Group Representations

Finite Groups

Let V be a finite dimensional vectorspace over a field K. By a representation of a group G on V is meant an action of G on v (written $(g,v) \mapsto gv$) satisfying the following.

$$\text{(i)} \qquad g(v+w) = gv + gw \quad g(\lambda v) = \lambda gv$$
$$\text{(ii)} \qquad (gh)v = g(hv)$$

The first part (i) means that $v \mapsto gv$ is a linear map. Thus we have a map

$$G \to \mathrm{G}l(V)$$

The second part (ii) means that this map is a group homomorphism. Thus a representation of G on V is nothing but a group homomorphism from G into $\mathrm{G}l(V)$. Thus the elements of G are represented by linear maps, and if a basis is chosen for V, also represented by matrices.

Ex 1. *The trivial representation.* If $G \to (1)$ we say that V is a trivial representation. This means that $gv = v, \quad \forall g$ thus all vectors are invariant. Clearly every vectorspace can be endowed with a trivial representation. Thus note that G does not have to be a subgroup of $\mathrm{G}l(V)$.

Any representation of G on a vector space V extends to linearity to the whole ring $K[G]$ - the so called group algebra, consisting of elements of the form $\sum_g a(g)g$, which is a vectorspace with the elements g of the group as a basis. A multiplication is defined by letting multiplication on the basis elements $g \times h = gh$ and then extend by linearity.

Thus a representation of G on V is the same thing as a $K[G]$-module structure on V. Thus a sub-representation is defined as a $K[G]$-submodule (and similarly for quotient representations).

Ex 2. *The regular representation* $K[G]$ is a module over itself (by multiplication on the left). This defines the vector space $K[G]$ as a representation called the regular representation. Note that in this case, the representation is faithul, i.e. if $gv = v \quad \forall v$ then $g = 1$ (check it on basis elements!), thus G appears as a subgroup of $\mathrm{G}l(K[G])$

Two representations V and W are said to be isomorphic iff they are isomorphic as $K[G]$ modules. This means that there is a K-linear map $\Phi : V \to W$ commuting with all the elements of $K[G]$ and *a fortiori* with G. This means

$$\Phi(gv) = g\Phi(v)$$

or equivalently for each $g \in G$ we have the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\Phi} & W \\ g\downarrow & & \downarrow g \\ V & \xrightarrow{\Phi} & W \end{array}$$

Ex 3. Two different embeddings of G in $\mathrm{G}l(V)$ define the same representation iff the two subgroups G_1, G_2 are conjugated. This means that there is an element $A \in \mathrm{G}l(V)$ such that $G_1 = AG_2A^{-1}$

Given representations V, W we may cook-up new representations on $V^*, V \otimes W, \mathrm{Hom}(V, W)$ etc. E.g. on $V \otimes W$ we define on the elements of type $v \otimes w$

$$g(v \otimes w) = gv \otimes gw$$

and on V^*

$$(gv^*)(v) = v^*(g^{-1}v)$$

As $\mathrm{Hom}(V, W) \equiv V^* \otimes W$ we get likewise a representation on that space. Explicitly

$$(gf)(v) = g(f(g^{-1}v))$$

Note that $V^* = \mathrm{Hom}(V, K)$ with K endowed with the trivial representation.

Ex 4. If $V = W$ then G both act on $\mathrm{Hom}(V, V)$ and can be thought of as a member of G (although not necessarily faithfully). Thus gf can mean two things. Either the action of g on the vector f or the composition of the two linear maps g and f! To distinguish the two meanings let us write $g \circ v$ for the former. Then we have

$$g \circ f = gfg^{-1} \tag{1}$$

To every representation V we can associate the submodule V_0 of invariant element. namely $v : gv = v \forall g$. This is clearly a sub-representation.

