} A_{ij,kl}x_ix_jy_iy_j$$

where the coefficents $A_{ij,kl}$ form a 3×3 matrix $\mathbb{A}$.

Setting $z_{ij}=x_iy_j$ show that F can be written as a quadratic form

$$\sum B_{ij,kl}z_{ij}z_{kl}$$

but that this form is only unique up to the Segre condition

$$z_{01}z_{10}=z_{00}z_{11}$$

and conclude that any bihomogenous curve of bidegree $(2,2)$ on a quadric (Q_1) is given by the intersection of another quadric (Q_2).

370 Preserving the notation of 369, show that

a) F vanishes at $(1,0;1,0)$ iff $A_{00,00}=0$ and is non- singular at that point iff the linear part $(A_{01,00},A_{00,01})\neq(0,0)$

b) Write down, in general, the tangent at a point (x,y) to $F=0$ considered as a curve in $\mathbb{C}^2$

c) Interpret the tangent $\mathbb{C}^2$ (as in b)) as a line in $\mathbb{P}^3$ using the coordinates z_{ij} of 369

371 (Still keeping the notation of 369) Assume that $A_{00,00}=0$ so that $p=(1,0;1,0)\in C=(F=0)$. Write down a cubic condition on z_{01},z_{10},z_{11} in order that the line spanned by $(1,0,0,0)$ and $(0,z_{01},z_{10},z_{11})$ intersect the intersection of the two quadrics Q_1 and Q_2 (see 369) in an additional point to p

372 By dehomogenization, a biquadratic form F can be considered as a quartic in $\mathbb{C}^2$, and by homogenization as a quartic in $\mathbb{P}^2$.

Show that this quartic will have two nodes at infinity, and hence by a suitable Cremona transformation be brought to a (smooth) cubic.

For a suitable equation of F do this Cremona transformation expliciyly.

373 Show that by projecting a quartic curve from a point onto a plane, we get (unless the quartic is contained in a plane) a cubic curve. Conclude that the cubic is either smooth or singular, and that it is rational (i.e. parametrisable by a $\mathbb{P}^1$) in the latter case.

Try also and show the converse to 372, namely that if the projection of a quartic is a smooth cubic, the quartic is given by the intersection of two quadrics.

374 Extending 373, show that if two elliptic quartics lying on the same quadric Q have projectively equivalent (cubic) projections, then they are cut out by additional quadrics which are projectively equivalent under transformations preserving Q. Or equivalently the corresponding biquadratic forms can be transformed into each other by bi-Möbius transformations on $\mathbb{P}^1\times\mathbb{P}^1$. I.e by transformations of the type $(x;y)\mapsto(Ax,By)$ (if necessary composed with conjugation $(x,y)\mapsto(y,x)$) where $A,B\in PGL(2,\mathbb{C})$

375 Given an elliptic curve E with zero (and hence addition defined), show that all involutions with rational quotients can be given under the form $x\mapsto a-x$ for some a, hence the collection of such involutions is isomorphic with E but not canonically isomorphic. Show that geometrically an isomorphism is given by considering the elliptic as a cubic and considering projections from points p on the cubic.

376 Given a rational involution ι on an elliptic quartic C, show directly that the surface traced out by the lines $< x, \iota(x) >$ is a smooth quadric.

In particular show that two such lines cannot meet (unless they coincide, or ι belongs to four special cases) and that any line L not meeting C intersects two such lines.

377 Given an elliptic quartic C and a point p_0 on it. Consider an involution defined by considering the pencil of planes through the tangent line T_{p_0} and its residual intersection with C. Show that this is equivalent to the one considered in the proof of Proposition.(cf 375 as well)

378 Let L be a line that intersects the basecurve C of a pencil Λ of quadrics in just one point. Show that to each point p on L there is a unique quadric Q_p in the pencil passing through p. (In case p lies on C this statement has to be modified somewhat, how can uniqueness be ensured?).

At each point p of L we can consider the two lines (from each ruling) of Q_p, letting Π be a plane not containing L those lines will trace out a curve on Π. Try and determine that curve.

