Pencil of Conics

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Introduction

All conics in the plane make up a P^5 as there are six quadratic monomials in three variables. The singular conics form a cubic hypersurface referred to as the discriminant, whose singular locus consists of all double lines and make up 2-dimensional subvariety of the discriminant, in fact isomorphic with P^2 via the Veronese embedding.

In fact there is a 1-1correspondence between trinary quadratic forms and symmetric 3×3 matrices. The singular ones correspond to singular matrices, given by the determinant, and hence by a cubic equation. It can be parametrized by $P^2 \times P^2$ as follows.

$$(x_0, x_1, x_2) \times (y_0, y_1, y_2) \mapsto \begin{pmatrix} x_0y_0 & \frac{x_0y_1+x_1y_0}{2} & \frac{x_0y_2+x_2y_0}{2} \\ \frac{x_0y_1+x_1y_0}{2} & x_1y_1 & \frac{x_1y_2+x_2y_1}{2} \\ \frac{x_0y_2+x_2y_0}{2} & \frac{x_1y_2+x_2y_1}{2} & x_2y_2 \end{pmatrix}$$

where the map factors through the obvious involution ι (switching x and y). Its singular locus is given by all symmetric matrices of rank one, and those can easily be parametrized by P^2 as follows

$$(x_0, x_1, x_2) \mapsto \begin{pmatrix} x_0^2 & x_0x_1 & x_0x_2 \\ x_0x_1 & x_1^2 & x_1x_2 \\ x_0x_2 & x_1x_2 & x_2^2 \end{pmatrix}$$

which also becomes the singular locus of $P^2 \times P^2/\iota$.

If we consider the complex and real cases, we can easily compute the euler-numbers and for finite fields actually the number of points.

	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{F}_q$
conics	6	1	$q^5 + q^4 + q^3 + q^2 + q + 1$
singular	6	1	$\frac{(q^2+q+1)(q^2+q+2)}{2}$
supersingular	3	1	$q^2 + q + 1$

Pencils

By a pencil is meant a 1-dimensional linear subspace of conics. If given two elements in the pencil Q_0 and Q_1 we can write the elements of the pencil in the form $t_0Q_0 + t_1Q_1$. The elements of the pencil are parametrized by P^1.

The intersection $Q_0 \cap Q_1$ the so called base locus constitute the so called base points and consists of four points over an algebraically closed field. The map $P^2 \to P^1$ given by Q_1/Q_0 is defined outside the base points. And its level curves will be the conics given by the quadrics in the pencil.

We see that through each point outside the base locus there is exactly one member of the pencil passing through it.

The generic pencil has four distinct base points over an algebraically closed field, and it has three distinct singular fibers. This is clear because the discriminant has degree three. It can also be seen geometrically. Given four points we can find six lines through them, which can be paired off two and two into three line-pairs, such that there are exactly one of each line pair through a given base point.

Thus each generic pencil determines four distinct point, conversely every set of four disjoint points no three of which lie on a line, determine a pencil, because passing through a point imposes one linear condition on the 5-dimensional space of conics.

Thus the pencil is given by all the conics which pass through its four base points.

Furthermore we note that the tangents at a base-point vary with the members. In fact if P is a base-point and L is a line through it, there is exactly one member in the pencil having L as the tangent-line through P. If the line L is made to pass through one additional base-point (there are $3 = 4 - 1$ such choices, the corresponding conic will be singular.)

Digression:

A non-singular conic can be parametrized by binary quadrics. The intersection of two conics hence give rise to a quartic equation.

Ex The conic $x^2 + y^2 = z^2$ is parametrized by $((s^2 - t^2), 2st, (s^2 + t^2))$ if we intersect it with the conic $2x^2 + xy + yz - z^2 = 0$ say we get the quartic $s^4 + 4s^3t - 6s^2t\neg 2 + t^4$. Finding its four roots will give us the explicit base points.