Ex 5. We have that

$$\mathrm{Hom}(V, W)_0 = \mathrm{Hom}_{K[G]}(V, W)$$

To every vector v we can associate the weighted-average $\frac{1}{|G|}\sum_{g \in G} gv$. This clearly is an invariant element. As we can write

$$\frac{1}{|G|}\sum_{g \in G} gv = ev$$

with $e = \frac{1}{|G|}\sum_{g \in G} g$ this defines a $K[G]$ linear map $V \to V_0$. As one easily computes that $e^2 = e$ (a so called idempotent) in the group-algebra, we conclude that this is a projection, and thus that $V = V_0 \oplus V_1$, where V_1 is the image of the projection $1 - e$ or equivalently the kernel of e.

Ex 6. If $p \in \mathrm{Hom}(V, V)$ is a projection so is $e \circ p$? In fact by (1) we have

$$(g \circ p)(g \circ p) = (gpg^{-1})(gpg^{-1}) = gp^2g^{-1} = gpg^{-1} = g \circ p$$

As e is just a linear combinations of the elements g we would like to conclude that the same holds for $e \circ p$ but this is not a linear condition. However if p is a projection onto an invariant subspace then $pgp = gp$ from which it easily follows that $e \circ p$ is an idempotent.

From this example we may draw the following important consequence.

Semi-simplicity If W is a sub-representation of a representation V, then there exists another representation U such that $V = W \oplus U$.

We simply choose an arbitrary projection $p : V \to W$ and consider the average ep which is a G-homomorphism, by the above example ep is also an idempotent, and thus the invariant complement is given b y $\ker(1 - ep)$.

Another argument can be construed as follows. Given a positive definite Hermitian form. I.e. a sesquilinear form $H(x, y)$ such that $H(x, y) = \overline{H(y, x)}$ and $H(x, x) \geq 0$ with equality iff $x = 0$ we can consider its average

$$eH = \frac{1}{|G|} \sum_g H(gx, gy)$$

. This average is non-zero, as for $x = y \neq 0$ the terms are strictly positive. We conclude that there always exists non-degenerate invariant sesquilinear forms, in fact positive definite Hermitian such.

We draw two consequences.

1) Every representation of a finite group G can be chosen unitary, i.e. one may conjugate the image of G to lie in $U(n, \mathbb{C})$.

2) Every submodule has an invariant complement, the orthogonality relation given by the Hermitian form, and its G-invariance assured by the G invariant of the form. In the case of real representations, we may instead look at positive definite symmetric forms, and conclude that finite subgroups of $Gl(n, \mathbb{R})$ can be conjugated into $O(n, \mathbb{R})$

Note: Every unitary matrix can be diagonalised over the complex numbers. Thus all the element $g \in \mathrm{G}l(n, \mathbb{C})$ can be chosen to have a basis of eigenvectors. We can also see this directly for elements of finite order by using the Jordan decomposition. Every matrix A can be written as $A = S + N$ where S can be diagonalised and N is nilpotent commuting with S. We want to show that $N = 0$ if A has finite order. If $N \neq 0$ we can choose v such that $Nv \neq 0$ and V an eigenvector for S but $N^2 v = 0$. Then $v = Iv = A^n v = v + nS^{n-1}v \neq v$

DEFINITION A representation is said to be irreducible iff it does not contain any proper sub-representations. Clearly every (finite-dimensional) representation contains an irreducible representation.

We now come to the central observation on which much of representation theory hinges.

Schurs Lemma Let V, W be two irreducible modules then either

$$\mathrm{Hom}_{K[G]}(V, W) = 0 \tag{0}$$

$$\mathrm{Hom}_{K[G]}(V, W) = \text{a skew-field over} K \tag{1}$$

With the case zero corresponding to non-isomorphic modules, and the case one corresponding to isomorphic modules.

Proof Given a map $\Phi : V \to W$ between irreducible modules, its kernel and cokernel will be submodules, thus either equal to zero or the whole thing. If there exists a non-trivial map we are in case 1, (as every non-zero map is an isomorphism and hence invertible) in which V and W in particular will be isomorphic (by any non-zero map), in the case of 0, V and W clearly are non-isomorphic. ♠

Note: $\mathrm{Hom}_{K[G]}(V,V)$ is of course a ring, sometimes denoted by $\mathrm{End}(V)$ the endomorphism ring, while $\mathrm{Hom}_{K[G]}(V,W)$ 'becomes' a ring only when an isomorphism between V and W is presented.