Preserving the setup of exericise 378 Consider a point p_0 such that Q_{p_0} is a cone, and a small circle γ around p_0 in $L \cong \mathbb{C}P^1$. For each t consider a line $l_{p(t)} \ni p(t)$ in $Q_{p(t)}$ with $p(t) = p_0 + e^{it}$ letting the lines vary "continously". Compare $l_{p(0)}$ with $l_{p2\pi)}$. If you prefer make the calculations explicit by suitable choice of C, L and p. The phenomena is refered to as *monodromy* .(Cf the choice of squareroot $\sqrt{z}$ when z rotates once around the origin)

379 Consider a (generic) pencil Λ of quadrics, and the collection of all lines contained in some member of the pencil.

Show that this collection is given by $C \times \mathbb{P}^1$ *Hint: Use exercise 378 to get a map onto C and to show that the ensuing fibration is trivial. (lots of disjoint sections)*

380 The lines of quadrics contained in a pencil will trace out planes in $\mathbb{P}^5$ Try and determine the union of those planes. In particular by fixing a 3-dimensional subspace appropriately, they will trace out a curve in that space, try and determine the curve.

381 Consider a real pencil of quadrics spanned by two quadrics of index zero, and such that the four cones are complex conjugate.

Show that all the reaL members of are smooth one-sheeted hyperbolas, and that their lines gives two families of real skewlines filling up $\mathbb{R}P^3$

382 Show that the projection of an elliptic quartic from a point neither on the quartic nor on any of the four cones associated to it is a quartic with two distinct singularities, which are either nodes or cusps. (Show that all the (three) possible combinations actually occur). What happens from a point not on a vertice of a cone, but on one of the cones?

383 Show that any elliptic quartic can be defined by equations of the form

$$X_0^2 = a_{11}X_1^2 + a_{12}X_2^2$$
$$X_3^2 = a_{21}X_1^2 + a_{22}X_2^2$$

Try and compute the $\jmath$-invariant of such an elliptic curve in terms of the determinant

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

384 Given an elliptic quartic C given as the baselocus of a pencil Λ of quadrics, and a point p on it. Show that the osculating plane to C at p can be constructed by choosing the one member containing the tangentline T_p to C at p, and its tangentplane at p

385 Given a point p on an elliptic quartic C, we may try and find the points p' such that there is a plane Π containg both p and p' such that restriction of the pencil of quadrics defining C yields a pencil of bitangent conics. Show that there are four such points p' and give a geometric construction of them.*Hint: Involve the four vertices associated to C*

386 Consider a pencil generated by a cone K with vertex p and a non-singular quadric Q with $p \in Q$. The tangent plane to Q at p may intersect K in either two lines or a double line. Show that those cases correspond to whether the intersection has a node or a cusp.

Involutions on twisted Cubics and nets of quadrics

Let C denote a twisted cubic (fixed from now on) i.e. a $\mathbf{P}^1$ embedded in $\mathbf{P}^3$ via a complete linear system of degree three.(cf exercise 387)

An involution T on a $\mathbf{P}^1$ is given by an unordered set of two distinct points.(The two fixed points of the involution) Thus the involutions are parametrised by $\mathbf{P}^2$ minus a conic. This can be made nicely explicit via Fregier points, in fact embedding $\mathbf{P}^1$ as a conic in $\mathbf{P}^2$ each point outside the conic defines an involution by projection, and conversely the lines joining any two points interchanged by T (a so called conjugate pair with respect to the involution) go throuh one point - the *Fregier* point of the involution.

Note that in this way it is clear that any two involutions share exactly one conjugate pair

One may now do someting similar for our twisted cubic C. In fact to each involution T we consider the surface that is sweeped out by the lines <p,Tp>. This surface turns out to be a non-singular quadric, and the lines constitute one of the rulings. And conversely all the non-singular quadrics that contain C occurs in this way. (As C must sit as a (1,2) curve)

Choosing a point P on C and projecting from P onto a plane Π C becomes a conic. The images of the lines joining conjugate pairs will hence need to pass through one point (The Fregier point), this point turns out to be exactly the intersection of Π with the line through P from the other ruling.

We thus see that to each twisted cubic C we can associate a net of quadrics. The singular quadrics will be parametrised by a conic in the net. Any twisted cubic lying on a singular quadric must pass through the vertex.