As a conic is isomorphic to P^1 its group of automorphism is isomorphic to $PSL(2, \mathbb{C})$ (or more generally $PSL(2, k]$ with base-field k). This is nothing else then the group of Moebius transformations. This group acts triply transitive, thus any three points on P^1 are projectively equivalent. This is not true for four points. Given four points, we can normalize three of them to $0, 1, \infty$. A permutation of those three points gives rise to a Moebius transformation, and thus we get a representation of S_3 into that group. This can easily be written down explicitly as

$$1, 1 - z, \frac{1}{z}, \frac{z}{z-1}, \frac{1}{1-z}, \frac{z-1}{z}$$

the value λ of the fourth point is only determined up to that orbit. However we can write down the j-invariant

$$j(\lambda) = \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(1-\lambda)^2}$$

which is invariant under those transformations and in fact gives a complete invariant as to when two sets of four points are actually projectively equivalent[0].

[0]The careful reader may wonder what happens if we chose another set of three points as normalized to $=, 1, \infty$ and can easily work out to his or her conviction that nothing happens. E.g.

This ties up with the notion of the cross-ratio of four points on a line, that goes back ultimately to euclid. Given $z_1, z_2; z_3, z_4$ for points and associate to it the double ratio

$$\frac{z_1 - z_3}{z_2 - z_3} \Big/ \frac{z_2 - z_3}{z_2 - z_4} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}$$

Permutation of the four variables will give permutation of the values according to the representation of S_3 above and hence a surjective homomorphism $S_4 \to S_3$ whose kernel is the Klein Viergruppe represented by the involutions $(z_1 z_2)(z_3 z_4), (z_1 z_4)(z_2 z_3)$ and $(z_1 z_3)(z_2 z_4)$.

Furthermore as we have seen, any four points in the plane give rise to three pairs of two lines, namely the three singular fibers of a pencil. Hence any permutation of the four points will automatically give a permutation of the three singular fibers. This peculiar fact is the reason why we can reduce the solution of an equation of fourth degree to one of three. In practice any fourth degree equation could be achieved through a pencil of conics, and given such we could find the three singular ones, which is equivalent to solving a third degree equation. Given the singular ones, it is only a question of solving linear equations to get the base points.

Ex: Suppressing projective notation for simplicity. Consider the parametrization $(1, t^2.t)$ which parametrizes $C_0 = (XY - Z^2 = 0)$. Consider the quadric $C_1 = (aX^2 + bXZ + cZ^2 + dYZ + Y^2 = 0)$ which gives rise to the polynomial $a + bt + ct^2 + dt^3 + t^4$. For what s is $C_0 + sC_1$ singular? The associated symmetric matrix associated to the conic is

$$\begin{pmatrix} sa & \frac{1}{2} & \frac{bs}{2} \\ \frac{1}{2} & s & \frac{ds}{2} \\ \frac{bs}{2} & \frac{ds}{2} & cs - 1 \end{pmatrix}$$

the determinant will be a cubic equation in s. Given a specific root we need only to decompose a singular quadric into two lines. An explicit example of a singular quadric is given by

$$x^2 - y^2 + 2z^2 + 3xz + yz$$

how do we factor it? Take the successive partials $2x + 3z, -2y + z, 3x + y + 4z$ they have a non-trivial common zero (the singularity of the conic) where the two legs of it intersect. It is easy to find $(3, -1, -2)$. Now look at the pencil of lines through that point, say e.g. spanned by $2x + 3z$ and $2y - z$ and find the intersection with the quadric. Look at $(2x + 3z) - s(2y - z) = 2x - 2sy + (3 + s)z$ and plug in $x = sy - \frac{(}{3} + s)2z$ and we get a binary quadric in y, z with coefficients in s. Explicitly

$$(s^2 - 1)y^2 - (s^2 - 1)yz + \frac{s^2 - 1}{4} z^2$$

from which we conclude that $s = \pm 1$ give the two lines $2x - 2y + 4z$ and $2x + 2y + 2z$ and the factorization $(x + y + z)(x - y + 2z)$

Now given a conic C in the pencil, there are two sets of four points on a P^1, namely the four base-points on C itself and the four points in the parameters space of the pencil, given by C itself and the three singular fibers. Those two sets of four give the same j-invariant!

The proof is easy. Chose one base-point P and let the point Q vary on C. Consider the line L passing through P and Q and let C_Q be the member of the pencil tangent to L at P. In this way we get a map from C to the pencil itself. When Q is one of the other base-points C_Q is singular, if it is P then the line L should be the tangent of C at P and thus P as the fourth point is mapped to C itself as a member of the pencil.