In the case of $K = \mathbb{C}$ we can strentgthen the conclusion in one to state.

If V, W isomorphic then $\mathrm{Hom}_{[G]}(V,W) = \mathbb{C}$, and in fact if $V = W$ every map is given by a homothety $x \mapsto \lambda x$.

Proof The only finite-dimensional skew-fields over $\mathbb{C}$ are $\mathbb{C}$ itself. In fact for any linear map $\Phi : V \to V$ let λ be an eigenvalue. Then the linear map $\Phi - \lambda I$ has non-trivial kernel, hence the kernel is the whole module. ♠

As a first corollary of Schurs lemma we will note that in any direct sum decomposition of an arbitrary module into a direct sum of irreducible components, the types of irreducible modules that appear are independant of the decomposition. In fact if $V = \bigoplus W_i$ is a decomposition into irreducible components $\mathrm{Hom}_G(V,W) = \bigoplus \mathrm{Hom}_G(W_i,W)$ where the lefthand side is independant of the decomposition.

We are now ready to make the central definition of a character of a representation.

Definition Given a representation V by the character χ_V we mean the map $\chi_V : G \to K$ defined by $\chi_V(g) = \mathrm{Tr}(g)$ where g is considered as the linear map $v \mapsto gv$.

Note that isomorphic representations will give rise to the same character. This follows from the fact that isomorphic representations give rise to conjugate linear maps (or matrices). Furthermore as $Tr(AB) = Tr(BA)$ which implies

$$Tr((AB)A^{-1}) = Tr(A^{-1}(AB)) = Tr(B)$$

. Thus we are done.

Let us note some simple but important properties of characters

a) A character is a class-function, i.e. it is constant on conjugacy classes.

This follows from the fact that trace is constant on conjugacy classes as we noted above. Alternatively we note that the trace is the sum of the eigenvalues, which are independant of the conjugacy class of a matrix.

b) We have $|\chi(g)| \le \dim V$ with $\chi(g) = \dim V$ iff g acts as the identity.

As $g^n = 1$ for some n, the eigenvalues are roots of unity, hence in particular of absolute value at most one. As noted above the trace is the sum of the eigenvalues.

c) A 1-dimensional character is a homomorphism $G \mapsto \mathbb{C}^*$ and conversely any homomorphism $G \mapsto \mathbb{C}^*$ (sometimes refered to as a character of G) is the character of a 1-dimensional representation (namely itself!).

In the 1-dimensional case $Gl(1,\mathbb{C})$ is simply $\mathbb{C}^*$ and the trace of a map can naturally be identified with the map.

Ex 7. The character of the regular representation is given by

$$\chi(1) = |G|$$
$$\chi(g) = 0 \quad g \neq 1$$

Given the character of a representation we can also compute the character of associated representations. More specifically, let χ_V, χ_W be the characters of V and W respectively. Then
a) $\chi_{V^*} = \overline{\chi_V}$
b) $\chi_{V\oplus W} = \chi_V + \chi_W$
c) $\chi_{V\otimes W} = \chi_V \chi_W$
d) $\chi_{\mathrm{Hom}(V,W)} = \chi_W \overline{\chi_V}$

Proof Given an element g we can choose a basis of eigenvectors. Thus let $ge_i = \lambda_i e_i$ where e_i constitute a basis for V and similarly $gf_j = \mu_j f_j$ where f_j is a basis for W. Now let e_i^* be a dual basis for e_i then

$$ge_i^*(e_j) = e_i^*(g^{-1}e_j) = \lambda_i^{-1} e_i^*(e_j)$$

thus e_i^* is also a basis of eigenvectors, but with eigenvalues λ_i^{-1}. The eigenvalues are roots of unity, in particular $|\lambda_i| = 1$ thus $\lambda_i^{-1} = \bar{\lambda}_i$ and thus the trace, being the sum of eigenvalues will be the conjugate.

Furthermore $(e_i, 0), (0, f_j)$ will constitute a basis of eigenvectors for the action of g on $V \oplus W$ with the eigenvalues λ_i, μ_j respectively hence b).