Given two involutions T_1 and T_2 they will correspond to two quadrics Q_1 and Q_2 respectively, and they will have one conjugate pair in common, thus there intersection will consist of C and the line joining that conjugate pair.

Q_1 and Q_2 will define a pencil, all of whose members are bitangent, the tangency points being exactly the common conjugate pair.

In that pencil there will be only two singular members (each occuring with multiplicity two) namely the two quadric cones we get by considering the chords of C through any of the two points in the common conjugate pair

This incidentally explains why one can always find a twisted cubic through six (*generic*) points in $\mathbf{P}^3$. Pick any two, and for each of those project onto a plane. In each case there will be a conic through the projections of the remaining five and the twisted cubic will be the residual intersection of the two cones with respect to the common line joing the two points

In the degenerate case however, the common conjugate pair may reduce to a fixed point, in that case all the quadrics in the pencil are hyperflexed at the fixed point, and there will be only one singular quadric (of multiplicity four) in the pencil.

Thus we have identified a subpencil of the net of quadrics containing C with a chord of C. The chords of C hence form a $\mathbf{P}^2$ dual to the net of quadrics containing it

Given a point P outside C there will be a unique chord to C through P. One sees that by projecting C from P obtaining a plane cubic with one singular point. The chords of C ($\mathcal{C}(C)$) form a $\mathbf{P}^2$ and hence we have established a rational map from $\mathbf{P}^3$ to $\mathbf{P}^2$ with fibers $\mathbf{P}^1$.

Given a plane Π in $\mathbf{P}^3$ it can be used to rationally parametrising the chords. In fact each chord to C determines a point on Π and conversely a point on Π determines a chord. This however only works birationally. Letting P_1,P_2 and P_3 be the three intersection points of C with Π (they may coincide), we see that the plane of chords is the Cremona of Π, the three lines joing the the points P_i are blown down, and to each P_i there will be a whole $\mathbf{P}^1$ of chords

The map from $\mathbf{P}^3$ to the plane $\mathbf{P}^2$ of chords is clearly given by the net of all quadrics containing C. This map blows up C. What is the inverse image?

The tangent chords will be parametrised by a conic in the net, the dual of the reduced discriminant of the net. Thus the exceptional divisor will map 2:1 onto the family of chords and be branched over a conic. Thus it will be a $\mathbf{P}^1$ x $\mathbf{P}^1$ i.e a quadric in its own right.

The exceptional divisor will have a natural section, namely the tangent chords. On the other hand to each point P on C we may consider the chords through P. On Π those trace out a conic (the projection of C) this conic clearly passes through the three fundamental points, thus it will correspond to a line on $\mathcal{C}(C)$. As there is only one tangent through each point, those lines must all be tangent to the conic in $\mathcal{C}(C)$ parametrising all tangentchords. The inverse image of that line will split up into two components in the quadric branched over the conic of tangent chords. One component will be naturally identified with C (through the residual intersections of the chords through P) the other will be identified with the constant P (hence P times a $\mathbf{P}^1$ so to speak) We have those identified the rulings of the exceptional divisor. Due to the existence of the section of tangent chords there will be a natural identification of points on either ruling.

The chords through a point P on C establish a 1:1 correspondence between the normal directions, with respect to C, through P and the curve C. In fact if L_1 and L_2 are two distinct non-tangential chords through P, the plane spanned by them cannot contain the tangent chord through P. (A line cannot intersect a conic in three points) Thus the exceptional divisor can be identified with the *ordered* pairs (P,Q), hence naturally identified with $C \times C$ and with the diagonal as distinguished section.

Given a pencil $\mathcal{L}$ of quadrics containing C (denoting the residual chord by L) this defines naturally a $\mathbf{P}^1$ bundle over the base $\mathbf{P}^1$ of $\mathcal{L}$, by considering to each quadric in the pencil the lines of the appropriate ruling. This Hirzebruch surface has one distinguished section, namely the line L which will occur in each fiber. Clearly we are looking at $\mathbb{F}_1$ and by blowing down the distinguished section we obtain $\mathcal{C}(C)$ = $\mathbf{P}^2$. Disjoint sections are simply gotten by fixing a quadric Q in the net not belonging to $\mathcal{L}$, and for each Q_t in $\mathcal{L}$ considering the residual chord L_t of Q and Q_t.