We can classify all the possible pencils over $\mathbb{C}$ as follows

if $0, \infty$ are left invariant then the transformation is of form $z \mapsto az$ if $\lambda \mapsto 1$ then $a = 1/\lambda$ and $1 \mapsto 1/\lambda$

type	base-locus	singular members
I	$1,1,1,1$	$s \quad s \quad s$
II	$2,1,1$	$2s \quad s$
III	$3,1$	$3s$
IV	$2,2$	$ss \quad s$
V	4	$2ss$

Where s means a singular conic (i.e. two distinct lines) and ss denotes a supersingular conic (i.e. a double line). In case I as noted the pencil intersects the discriminant transversally away from the singular locus, in case II it is tangent at one point to the discriminant in case III it is flexed to it while in IV it intersects the singular locus of the discriminant transversally and in the last case V it is actually tangent to it.

To complete the story one also needs to list the cases when the pencil only consists of singular members. In one case we have a secant of the supersingular locus, the base locus then consists of a single point, the common singular point of all the members, and there will be exactly two supersingular conics in it. In the second case we have a fixed line, and a pencil of lines, and we have two cases whether or not the base point of that pencil is lying on the fixed line or not. In the first case we have exactly one double line in the second none.

Let us look at the non-generic pencils a bit more closely.

Non-generic pencils

In case II we have a double point P of the base locus $2P, Q, , R$. Through this point all the members of the pencil are tangent, and one member C_0 actually has a singular point there, and is one of the two singular members. Thus C_0 consists of the two lines PQ, PR, while the other singular member C_1 consists of QR and the common tangent T to all the members of the pencil.

If we perturb this pencil slightly so P splits up in nearby P, P' the fiber C_0 splits up in two nearby fibers $PQ, P'R$ and $PR, P'Q$ thus we see that C_0 is a double fiber and corresponds to the tangent point on the discriminant locus.

Thus a type II pencil is simply a line in P^5 that is tangent to the smooth part of the discriminant locus at one point (C_0).

Pencils of type II are given by three points along with a line through one of the points, and its members the conics passing to the three point tangent to the given line at the assigned double point.

Ex: We may choose $C_0 = xy$ and $C_1 = (x-y)(x+y+z)$ the three base points will be $(0,0,1)$ with multiplicity 2 and $(-1, 0.1], (0,-1,1)$ and the common tangent given by $x = y$.

In case III we have a triple point P of the base locus $3P, Q$. Through this point all the conics will be flexed to each other, and there can only be one singular fiber C_0 corresponding to the common tangent T to all the conics, and the line PQ.

The pencil will be flexed to the discriminant locus at the point C_0

Ex: We may choose a conic $C = xy - z^2$ which is tangent to the line $x = 0$ at the point $(0,1,0)$ then chose $C_0 = x(x-z)$. The conics of the pencil will all be flexed to each other at $(0,1,0)$ explicitly we have that C and $D = x^2 - xz + yz - z^2$ are flexed to each other.

In case IV we have two double points P, Q of the base locus $2P, 2Q$. The conics are all tangent to each other at P, Q. There will be two singular fibers. C_1 will correspond to the union of the two tangents at P, Q respectively, while C_0 will correspond to the double line passing through P, Q and will thus be a super-singular conic. If we wiggle P and Q and getting two nearby points P, P' and Q, Q' respectively we see how two regular singular fibers come together to form the double line.

The pencil will be a line that passes through the singular locus of the discriminant.

Ex: We may chose $C_1 = xy$ and $C_0 = (x+y+z)^2$ the two non-singular conics $D_\pm = x^2 + y^2 + z^2 + 2yz + 2xz + (\pm 1)xy$ are hence bitangent at the points $(0,-1,1), (-1,0.1)$ respectively with the x, y co-ordinate axi as common tangents.

In case V we have one point P with multiplicity four in the base locus $4P$. There is only one singular fiber C_0 which is a double line, where the line is tangent to all the conics in the pencil. All the conics are super-flexed at their one common point P.