We also note that $e_i \otimes f_j$ will be a basis of eigenvectors for the action on $V \otimes W$ as

$$g(e_i \otimes f_j) = ge_i \otimes gf_j = \lambda_i \mu_j (e_i \otimes f_j)$$

from which c) follows.

As $\mathrm{Hom}(V, W) \equiv V^* \otimes W$ a) and c) implies d) ♠

For future references it may also be useful to note that the module of sesquilinear forms also gets, as we have already observed, a natural action. Given a basis e_i as above we get

$$H(ge_i, ge_j) = \lambda_i \bar{\lambda}_j H(e_i, e_j)$$

from which we conclude that the corresponding character is given by $|\chi_v|^2$, which incidentally is the same character as for $\mathrm{Hom}(V, V)$.

Now we want to establish the following formula

$$\frac{1}{|G|} \sum_{g\in G} \chi_V(g) = \dim V_0$$

Proof Recall the idempotent $e = \frac{1}{|G|}\sum_{g\in G} g \in K[G]$. Its trace can be computed in two different ways. Either as $\dim \mathrm{Im}(e) = \dim V_0$ or simply as the sum on the lefthand side. ♠

Now we apply this to the module $\mathrm{Hom}(V, W)$ obtaining the following orthogonality relation between irreducible characters

$$\frac{1}{|G|} \sum_{g\in G} \chi_W(g)\overline{\chi_V(g)} = \dim \mathrm{Hom}_G(V, W)$$

Let us define the Hermitian innerproduct on $\mathbb{C}[G]$ (i.e. on complex-valued functions on G) as follows

$$< \psi, \chi >= \frac{1}{|G|}\sum_{g\in G}\psi(g)\chi(g^{-1}) = \frac{1}{|G|}\sum_{g\in G}\psi(g)\overline{\chi(g)}$$

We can thus state for irreducible representations over $\mathbb{C}$

$$< \psi, \chi >= \begin{cases} 0 & \psi, \chi \quad \text{different representations} \\ 1 & \psi, \chi \quad \text{, the same representation} \end{cases}$$

Let us first note a startling consequence, namely the character of an irreducible representation (and hence any representation) determines the representation.

Thus in particular if a finite group G is mapped into $Gl(V)$ in two different ways such that elementwise the matrices are conjugate (*a priori* by different matrices) then there exists a fixed (common) linear transformation performing all the conjugations!

Also we note that on a fixed vector-space (i.e. fixing the dimension $= n$) only a finite number of different representations can be induced. Namely the group being finite there is a common exponent N say. Thus there is only N possibilities for the eigenvalues, and thus at most N^n possibilities for the traces of any given element. The number of element being finite, we note that there will only be a finite number of possible characters. Furthermore as we have noted above, the dimension of an irreducible representation is at most the order of the group, thus there can only be a finite number of irreducible characters and hence irreducible representations. The estimates that we have given are, however, ridiculously crude

We finally note that as the module of sesquilinear forms is isomorphic to the endomorphism module, there will for irreducible modules only be one, up to multiplication by scalar, invariant form. As one example is given by a positive definite Hermitian, that invariant form will in fact be positive definite Hermitian!

Now let V be an arbitrary representation and let $\bigoplus m_i V_i$ be a decomposition of V into irreducible representations. Then the m_i and the irreducible components are uniquely determined by V. In fact we have

$$m_i =< V, V_i >$$

Proof this is an immediate consequence of the orthogonality relations. The characters of irreducible components span the space of generated by all characters, and being an orthonormal system, they provide in fact a basis, thus every character can be uniquely written as a linear combination. The coefficients are in fact given by the formula above. ♠

Given an arbitrary representation $\chi = \sum_i m_i \chi_i$ we compute

$$< \chi, \chi >= \sum_i m_i^2$$

Thus we see that a character χ corresponds to an irreducible representation iff $< \chi, \chi >= 1$.