Exercises

387 Show that by proper choice of co-ordinates any twisted cubic can be parametrised as follows

$$(s^3, s^2t, st^2, t^3)$$

(from now on refered to as the standard parametrisation) and conclude that any two twisted cubics are projectively equivalent.

388 Given a point (a_0, a_1, a_2, a_3) in $\mathbf{P}^3$ find the secant to the "standard" twisted cubic passing through it.

389 Given the standard twisted cubic, find the net of all quadrics containing it

390 Consider the 2×2 minors of the matrix

$$\begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_2 & X_3 \end{pmatrix}$$

Show that those define a net of quadrics with a twisted cubic as baselocus. Determine the twisted cubic explicitly.

391 The family $\mathcal{C}(C)$ of secants to C form a $\mathbf{P}^2$ together with a distinguished conic K corresponding to the tangents. The points of K are naturally identified by the points of C. Show

i) The secants through a point p on C corresponds to the tangentline of K at p

ii)The secant through p and q ($< p, q >$) correspond to the intersection of the tangents to K at p and q, hence to the polar of the line p, q

Furthermore show that

iii) A line L in $\mathcal{C}(C)$ defines an involution on K (the dual of the *Fregier* construction, and the secants corresponding to L trace out a quadric containing C, and that the corresponding involution (defined by the quadric) is the same as defined by the line L

This gives a way of identifying the net of quadrics containing C with the dual of the projective space $\mathcal{C}(C)$; as we already have an identification with $\mathcal{C}(C)$ itself we get a canonically defined quadric. Show

iv) This quadric is just K

392 Given the parametrisation of C given in *387*, write down explicitly the Plücker co-ordinates of a given secant in terms of points of $\mathbf{P}^2$ *Thus a point* (a_0, a_1, a_2) *corresponds to a binary form* $a_0s^2 + a_1st + a_2t^2$ *whose two zeroes give the endpoints of the secant*

In particular consider the restriction of this parametrisation of $\mathcal{C}(C)$ to the conic K of tangents.

393 Show that the secants of C that meet a given line L form a quadric in $\mathcal{C}(C)$ and that this quadric splits into two lines (one of which necessarily is a tangent to K) iff L meets C. (What happens if L is a secant of C?)

394 The tangents of C sweep out a surface $\mathcal{T}(C)$, called the tangent developable of C, show that this surface is a quartic, and that its singular locus consists of C. Furthermore show that the intersection of $\mathcal{T}(C)$ with a generic plane is a tricuspidal quartic (a *Steiner quartic*)

395 Given the standard twisted cubic find the equation of its tangent developable explicitly.

396 Among the planes that contains the tangent at a point, there is a distinguished one, whose intersection at the point has multiplicity three. This is caled the osculating plane. Show that in the case of a twisted cubic, a plane is osculating iff it can be written as

$$A_0s^3 + A_1s^2t + A_2st^2 + A_3t^3 = \lambda(\alpha_0 s - \alpha_1 t)^3$$

use this to show that the osculating planes of a twisted cubic form a twisted cubic in the dual space

397 Using the fact that the Grassmanian of lines in $\mathbf{P}^3$ and that of its dual are canonically isomorphic, we can to a twisted cubic associate the tangent developable to its dual (according to the construction of *396* via osculating planes) and consider it in the same $\mathbf{P}^3$. Thus to each twisted cubic we can associate a dual cubic (the singular locus of the tangent developable of the dual cubic in $\mathbf{P}^{3*}$). Show that the dual cubic lies on the tangent developabel of the original one (and vice versa). Do the two cubics coincide?

398 Show that three distinct osculating planes to a twisted cubic C cannot belong to a pencil, hence that they determine a point. Conversely show that through a point (outside C) there are exactly three osculating planes passing through it.

Thus a point p in $\mathbf{P}^3$ determines three points on C and conversely. Show that $p \in C$ iff the three points coincide, and p lies outside the tangentdevelopable iff the three points are distinct.