The pencil is a line which is tangent to the singular strata of the discriminant locus, which it only hits once.

Ex: Let $C_0 = x^2$ and $C = xy - z^2$ all the conics $tx^2 + xy - z^2$ are all superflexed to each other.

As to the pencils only consisting of singular fibers we have just one with a finite base locus, namely the pencil given by a secant to the singular locus. Such a secant is necessarily contained in the discriminant locus, as a line can only meet in three points unless contained.

Ex: Let $C_0 = x^2$ and $C_1 = y^2$ the conics $sx^2 + ty^2$ always split into two lines. The base point has multiplicity four.

In all the other cases there will be a fixed line as a component. The pencil hence decomposes as a fixed line and a so called moving part which is a pencil of lines. A pencil of lines is given by all the lines through a fixed point. We have two cases, whether that fixed point lies on the fixed component or not. The latter case is characterized by there being a double line in the pencil.

Fixing a line and moving a pencil gives a linear map from P^1 into the discriminant locus. All the lines of the discriminant locus are of this type. Some of those lines meet the singular locus in a single point, most do not.–

Blowing up

The Real case

In the real case not all base points have to be real, but can split up in conjugate pairs. Also there will be two types of singular conics. Those consisting of two real lines, and those consisting of two conjugate lines, in the latter case only the singular point will be visible as real.

Thus our complex classification will split up into sub cases. Those are easy to work out and can most conveniently be listed in a table. The notation should be self-explanatory (P,$\bar{\text{P}}$) denotes a conjugate pair, $\mathbb{I}$ denotes a double line, X a union of two distinct real lines, o a union of two conjugate lines meeting in a real singular point, and $o \leftrightarrow o$ means that the two singular fibers are complex conjugate, as will be their singular points, and hence completely invisible over the reals.

type	branch locus	# b.points	singular fibres
Ia	P,Q,R,S	4	$X \quad X \quad X$
Ib	(P,$\bar{\text{P}}$),R,S	2	$X \quad o \leftrightarrow o$
Ic	(P,$\bar{\text{P}}$),(R,$\bar{\text{R}}$)	0	$X \quad o \quad o$
IIa	2P,Q,R	3	$2X \quad X$
IIb	2P,(Q$\bar{\text{Q}}$)	1	$2o \quad X$
III	3P,Q	2	$3X$
IVa	2P,2Q	2	$\mathbb{I} \quad X$
IVb	(2P,2$\bar{\text{P}}$)	0	$\mathbb{I} \quad o$
V	4P	1	$2\mathbb{I}$

Note that when there are no real base points we can form the rational function $\Phi = Q_1/Q_0$ which will be defined on $\mathbb{R}P^2$ and the members of the pencil will simply be the level curves.

Of particular interest is Ic. The X singular fiber disconnects the real projective plane into two connected open sets, each of which can be thought of as a pointed Moebius strip. Namely instead of taking a rectangle and identify the short ends after a twist, we can make those pointed and do the same, i.e. twisting and identifying the two points. If we normalize the function Φ (which amounts to composing with a Moebius transformation) by letting the X fiber correspond to the level curve of zero (we chose Q_1 to define that singular quadric), the function will be positive on one component and negative on the other. The two o fibers correspond to maximum and minima respectively, so in particular we see that the image of Φ cannot be the entire real projective line, but a compact subinterval. While the singular point of Q_1 will of course be a saddle point. Examples are given by choosing two disjoint ellipses, the extremal points will lie in the respective interiors of those.

A circle can be defined as a conic passing through the two so called circular points $(1, \pm i, 0)$ at the line at infinity. Thus circles make up a linear system, the degenerate members of which are the line at infinity and an arbitrary line (circle with infinite radius) and conjugate lines passing through them, which can identified with points in the plane (circles with radius zero).

Any pencil spanned by two circles will consist of circles, except the singular ones.

If the two circles are disjoint we will get case Ic above or IVb in case the circles are concentric, with the line at infinity as double line.

Ex: Let us work out an explicit example.

Two tangent circles give rise to a pencil of type IIb the X fiber is given by the common tangent and the line at infinity. Blowing up the base point we get a Klein bottle and a rational map to a circle, with circular fibers except one.