Ex 8. The decomposition of the regular representation is given by

$$\mathbb{C}[G] = \bigoplus m_i V_i$$

where $m_i = \dim V_i$

Proof Let χ be the regular character, and χ_i the character corresponding to the irreducible representation V_i, then

$$< \chi, \chi_i >= \frac{1}{|G|} \sum_g \chi(g) \chi_i(g^{-1}) = \chi_1(1) = \dim V_i$$

♠

As a corollary we note that every irreducible representation V_i occurs in the regular representation (with a multiplicity given by its dimension). This gives a much sharper bound on the number of irreducible representations than the one above. In particular if we exploit the identity

$$\chi = \sum_i m_i \chi_i$$

we get

$$\sum_i m_i^2 = |G|$$

$$\sum_i m_i \chi_i(g) = 0 \quad g \neq 1$$

The subrepresentations $W_i = \bigoplus_{k=1}^{m_i} V_i$ are refered to as the isotopic components. Those are unique components of V. However each isotopic component can be represented as a sum of its irreducible components (all isomorphic to the irreducible 'type') in many different ways.

Ex 9. The isotopic component corresponding to the trivial irreducible representation has been refered to above as V_0. Its dimension is equal to the number of times the trivial irreducible representation appears. The latter is given by $\frac{1}{|G|} \sum_g \chi(g)$

Ex 10. The group $\mathbb{Z}_2$ has only two types of irreducible representations, namely the two maps of $Z_2 \to \mathbb{C}^*$. The non-trivial representation is given by the unique embedding of Z_2 as ± 1. A general representation is determined by the action of the non-trivial element ι which is an involution (i.e. $\iota^2 = 1$). The two isotypic components are determined by $\iota v = v$ and $\iota v = -v$ respectively. Projections onto each component are given respectively by

$$v \mapsto \frac{1}{2}(v + \iota v)$$

$$v \mapsto \frac{1}{2}(v - \iota v)$$

Maps of $\mathbb{Z}_2$ into $Gl(V)$ are classified up to conjugacy by the number of $+1$ eigenvalues and -1 eigenvalues. Which also is the classification of involutions up to conjugacies. Fixing the dimension to n the trace will take every other value from $-n$ to n a total number of possibilities given by $n+1$. Involutions can be thought of reflections in the linear subspace V_0, those reverse the orientation iff there is an odd number of -1 eigenvalues.

We have noted that characters are constant on conjugacy classes. A function constant on conjugate classes is called a classfunction. Those make up a subring of the groupalgebra $\mathbb{C}[G]$ and its dimension is clearly equal to the number of conjugacy classes. As irreducible characters are linearly independant, their numbers is bounded by the number of conjugacy classes. We would like to show that it is in fact equal to the number. To do so, we would like to show that a classfunction orthogonal to all characters is equal to zero.

Consider for each classfunction f the element $\tilde{f} = \sum_g f(g)g \in \mathbb{C}[G]$. Then $\tilde{f}$ is invariant under conjugation (i.e. $g\tilde{f}g^{-1} = f$. This means that $\tilde{f}$ commutes with all elements g and thus belongs to the center of the groupalgebra. In particular multiplication by f is a G homomorphism, and thus its restriction on an irreducible module V is given by a homethety $v \mapsto \lambda v$. The trace of this homethety is given by on one hand $\lambda \dim V$ on the other hand by $\sum_g f(g)\chi(g)$. If f is assumed orthogonal to all irreducible components, it restricts to zero on all irreducible modules in particular on the regular representation. But $\tilde{f}1 = \tilde{f}$ thus $\tilde{f} = 0$ which means (as g constitute a basis for $\mathbb{C}[G]$) that all the $f(g)$ vanish.

If we apply the above argument to the classfunctions χ themselves we can state that the element $e_\chi = \frac{\chi(1)}{|G|}\sum_{g\in G}\overline{\chi(g)}g$ is an idempotent, and that the map $v \mapsto e_\chi v$ is a projection onto the isotopic component V_χ

PROOF Let us restrict e_χ to the irreducible components. Being a G-homomorphism it is a homothety, its value being given by the trace. Let us choose an irreducible component W with character ψ. The trace will be given by

$$\frac{\chi(1)}{|G|}\sum_{g\in G}\overline{\chi(g)}\psi(g) = \chi(1) < \psi, \chi >$$

Thus we note it will act as the identity on components isomorphic to W and as zero on others. ♠

Thus we are able to each group G associate a character-table. To fix ideas we let the horizontal row denote the different conjugacy classes c, each by a multiplicity μ_c given by the number of elements in it, while the vertical column will list the different irreducible representations. Thus each horizontal row will encode a character, its values given on the different conjugacy classes, while each vertical row will denote the traces of the possible irreducible representations of elements in a given conjugacy class.