399 The conics of $\mathcal{C}(C)$ corresponding to lines in $\mathbf{P}^3$ form a 4-dimensional subvariety ($\mathcal{G}$), in fact a quadratic hypersurface of the space $\mathbf{P}(S^2(\mathbb{C}^3)) = \mathbf{P}^5$ of conics. Show that to each four points of K there corresponds two conics in $\mathcal{G}$ *Hint: To four skew lines there exists exactly two lines meeting them*, and that this exhibits a double cover of $\mathcal{G}$ onto $\mathbf{P}^4$ (sets of four points on K) through projection from K

400 Show that there is a natural embedding of $\mathcal{C}(C)$ into the Grassmanian of lines $\mathcal{G}(2,4)$ in $\mathbf{P}^3$, and that the composite

$$\mathcal{C}(C) \to \mathcal{G}(2,4) \to \mathbf{P}^5$$

exhibits $\mathcal{C}(C)$ as the Veronese embedding of $\mathbf{P}^2$. Thus the conics of $\mathcal{C}(C)$ corresponds to the secants lying in a linear complex (the intersection of $\mathcal{G}(2,4)$ with a hyperplane, and the special conics to tangent hyperplanes to $\mathcal{G}(2,4)$)

401 Considering the setup of *392* write down the conditions of the coefficients of conics corresponding to specail conics (those whose secants all intersect a fixed line). Try to find a purely geometric characterization of such conics.

402 Corresponding to a conic X in $\mathcal{C}(C)$ we obtain a surface $\mathcal{S}(X)$ in $\mathbf{P}^3$ traced out by the secants. Find the degree of $\mathcal{S}(X)$ and determine its singular locus. And finally determine when two such surfaces are projectively equivalent.

403 Let $< Q_0, Q_1, Q_2 >$ span a net with a twisted cubic C as baselocus, in the product $\mathbf{P}^{3,2} = \mathbf{P}^3 \times \mathbf{P}^2$ consider the variety V cut out by the three bihomogenous polynomials $y_iQ_j(x) - y_jQ_i(x)$. Show that

i) V is the closure of Γ the graph given by the map defined by the net

ii)V is the blow up of $\mathbf{P}^3$ along the twisted cubic C

iii) If $\pi_2 : \mathbf{P}^{3,2} \to \mathbf{P}^2$ is projection onto the second factor (and π_1 defined similarly). Then for any conic X the map $\pi_1 : \pi_2^{-1}(X) \cap V \to \mathbf{P}^3$ exhibits a so called resolution of $\mathcal{S}(X)$

404 Consider the linear space V^3 of all cubic forms that vanish on a fixed twisted cubic C.

i) Show that $\dim V^3 = 10$ and any $v \in V^3$ can be written under the form

$$L_0Q_0 + L_1Q_1 + L_2Q_2$$

where L_i are linear forms, and Q_i span the net of quadrics vanishing on C

ii) Conclude that the linear map $\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4 \to V^3$ given by

$$(L_0, L_1, L_2) \mapsto L_0Q_0 + L_1Q_1 + L_2Q_2$$

is surjective and has a 2-dimensional kernel K

iii) Show that if $(L_0, L_1, L_2), (M_0, M_1, M_2)$ span K than the line $L_0 = M_0 = 0$ coincide with the residual intersection line of Q_1 and Q_2 and etc. by symmetry

iv) Given L_0 subjected to the condition of iii) show that the planes $L_1 = 0$ and $L_2 = 0$ giving $L_0Q_0 + L_1Q_1 + L_2Q_2 = 0$ are unique and describe how to get them

v) Determine K explicitly in the case of the net given in *390*

405 Consider the 15-dimensional group $PGL(4, \mathbb{C})$ of projective transformations of $\mathbf{P}^3$. Show that it operates transitively on twisted cubics (cf *387*) and that its stabilizer S at each cubic is isomorphic with $PGL(2, \mathbb{C})$

406 Conserving the notation of *405* consider the orbits of S operating on $\mathbf{P}^3$. Show that there are exactly three orbits. (cf *398*)

407 Consider a real twisted cubic C in $\mathbf{RP}^3$ show that the secants do not fill out space, but that the tangent developable subdivides it into two parts, one which consists of the union of all secants. (*Note; Real secants corresponds to the outside of the conic K in $\mathcal{C}(C)$ considered as a real projective space*)

By introducing a plane at infinity we can talk about betweeness and segments and hence convexity. Show that the union of all secants can never be convex (in fact that the convex hull of a real twisted cubic is always the whols space). But that the "empty" space is convex?