Ex 11. the horizontal row corresponding to the trivial (irreducible) representation will consist of just 1's; while the vertical row corresponding to the unique conjugacy class of the identity, will list the various dimensions.

The square array of numbers will satisfy a number of numerical relations. The horizontal rows will constitute an orthonormal basis under the innerproduct given by

$$< \chi, \psi >= \frac{1}{|G|}\sum_c \chi(c)\overline{\psi(c)}\mu_c$$

while the vertical columns will likewise constitute an orthonormal basis under the innerprocuct given by

$$< c, d >= \frac{1}{|G|}\sum_\chi c(\chi)d(\chi) = \frac{1}{|G|}\sum_\chi \chi(c)\chi(d)$$

Note: For the case of $c = \{1\}$ this has already been established above in connection with the regular representation. In general let f_c be a class-function which is one on c and zero on the other classes. We can write $f_c = \sum_\chi \alpha_c \chi$ where the coefficients α_c are given

$$\alpha_c =< f_c, \chi >= \frac{1}{|G|}\chi(c)$$

Thus

$$f_c(d) = \frac{1}{|G|}\sum_\chi \chi(c)\chi(d)$$

from which we can read it off.

Ex 12. The trivial character table is given simply by the number 1. The character table for $\mathbb{Z}_2 = \{\pm 1\}$ is given by the 2×2 array

	1	-1
1	1	1
-1	1	-1

While finally the character table for the group S_3 (given by permutation on three letters) is given as follows.

	$()$	(12)	(123)
1	1	1	1
-1	1	-1	1
θ	2	0	-1

The reader is encouraged to check that all the numerical conditions are satisfied!

A group is abelian iff all of its irreducible representations are 1-dimensional.

PROOF If two matrices A, B commute, then they preserve each others eigenspaces. In fact if $Av = \lambda v$ then $A(Bv) = BAv = B\lambda v = \lambda Bv$. Thus any two commuting matrices can be simulaneously diagonalised. This clearly generalises to any set of pair-wise commuting matrices. Thus the common eigenspaces constitute an irreducible decomposition.

Conversely if all representations are 1-dimensional, then there must be $G = \sum 1$ different representations, hence the same number of different conjugacy classes, i.e. each conjugacy class reduces to one element, thus G is abelian. (This argument is incidentally reversible, showing that by numerology each irreducible representation of an abelian group is 1-dimensional) ♠

Note that any representation of an abelian group can be effected by diagonal matrices, the entries of which are given by the (1-dimensional) characters. Thus it is easy to read off from the character table explicit matrices for a representation.

Ex 13. Every non-commutative group has at least one multi-dimensional irreducible representation. If the group G has an abelian subgroup A of index d then any irreducible representation W of G has dimension at most d. In fact let $W_0 \subset W$ be a 1-dimensional A-submodule generated by w say. Then the space spanned by the d vectors gAw is invariant under G.

Real representations

There are three types of irreducible representations W over the reals, classified by the skew-field $\mathrm{Hom}_{\mathbb{R}[G]}(W,W)$. The three cases are summarized below

skewfield	$<\chi,\chi>$
real	1
complex	2
quaternions	4

where the numbers in the right column are just the dimensions (over $\mathbb{R}$) of the corresponding skew-fields.

Taking an irreducible real representation W and complexifying it to $W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$ we get a complex representation which may or may not be irreducible. Clearly complexifying a real representation does not change the character (we are simply looking at the real representation with real matrics and everything and just considering them to be complex numbers). Thus we can conclude

type	$<\chi,\chi>$	decomposition		
real	1	$W_{\mathbb{C}}$		
complex	2	V	$\oplus$	V^*
quaternions	4	V	$\oplus$	V

Thus to every real irreducible representation corresponds either an irreducible complex representation, or a pair of two conjugate irreducible complex representations. The first case (one and four) means that the complex irreducible character is real, while the second case (two) corresponds to non-real characters.

Note that case one means that the irreducible complex representation $W_{\mathbb{C}}$ is defined over $\mathbb{R}$. But not every complex representation $G \to Gl(n,\mathbb{C})$ can be realised over $\mathbb{R}$. A necessary condition for this to be possible is that the characters are real, or equivalenty that $\bar{\chi} = \chi$. This can also be reformulated as $V \equiv V^*$ which means that there is a non-degenerate bilinear form B invariant under G which defines an isomorphism via $x \mapsto B(x,*)$. If V is irreducible there can, up to homethety, only exist one such form B.

We can compute the dimension of the invariant part of $V \otimes V$ (the space of bilinear forms) as given by

$$\frac{1}{|G|}\sum_g \chi^2(g) =<\chi,\bar{\chi}>$$

If χ is real then $\chi,\bar{\chi} >=< \chi,\chi >= 1$ otherwise it is zero by orthogonality of distinct irreducible representations. In the case of non-real characters, there will be a unique sesqui-linear form, actually a postive definite Hermitian as we have observed above.

On the space $V \times V$ of bilinear forms there is an action of $\mathbb{Z}_2$ given by $v \otimes w \mapsto w \otimes v$ and extended by linearity. The isotopic components correspond to symmetric and alternating forms respectively, denoted by $S(V)$ and $A(V)$. This action of $\mathbb{Z}_2$ commute with the action of G, so both of those are also G modules. By setting $vw = \frac{1}{2}(v \otimes w + w \otimes v)$ and $v \wedge w = \frac{1}{2}(v \otimes ww \otimes v)$, a basis e_i of eigenvectors for g with eigenvalues λ_i gives a both a basis for $S(V)$ via e_ie_j with $i \leq j$ and for $A(V)$ via $e_i \wedge e_j$ with $i < j$ with eigenvalues $\lambda_i\lambda_j$. Adding up we get the characters χ_s

and χ_a as follows

$$\chi_s(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2))$$
$$\chi_a(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2))$$

as the eigenvalues for g^2 are given by λ_i^2 If V is irreducible, then the non-zero invariant bilinearform is either symmetric or alternate (never both). In fact by considering the sum

$$\frac{1}{|G|}\sum_g(\chi_s(g) - \chi_a(g)) = \frac{1}{|G|}\sum_g \chi(g^2)$$

which will take values $1, 0, -1$ we can from the character itself distinguish the different cases, corresponding to the existence of either an invariant symmetric form, an invariant Hermitian form, or an invariant alternate form.

We can also make another division into three classes of irreducible complex representations. Namely each n-dimensional complex representation V becomes a $2n$-dimensional real representation by restriction of scalars (i.e. forgetting that we can multiply with i). Let us denote the real representation by $V_{\mathbb{R}}$. As before, starting with an irreducible complex representation V there is no reason why $V_{\mathbb{R}}$ should be an irreducible real representation.

The first thing we need to know is to compute the real character χ from the complex character ψ. Recall that multiplication by the complex number $a + ib$ on $\mathbb{C}$ is rendered by the matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ when $\mathbb{C}$ is identified with $\mathbb{R}^2$ via the natural basis $1, i$. From this we conclude that the trace $a+ib$ of the complex multiplikation is turned into $2a$. From this we conclude that

$$\chi = 2\text{Re}\psi$$

The second thing we should observe is that either is $V_{\mathbb{R}}$ irreducble (over $\mathbb{R}$) or there is a maximal real-submodule $W \neq V$. Now $W \cap iW$ is a complex submodule, hence equal to zero, thus $W \oplus iW$ is a complex submodule of V, the letter being irreducible we note $V = W \oplus iW$. Also any real submodule $W_0 \subset W$ will give rise to a complex submodule $W_0 \oplus iW_0$ hence W is real irreducible. Not also that multiplication by i commutes with G so iW and W are isomorphic, and thus V as a real module is isotopic.

Now if ψ is real, we get that $\chi = 2\psi$ and thus $<\chi, \chi>= 4 <\psi, \psi>= 4$. If $V_{\mathbb{R}}$ is irreducible, it is thus of quaternionic type, and if it is reducible, we have $\chi_W = \psi$ and thus they define the same module, $V = W \otimes_{\mathbb{R}} \mathbb{C}$ and W is of real type.

But if ψ is not real than at least for some g we have $|\chi(g)|^2 < 4|\psi(g)|^2$ from which we conclude that $<\chi, \chi>< 4$ thus only the cases $<\chi, \chi>= 1, 2$ occurs. Also we may not form $\chi/2$ (the putative character of a summand W) so that $V_{\mathbb{R}}$ has to be irreducible. If $<\chi, \chi>= 1$ then $V \otimes_{\mathbb{R}} \mathbb{C}$ is irreducible, as we have observed above, but that is impossible as V itself is a submodule, and so we are left with $<\chi, \chi>= 2$ - the complex case.

We have now shown how we can go back and forth between irreducible complex modules and real. The real irreducible are classified into type $1, 2, 4$ (real, complex, quaternionic respectively), while we have a similar classification of complex irreducible in terms of real (real character and split over real), quaternionic (real character and irreducible over reals) and finally complex (non-real character). Thus real corresponds to real. (A real real gives a real complex upon extension of scalars, and a real complex gives rise to a real real when restricting). Similarly a complex real give rise to a pair of conjugate complex complex, each of which when restricted give a complex real). Finally a quaternionic real give rise to a quaternionic complex, which in its turn becomes a quaternionic real when scalars are restricted.

What we finally need to know to tie things together is to connect this classification of complex irreducibles to the one pertaining to invariant forms. One part is clear. Real and Quaternionic are characterized by the existence of invariant bilinear forms, and we need only to make the right connection between symmetric and alternate.

First we note that if V can be realised over the reals, then we can cook up an invariant symmetric form, by taking the average of an arbitrary positive definite quadratic form on the real part (as we have noted above). This survives by tensoring over the complexes.

Now assume that the irreducible complex representation is quaternionic. This allows us multiplication by j and k in addition to i, and in fact real linear combinations of those (or equivalently complex multiplications of j) make up elements α such that $\alpha i = -i\alpha$. Now pick an invariant Hermitian form $H(x,y)$ and set $B(x,y) = H(x,\alpha y)$. Clearly $B(x,y)$ is invariant bilinear. Furthermore the form $H(\alpha x, \alpha y)$ is anti-linear in the first variable, thus by uniqeuness of H we have $H(\alpha x, \alpha y) = \lambda \overline{H(x,y)}$, normalizing α so that $\alpha^2 = -1$ we note by repeated application of the identity that $|\lambda| = 1$. By multiplying α with a suitable complex number of lenngth one, we may assume $\lambda = 1$, it is then natural to set $j = \alpha$. Now

$$-H(jx,x) = H(jx,-x) = \overline{H(x,jx)} = H(jx,x)$$

where the last equality follows from H being Hermitian. This implies that $B(x,x) = 0$ and thus B is alternating.

This allows us to identify quaternionic with alternating forms, real with symmetric forms (and complex with Hermitian), and the picture is complete. It is now a straightforward exercise to produce the irreducible real characters from having the complex character table.

Ex 14. Real irreducible representations of Abelian groups are either real or complex (i.e. quaternionic do not appear). Thus they are either 1-dimensional (in case of real characters (taking the values ± 1)) or 2-dimensional (being the sum of two conjugate characters).

Ex 15. The quaternionic group $\{\pm 1, \pm i, \pm j, \pm k\}$ of eight elements has five representations. Its center is given by ± 1, and the quotient is simply $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ which gives rise to four 1-dimensional representations. Remains a 2-dimensional representation. This is not defined over the reals, but the sum of itself, gives rise to a quaternionic real representation. In fact the quaternions, and *a fortiori* the quaternionic group, can be represented by 4×4 real matrices.

It is a nice exercise to show directly that the existence of a symmetric invariant form defines a real subspace, as well as the existence of an alternating form allows a quaternionic structure.